Filomat 30:13 (2016), 3525–3546
DOI 10.2298/FIL1613525L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Comultiplication Structures for a Wedge of Spheres

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Abstract. In this paper, we consider the various sets of comultiplications of a wedge of spheres and provide some methods to calculate many kinds of comultiplications with different properties. In particular, we concentrate on studying to compute the number of comultiplications, associative comultiplications, commutative comultiplications, and comultiplications which are both associative and commutative of a wedge of spheres. The more spheres that appear in a wedge, the more complicate the proofs and computations become. Our methods involve the basic Whitehead products in a wedge of spheres and the Hopf-Hilton invariants.

1. Introduction

In this paper we assume that all spaces are based and have the based homotopy type of based and connected CW-complexes. All maps and homotopies preserve the base point. Unless otherwise stated, we do not distinguish notationally between a map and its homotopy class. Thus equality of maps means equality of homotopy classes of maps; that is, we use '=' for the homotopy '$\simeq$'.

A pair (Y, φ) consisting of a space Y and a function $\varphi : Y \to Y \vee Y$ is called a *co-H-space* if $q_1\varphi = 1$ and $q_2\varphi = 1$, where q_1 and q_2 are the projections $Y \vee Y \to Y$ onto the first and second summands of the wedge product and 1 is the identity map of Y. In this case the map $\varphi : Y \to Y \vee Y$ is called a *comultiplication*. Equivalently, (Y, φ) is a co-H-space if $j\varphi = \Delta : Y \to Y \times Y$, where Δ is the diagonal map and $j : Y \vee Y \to Y \times Y$ is the inclusion. A comultiplication φ is called *associative* if $(\varphi \vee 1)\varphi = (1 \vee \varphi)\varphi : Y \to Y \vee Y \vee Y$, and φ is called *commutative* if $T\varphi = \varphi$, where $T : Y \vee Y \to Y \vee Y$ is the switching map. A *left inverse* for φ is a map $L : Y \to Y$ such that $\nabla(L \vee 1)\varphi = 0$ and a *right inverse* is a map $R : Y \to Y$ such that $\nabla(1 \vee R)\varphi = 0$, where $\nabla : Y \vee Y \to Y$ is the folding map and 0 is the constant map.

Co-H-spaces, dual of H-spaces, play a fundamental role in homotopy theory. If the comultiplication $\varphi : Y \to Y \vee Y$ is homotopy associative and has a right and left inverse, then for every space Z the set $[Y, Z]$ of homotopy classes from Y to Z becomes a group with the group operation depending on the comultiplication of a space. The important examples of co-H-spaces are all n-spheres, $n \geq 1$ and suspensions of a based space. It is easily seen that the wedge of two co-H-spaces is a co-H-space, and therefore it is natural to

2010 *Mathematics Subject Classification*. Primary 55Q25; Secondary 55Q15, 55P40, 52C45, 68U05
Keywords. Co-H-spaces; comultiplications; homotopy associativity; homotopy commutativity; basic Whitehead products; the Hopf-Hilton invariants; the Hilton's theorem
Received: 22 October 2014; Accepted: 03 March 2015
Communicated by Ljubiša D.R. Kočinac
This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2015R1D1A1A09057449).
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ask about the comultiplications on a wedge of spheres. It turns out that this set of comultiplications is complicated - there are usually many comultiplications (sometimes infinitely many) with many different properties. For example, $S^2 \vee S^5$ has infinitely many homotopy classes of comultiplications and commutative comultiplications. However, it has only 2 homotopy associative comultiplications [5]. Some indication of this complexity appeared in an early paper of Ganea [10, pp. 194-196] who gave an intricate argument to show that $S^3 \vee S^{15}$ has at least 72 associative comultiplications and at most 56 homotopy classes of suspension comultiplications.

The study of comultiplications of a co-H-space has been carried out by several authors (see [6], [7], [3], [14], [4] and [15]). One can find interesting examples of a co-H-space which is not homotopically equivalent to a suspension [9] and of a co-H-space $X = S^3 \cup_f e^{2p+1}$ which does not admit an associative comultiplication [8], where $f \in \pi_{2p}(S^3)$ and p is an odd prime. In [11] the set of homotopy classes of comultiplications of a wedge of two Moore spaces was investigated as a natural extension of the earlier work [2]. It is worth while examining the comultiplication structures on the wedge of Moore spaces for further studies.

In this paper we study the set of comultiplications on a wedge of k-spheres for $k \geq 2$. In particular, we focus on the study of the properties and cardinalities of comultiplications, associative comultiplications, commutative comultiplications, and comultiplications which are both associative and commutative of a wedge of spheres which is much more complicate than a wedge of two spheres. More precisely, let $|S|$ be the number of elements in a set S, and let $\mathcal{A}(Y)$ and $\mathcal{CO}(Y)$ be the sets of homotopy classes of associative comultiplications and commutative comultiplications, respectively, of a wedge of spheres $Y = S^n \vee S^m \vee S^p$ with $m \leq 2n-2$ and $2 \leq n < m < p \leq 4n-4$. Then we can show in Theorem 3.15 and Theorem 4.7 that if m and n are even, then

$$|\mathcal{A}(Y)| = |\pi_p(S^{2m-1})| \times |\pi_p(S^{2n-1})| \times |\pi_p(S^{m+n-1})|^2 \times |\pi_p(S^{2m+n-2})| \times |\pi_p(S^{m+2n-2})|;$$

and

$$\begin{aligned} |\mathcal{CO}(Y)| &= |\pi_p(S^{2m-1})| \times |\pi_p(S^{2n-1})| \times |\pi_p(S^{3m-2})| \times |\pi_p(S^{3n-2})| \\ &\quad \times |\pi_p(S^{m+n-1})| \times |\pi_p(S^{2m+n-2})|^3 \times |\pi_p(S^{m+2n-2})|^2. \end{aligned}$$

The main results of this paper also show that the cardinalities of $\mathcal{A}(Y)$ and $\mathcal{CO}(Y)$ depend on the homotopy structures of a wedge of spheres and the parity of m or n, or both. For examples, $|\mathcal{A}(S^8 \vee S^{12} \vee S^{27})| = 32$ and $|\mathcal{CO}(S^8 \vee S^{12} \vee S^{27})| = 16$.

The paper is organized as follows: In Section 2, we introduce the Hilton's theorem and the Hopf-Hilton invariants. We establish basic facts concerning the various types of comultiplications on a wedge of k-spheres and give general conditions for comultiplications to be associative or commutative in the case of a wedge of three spheres. In Section 3, we define certain general comultiplications on a wedge of three spheres and determine when they are associative. In Section 4.7, we prove the fundamental properties of the commutative comultiplications which are defined in Section 3. The examples will be listed at the end of each section.

2. Comultiplication Structures

We first consider a wedge of spheres $Y = S^{n_1} \vee \cdots \vee S^{n_k}$. Let $\zeta_j : S^{n_j} \to Y$ be the inclusion for $j = 1, \ldots, k$. We define and order the *basic Whitehead products* as follows: Basic products of weight 1 are $\zeta_1, \ldots, \zeta_k$ which are ordered by $\zeta_1 < \cdots < \zeta_k$. Assume that the basic products of weight $< n$ have been defined and ordered so that if $r < s < n$, any basic product of weight r is less than all basic products of weight s. We define inductively a basic product of weight n by a Whitehead product $[a, b]$, where a is a basic product of weight k and b is a basic product of weight l, $k + l = n$, $a < b$, and if b is a Whitehead product $[c, d]$ of basic products c and d, then $c \leq a$. The basic products of weight n are then ordered arbitrarily among themselves and are greater than any basic product of weight $< n$. Suppose ζ_j occurs l_j times, $l_j \geq 1$ in the basic product w_v. Then the *height* h_v of the basic product w_v is $\sum l_j(n_j - 1) + 1$.

There is the Hilton's theorem [12] concerning the basic Whitehead products as follows:

Theorem 2.1. *Let the basic products of $Y = S^{n_1} \vee \cdots \vee S^{n_k}$ be $w_1, w_2, \ldots, w_v, \ldots$ (in order) with the height of w_v being h_v. Then for every m,*

$$\pi_m(Y) \cong \bigoplus_{v=1}^{\infty} \pi_m(S^{h_v}).$$

The isomorphism $\theta : \bigoplus_{v=1}^{\infty} \pi_m(S^{h_v}) \to \pi_m(Y)$ is defined by

$$\theta|\pi_m(S^{h_v}) = w_{v*} : \pi_m(S^{h_v}) \to \pi_m(Y).$$

Let $\varphi : S^n \to S^n \vee S^n$ be the standard comultiplication. Then we have the induced homomorphism $\varphi_* : \pi_m(S^n) \to \pi_m(S^n \vee S^n)$ between homotopy groups and by the Hilton's theorem

$$\pi_m(S^n \vee S^n) \cong \pi_m(S^n) \oplus \pi_m(S^n) \oplus \pi_m(S^{2n-1}) \oplus \pi_m(S^{3n-2}) \oplus \cdots .$$

Let h_v be the height of the vth basic product $w_v \in \pi_{h_v}(S^n \vee S^n)$ and let $pr_v : \pi_m(S^n \vee S^n) \to \pi_m(S^{h_v})$ be the projection onto the vth summand for $v = 1, 2, 3, \ldots$.

We define the *Hopf-Hilton invariant* $H_t : \pi_m(S^n) \to \pi_m(S^{h_{t+3}})$ by

$$H_t = pr_{t+3} \circ \varphi_*, \quad t = 0, 1, 2, \ldots$$

as in [12]. We rewrite these invariants as follows: Set

$$\begin{aligned} H_1^1 &= H_0 : \pi_m(S^n) \to \pi_m(S^{2n-1}) \\ H_1^2 &= H_1, H_2^2 = H_2 : \pi_m(S^n) \to \pi_m(S^{3n-2}) \end{aligned}$$

and so on. Thus we have the Hopf-Hilton invariants

$$H_1^l, H_2^l, \ldots, H_{t_l}^l : \pi_m(S^n) \to \pi_m(S^{(l+1)n-l}),$$

for $l = 1, 2, 3, \ldots$, where t_l is the number of basic products of length l.

We now describe the Hilton's formulas [12, Theorems 6.7 and 6.9] as follows:

(1) If Z is any space and $\beta, \gamma \in \pi_r(Z)$ and $\alpha \in \pi_m(S^r)$, then

$$\begin{aligned} (\beta + \gamma)\alpha \quad &= \beta\alpha + \gamma\alpha + [\beta, \gamma]H_1^1(\alpha) \\ &\quad + [\beta, [\beta, \gamma]]H_1^2(\alpha) + [\gamma, [\beta, \gamma]]H_2^2(\alpha) + \ldots . \end{aligned}$$

(2) Let $a \in \pi_n(S^r)$, $b \in \pi_r(Z)$, and let m be an integer. Then, if r is odd,

$$mb \circ a = \begin{cases} m(b \circ a) & \text{for m = 4s or 4s + 1,} \\ m(b \circ a) + [b, b] \circ H_1^1(a) & \text{for m = 4s − 2 or 4s − 1.} \end{cases}$$

If r is even,

$$\begin{aligned} mb \circ a \quad &= m(b \circ a) + \tfrac{m(m-1)}{2}[b, b] \circ H_1^1(a) \\ &\quad + \tfrac{(m+1)m(m-1)}{3}[b, [b, b]] \circ H_1^2(a). \end{aligned}$$

From now on, we let $X = S^n \vee S^m$ and $Y = S^n \vee S^m \vee S^p$ with $2 \leq n < m < p$ unless otherwise stated, and we fix the following notations:

- $i_1, i_2 : S^n \to S^n \vee S^n$ are the first and the second inclusions, respectively.
- $j_1, j_2 : S^m \to S^m \vee S^m$ are the first and the second inclusions, respectively.
- $\alpha_1, \alpha_2 : S^n \to S^n \vee S^m \vee S^n \vee S^m (= X \vee X)$ are the first and the third inclusions, respectively.
- $\beta_1, \beta_2 : S^m \to S^n \vee S^m \vee S^n \vee S^m (= X \vee X)$ are the second and the fourth inclusions, respectively.
- $r : S^n \to Y$, $s : S^m \to Y$, $t : S^p \to Y$ and $A : S^n \vee S^m \to Y$ are the inclusions.

- $pr_1, pr_2 : (S^n \vee S^m) \vee (S^n \vee S^m) \to S^n \vee S^m$ are the first and the second projections, respectively.
- $\iota_1, \iota_2 : Y \to Y \vee Y$ are the first and the second inclusions, respectively.
- $q_1, q_2 : Y \vee Y \to Y$ are the first and the second projections, respectively.
- $I_1, I_2, I_3 : Y \to Y \vee Y \vee Y$ are the first, the second and the third inclusions, respectively.
- $J_{12}, J_{23} : Y \vee Y \to Y \vee Y \vee Y$ are the maps defined by $J_{12}(z) = (z, *)$ and $J_{23}(z) = (*, z)$, where $z \in Y \vee Y$.
- $T : Y \vee Y \to Y \vee Y$is the switching map.

Then the following are straightforward:

- $(r \vee r)i_1 = \iota_1 r,\ (r \vee r)i_2 = \iota_2 r,\ (s \vee s)j_1 = \iota_1 s,\ (s \vee s)j_2 = \iota_2 s.$
- $(A \vee A)\alpha_j = \iota_j r,\ (A \vee A)\beta_j = \iota_j s$ for $j = 1, 2$.
- $J_{12}\iota_j = I_j$ for $j = 1, 2$.
- $(\varphi \vee 1)\iota_1 = J_{12}\varphi$ and $(\varphi \vee 1)\iota_2 = I_3$.
- $J_{23}\iota_1 = I_2$ and $J_{23}\iota_2 = I_3$.
- $(1 \vee \varphi)\iota_1 = I_1$ and $(1 \vee \varphi)\iota_2 = J_{23}\varphi$.

We now order the basic products of weight 1 as follows:

- $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$ in $S^n \vee S^m \vee S^n \vee S^m$.
- $\iota_1 r < \iota_1 s < \iota_2 r < \iota_2 s$ in $Y \vee Y$.
- $I_1 r < I_1 s < I_2 r < I_2 s < I_3 r < I_3 s$ in $Y \vee Y \vee Y$.

For the moment, more generally, let $Y = S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k}$, $X = S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_{k-1}}$, $2 \le n_1 < n_2 < \cdots < n_k$ and let $\zeta_i : S^{n_i} \to Y$ be the inclusion for each $i = 1, 2, \ldots, k$ and $pr_1, pr_2 : X \vee X \to X$ the first and the second projections, respectively. Then every comultiplication $\varphi : Y \to Y \vee Y$ has the following form

$$\begin{cases} \varphi|_{S^{n_1}} &= \iota_1\zeta_1 + \iota_2\zeta_1, \\ \varphi|_{S^{n_2}} &= \iota_1\zeta_2 + \iota_2\zeta_2 + P_{n_2}, \\ \quad\vdots & \\ \varphi|_{S^{n_k}} &= \iota_1\zeta_k + \iota_2\zeta_k + P_{n_k} \end{cases}$$

for some $P_{n_i} \in \pi_{n_i}(Y \vee Y)$ such that $q_1 P_{n_i} = q_2 P_{n_i} = 0$ for each $i = 2, 3, \ldots, k$. Conversely, if $P_{n_i} : S^{n_i} \to Y \vee Y$ is any map such that $q_1 P_{n_i} = q_2 P_{n_i} = 0$ for each $i = 2, 3, \ldots, k$, then φ defined above is a comultiplication.

Therefore, we can define a comultiplication as follows:

Definition 2.2. Let $P_{n_i} \in \pi_{n_i}(Y \vee Y)$ be any map such that $q_1 P_{n_i} = 0 = q_2 P_{n_i}$ for each $i = 2, 3, \ldots, k$. We define a comultiplication $\varphi : Y \to Y \vee Y$ by

$$\begin{cases} \varphi|_{S^{n_1}} &= \iota_1\zeta_1 + \iota_2\zeta_1, \\ \varphi|_{S^{n_2}} &= \iota_1\zeta_2 + \iota_2\zeta_2 + P_{n_2}, \\ \quad\vdots & \\ \varphi|_{S^{n_k}} &= \iota_1\zeta_k + \iota_2\zeta_k + P_{n_k}. \end{cases}$$

The element $P_{n_i} \in \pi_{n_i}(Y \vee Y)$ is called the *ith perturbation* of φ for $i = 2, 3, \ldots, k$. We call $P = (P_{n_2}, P_{n_3}, \ldots, P_{n_k})$ the *perturbation* of φ.

It can be seen that the comultiplication φ in Definition 2.2 is commutative if and only if $TP_{n_i} = P_{n_i}$ for each $i = 2, 3, \ldots, k$; and it is associative if and only if

$$J_{12}P_{n_i} + (\varphi \vee 1)P_{n_i} = J_{23}P_{n_i} + (1 \vee \varphi)P_{n_i}$$

for each $i = 2, 3, \ldots, k$.

Let $A : X \to Y$ be the inclusion. Then we have

Lemma 2.3. *If $P_{n_k} \in \pi_{n_k}(Y \vee Y)$ satisfies $q_1P_{n_k} = 0 = q_2P_{n_k}$, then there exists a unique $W \in \pi_{n_k}(X \vee X)$ such that $P_{n_k} = (A \vee A)W$ and $pr_1W = 0 = pr_2W$.*

Proof. We consider the following cofibration sequence:

$$X \vee X \xrightarrow{A \vee A} Y \vee Y \xrightarrow{q \vee q} S^{n_k} \vee S^{n_k} ,$$

where $q : Y \to S^{n_k}$ is the projection. Since $X \vee X$ is $(n_1 - 1)$-connected and $S^{n_k} \vee S^{n_k}$ is $(n_k - 1)$-connected, by the Blakers-Massey Theorem (see [13] and [17]), the top line in the following commutative diagram is exact

$$\begin{tikzcd}
\pi_{n_k}(X \vee X) \arrow[r, "(A\vee A)_*"] \arrow[d, "f_*"] & \pi_{n_k}(Y \vee Y) \arrow[r, "(q\vee q)_*"] \arrow[d, "g_*"] & \pi_{n_k}(S^{n_k} \vee S^{n_k}) \arrow[d, "e_*", "\cong"'] \\
\pi_{n_k}(X \times X) \arrow[r, "(A\times A)_*"] & \pi_{n_k}(Y \times Y) \arrow[r, "(q\times q)_*"] & \pi_{n_k}(S^{n_k} \times S^{n_k}),
\end{tikzcd}$$

where f_*, g_* and e_* are homomorphisms induced by inclusion maps. We can see that e_* is an isomorphism. If $P_{n_k} \in \pi_{n_k}(Y \vee Y)$ is such that $q_1P_{n_k} = 0 = q_2P_{n_k}$, then $g_*(P_{n_k}) = 0$ and so $e_*(q \vee q)_*P_{n_k} = 0$. Therefore $(q \vee q)_*P_{n_k} = 0$ and hence, by exactness, there exists a unique $W \in \pi_{n_k}(X \vee X)$ such that $P_{n_k} = (A \vee A)_*W$ since $(A \vee A)_*$ is a monomorphism. Finally, since $(A \times A)_*f_*W = 0$ and $(A \times A)_*$ is a monomorphism, it follows that $f_*W = 0$. Thus we have $pr_1W = 0 = pr_2W$ since $\pi_{n_k}(X \times X) \cong \pi_{n_k}(X) \oplus \pi_{n_k}(X)$. □

We note that there are various types of wedges of spheres and commutative diagrams satisfying the statements in the proof of Lemma 2.3. Fortunately, all kinds of wedges of spheres and commutative diagrams are condensed into the above case by using the fact that the inclusion map from a wedge of spheres to another induces a monomorphism between homotopy groups.

The computation of comultiplications on a wedge of spheres is very complicated. So we restrict our investigation to a wedge of three spheres which is much more complicate than the wedge of two spheres. From now on, in the case of a wedge of three spheres, we use the notation $Y = S^n \vee S^m \vee S^p$ instead of $S^{n_1} \vee S^{n_2} \vee S^{n_3}$, *i.e.*, $n = n_1, m = n_2, p = n_3$. And we also use the inclusions $r : S^n \to Y$, $s : S^m \to Y$ and $t : S^p \to Y$ instead of ζ_1, ζ_2 and ζ_3, respectively, as previously mentioned.

Let W be the set of all basic Whitehead products of $S^n \vee S^m \vee S^n \vee S^m$ generated by $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$. Let W_1 denote the set of basic Whitehead products generated by $\{\alpha_1, \beta_1\}$ or by $\{\alpha_2, \beta_2\}$, and let $W_2 = W - W_1$. That is, W_1 is the set of basic Whitehead products like

$$[\alpha_1, \beta_1], [\alpha_2, \beta_2], [\alpha_2, [\alpha_2, \beta_2]], [[\alpha_1, \beta_1], [\alpha_1, [\alpha_1, \beta_1]]], \cdots$$

and W_2 is the set of basic Whitehead products such as

$$[\alpha_1, \alpha_2], [\alpha_1, \beta_2], [\alpha_2, [\alpha_1, \beta_1]], [[\alpha_1, \beta_1], [\alpha_2, [\alpha_2, \beta_2]]], \cdots .$$

Indeed, W_2 is the set of basic Whitehead products containing at least a '1' and at least a '2' in the subscript of the basic Whitehead products.

Lemma 2.4. *Let $pr_{1_*}, pr_{2_*} : \pi_p(S^n \vee S^m \vee S^n \vee S^m) \longrightarrow \pi_p(S^n \vee S^m)$ be the homomorphisms induced by the first and the second projections pr_1 and pr_2, respectively. Then*

$$Ker(pr_{1_*}) \cap Ker(pr_{2_*}) \cong \bigoplus_{\omega \in W_2} \pi_p(S^{h_\omega}),$$

where h_ω is the height of the basic Whitehead products $\omega \in W_2$.

Proof. If $x \in \bigoplus_{\omega\in W_2} \pi_p(S^{h_\omega})$, then the Hilton's theorem shows that x can be written as an element of $\pi_p(S^n \vee S^m \vee S^n \vee S^m)$ as follows:

$$\begin{aligned} x \; &= [\alpha_2,[\alpha_1,\beta_1]]\psi_1^1 + [\beta_2,[\alpha_1,\beta_1]]\psi_2^1 + [\alpha_2,[\alpha_2,[\alpha_1,\beta_1]]]\psi_3^1 + \dots \\ &+[\alpha_1,\alpha_2]\psi_1^2 + [\alpha_1,[\alpha_1,\alpha_2]]\psi_2^2 + [\beta_1,[\alpha_1,\alpha_2]]\psi_3^2 + \dots \\ &+[\alpha_1,\beta_2]\psi_1^3 + [\alpha_1,[\alpha_1,\beta_2]]\psi_2^3 + [\beta_1,[\alpha_1,\beta_2]]\psi_3^3 + \dots \\ &+[\beta_1,\alpha_2]\psi_1^4 + [\beta_1,[\beta_1,\alpha_2]]\psi_2^4 + [\beta_2,[\beta_1,\alpha_2]]\psi_3^4 + \dots \\ &+[\beta_1,\beta_2]\psi_1^5 + [\beta_1,[\beta_1,\beta_2]]\psi_2^5 + [\beta_2,[\beta_1,\beta_2]]\psi_3^5 + \dots \\ &+\dots \end{aligned}$$

for $\psi_i^j \in \pi_p(S^{h_\omega})$ and $\omega \in W_2$. Here, i and j are finite for each p since $h_\omega \to \infty$, and the summand $\pi_p(S^{h_\omega})$ is embedded into the sum via composition with $\omega : S^{h_\omega} \to S^n \vee S^m \vee S^n \vee S^m$. By applying pr_{1_*} and pr_{2_*} on both sides, we have

$$pr_{1_*}(x) = 0 = pr_{2_*}(x)$$

because all the basic Whitehead products describe above are the elements of W_2. Indeed, all the basic Whitehead products in W_2 are killed off by pr_{1_*} and pr_{2_*}. For example,

$$\begin{aligned} pr_{1_*}([\alpha_1,[\alpha_1,\beta_2]]) \; &= [pr_1\alpha_1,[pr_1\alpha_1,pr_1\beta_2]] \\ &= [pr_1\alpha_1,[pr_1\alpha_1,0]] \\ &= 0, \end{aligned}$$

and similarly for pr_{2_*}. We can also check the reverse inclusion. □

Theorem 2.5. *Every comultiplication $\varphi : Y \to Y \vee Y$ on $Y = S^n \vee S^m \vee S^p$ can be uniquely written in the following form*

$$\begin{cases} \varphi|_{S^n} = \iota_1 r + \iota_2 r, \\ \varphi|_{S^m} = \iota_1 s + \iota_2 s + (r \vee r)_* W_P, \\ \varphi|_{S^p} = \iota_1 t + \iota_2 t + (A \vee A)_* W_Q. \end{cases}$$

Here, (1) $W_P = \sum_{u=3}^{\infty} w_u\gamma_u \in \pi_m(S^n \vee S^n)$*, w_u is the uth basic product in $S^n \vee S^n$ and $\gamma_u \in \pi_m(S^{h_u})$ for $u = 3,4,\dots$, and* (2) $W_Q \in \pi_p(S^n \vee S^m \vee S^n \vee S^m)$ *is*

$$W_Q = \sum_{\omega\in W_2} \omega\psi_\omega,$$

where ω runs over all the basic Whitehead products in W_2, $\psi_\omega \in \pi_p(S^{h_\omega})$, and h_ω is the height of the basic Whitehead products $\omega \in W_2$.

Proof. We prove the result for W_Q. By Lemma 2.3, φ has the form stated in this theorem for some $W_Q \in \pi_p(S^n \vee S^m \vee S^n \vee S^m)$ with $pr_1 W_Q = 0 = pr_2 W_Q$. Let $x \in \bigoplus_{\omega\in W_2} \pi_p(S^{h_\omega})$ as in the proof of Lemma 2.4 and $y \in \bigoplus_{\omega\in W_1} \pi_p(S^{h_\omega})$. Then, by using the Hilton's theorem, we have

$$W_Q = \alpha_1\rho_1 + \beta_1\rho_2 + \alpha_2\rho_3 + \beta_2\rho_4 + x + y$$

for $\rho_1,\rho_3 \in \pi_p(S^n)$ and $\rho_2,\rho_4 \in \pi_p(S^m)$. By Lemmas 2.3 and 2.4, and by applying pr_{1_*} (and then pr_{2_*}) on the above equation, we get

$$\begin{aligned} 0 \; &= pr_{1*}W_Q \\ &= pr_1(\alpha_1\rho_1) + pr_1(\beta_1\rho_2) + pr_1(\alpha_2\rho_3) + pr_1(\beta_2\rho_4) + pr_{1_*}(x) + pr_{1_*}(y) \\ &= \xi(n)\rho_1 + \xi(m)\rho_2 + 0 + 0 + 0 + pr_{1_*}(y), \end{aligned}$$

where $\xi(n) : S^n \to S^n \vee S^m$ and $\xi(m) : S^m \to S^n \vee S^m$ are the inclusion maps. By the Hilton's theorem, $\xi(n)\rho_1 = 0 = \xi(m)\rho_2$ and $pr_{1_*}(y) = 0$. Since the inclusions $\xi(n)$ and $\xi(m)$ induce monomorphisms between homotopy groups, we have $\rho_1 = 0 = \rho_2$. Similarly, by applying pr_{2_*} on W_Q, we have $\rho_3 = 0 = \rho_4$.

We now consider the case of basic Whitehead products in W_1, namely,

$$[\alpha_1, \beta_1], [\alpha_1, [\alpha_1, \beta_1]], [\alpha_2, \beta_2], [\alpha_2, [\alpha_2, \beta_2]]$$

and so on. By the Hilton's theorem, and Lemmas 2.3 and 2.4, $y \in \bigoplus_{\omega \in W_1} \pi_p(S^{h_\omega})$ must satisfy $pr_{1_*}(y) = 0 = pr_{2_*}(y)$, and thus it has to be the following type

$$[\alpha_1, \beta_1]0 + [\alpha_1, [\alpha_1, \beta_1]]0 + \ldots + [\alpha_2, \beta_2]0 + [\alpha_2, [\alpha_2, \beta_2]]0 + \ldots.$$

Therefore

$$W_Q = \sum_{\omega \in W_2} \omega \psi_\omega,$$

where $\psi_\omega \in \pi_p(S^{h_\omega})$, as required. The proof in the case of W_P goes to the same way. □

By [1, p. 1167] we can determine the number of comultiplications of a co-H-space. By using Theorem 2.5, we now get a concrete clue how to calculate the number of comultiplications as follows:

Corollary 2.6. *The number of comultiplications of $Y = S^n \vee S^m \vee S^p$ is*

$$\prod_{u=3}^{\infty} |\pi_m(S^{h_u})| \times \prod_{\omega \in W_2} |\pi_p(S^{h_\omega})|.$$

By using the above results and the homotpy groups of spheres [16], we now provide an example.

Example 2.7. The cardinality of the set of comultiplications of $S^8 \vee S^{12} \vee S^{27}$ is 512. Indeed, we let $|C(Y)|$ be the number of comultiplications of Y. Then $|C(S^8 \vee S^{12} \vee S^{27})| = |\pi_{27}(S^{26})| \times |\pi_{27}(S^{15})| \times |\pi_{27}(S^{19})|^2 \times |\pi_{27}(S^{23})| \times |\pi_{27}(S^{22})|^2 \times |\pi_{27}(S^{26})|^2 \times |\pi_{27}(S^{26})|^2 \times 1 \times 1 \times \ldots = 2 \times 1 \times 16 \times 1 \times 1 \times 4 \times 4 \times 1 \times 1 \times 1 \times \ldots = 512$.

3. Associativity on a wedge of three spheres

In this section, we concentrate our attention on the associative comultiplications on a wedge of three spheres.

Definition 3.1. Let $Y = S^n \vee S^m \vee S^p$. We then define a comultiplication $\varphi = \varphi_{P,Q} : Y \to Y \vee Y$ by

$$\begin{cases} \varphi_{P,Q}|_{S^n} = \iota_1 r + \iota_2 r, \\ \varphi_{P,Q}|_{S^m} = \iota_1 s + \iota_2 s + P, \\ \varphi_{P,Q}|_{S^p} = \iota_1 t + \iota_2 t + Q, \end{cases}$$

where $P = (r \vee r)[i_1, i_2]\gamma$, $\gamma \in \pi_m(S^{2n-1})$, and $Q : S^p \to Y \vee Y$ is a map such that $q_1 Q = 0 = q_2 Q$.

We are especially interested in studying the homotopy classes $Q \in \pi_p(Y \vee Y)$ in order to get some information for the given comultiplication $\varphi : Y \to Y \vee Y$. We first consider the perturbations $Q_j, j = 1, 2, \ldots, 5$ of comultiplications $\varphi = \varphi_{P,Q_j}, j = 1, 2, \ldots, 5$ constructed below, and then define a perturbation $Q \in \pi_p(Y \vee Y)$ of a comultiplication $\varphi : Y \to Y \vee Y$ by

$$Q = \sum_{j=1}^{5} Q_j$$

in order to get some information about comultiplications, associative comultiplications, commutative comultiplications, and comultiplications which are both associative and commutative.

Notation In Sections 3 and 4, we will make use of the symbols $\bar{\alpha}_1, \bar{\alpha}_2 : S^n \to Y \vee Y$, $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3 : S^n \to Y \vee Y \vee Y$, $\bar{\beta}_1, \bar{\beta}_2 : S^m \to Y \vee Y$, and $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3 : S^m \to Y \vee Y \vee Y$ to denote $\iota_1 r, \iota_2 r, I_1 r, I_2 r, I_3 r, \iota_1 s, \iota_2 s$, and $I_1 s, I_2 s, I_3 s$, respectively.

Definition 3.2. For any homotopy classes $a_1 \in \pi_p(S^{2m-1}), a_2, a_3 \in \pi_p(S^{3m-2}), b_1 \in \pi_p(S^{2n-1}), b_2, b_3 \in \pi_p(S^{3n-2}), c_1 \in \pi_p(S^{n+m-1}), c_2, c_3 \in \pi_p(S^{m+2n-2})$ and $c_4, c_5 \in \pi_p(S^{2m+n-2})$, we define $Q_1 \in \pi_p(Y \vee Y)$ by

$$\begin{aligned} Q_1 &= [\bar{\beta}_1, \bar{\beta}_2]a_1 + [\bar{\beta}_1, [\bar{\beta}_1, \bar{\beta}_2]]a_2 + [\bar{\beta}_2, [\bar{\beta}_1, \bar{\beta}_2]]a_3 \\ &\quad + [\bar{\alpha}_1, \bar{\alpha}_2]b_1 + [\bar{\alpha}_1, [\bar{\alpha}_1, \bar{\alpha}_2]]b_2 + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\alpha}_2]]b_3 \\ &\quad + [\bar{\alpha}_1, \bar{\beta}_2]c_1 + [\bar{\alpha}_1, [\bar{\alpha}_1, \bar{\beta}_2]]c_2 + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\beta}_2]]c_3 \\ &\quad + [\bar{\beta}_1, [\bar{\alpha}_1, \bar{\beta}_2]]c_4 + [\bar{\beta}_2, [\bar{\alpha}_1, \bar{\beta}_2]]c_5. \end{aligned}$$

We then define a comultiplication $\varphi = \varphi_{P,Q_1} : Y \to Y \vee Y$ by substituting Q_1 for Q in Definition 3.1.

Proposition 3.3. *Let $m \leq 2n-2$ and $n < m < p \leq 8n-8$. If the Hopf-Hilton invariants $H_1^1(x) = 0 = H_1^2(y)$, where $x = a_i, b_i, c_j$ for $i = 1, 2, 3$, $j = 1, 2, \ldots, 5$ and $y = a_1, b_1, c_1$, then $\varphi = \varphi_{P,Q_1}$ is associative if and only if* (1) *m and n are even, and $a_2 = a_3 = b_2 = b_3 = c_2 = c_3 = c_4 = c_5 = 0$, or* (2) *$m$ is even and n is odd, $a_2 = a_3 = c_2 = c_3 = c_4 = c_5 = 0$, $b_2 = -b_3$ and $3b_2 = 0$, or* (3) *m is odd and n is even, $b_2 = b_3 = c_2 = c_3 = c_4 = c_5 = 0$, $a_2 = -a_3$ and $3a_2 = 0$, or* (4) *m and n are odd, $c_2 = c_3 = c_4 = c_5 = 0$, $a_2 = -a_3$, $b_2 = -b_3$ and $3a_2 = 0 = 3b_2$.*

Proof. We recall that a necessary and sufficient condition for the comultiplication $\varphi = \varphi_{P,Q_1}$ to be associative is that

$$J_{12}P + (\varphi \vee 1)P = J_{23}P + (1 \vee \varphi)P$$

and

$$J_{12}Q_1 + (\varphi \vee 1)Q_1 = J_{23}Q_1 + (1 \vee \varphi)Q_1.$$

It suffices to prove the second equation because $P = [\iota_1 r, \iota_2 r]\gamma = 0$. We compute $J_{12}Q_1$:

$$\begin{aligned} J_{12}Q_1 &= J_{12}\Big([\bar{\beta}_1, \bar{\beta}_2]a_1 + [\bar{\beta}_1, [\bar{\beta}_1, \bar{\beta}_2]]a_2 + [\bar{\beta}_2, [\bar{\beta}_1, \bar{\beta}_2]]a_3 \\ &\quad + [\bar{\alpha}_1, \bar{\alpha}_2]b_1 + [\bar{\alpha}_1, [\bar{\alpha}_1, \bar{\alpha}_2]]b_2 + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\alpha}_2]]b_3 \\ &\quad + [\bar{\alpha}_1, \bar{\beta}_2]c_1 + [\bar{\alpha}_1, [\bar{\alpha}_1, \bar{\beta}_2]]c_2 + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\beta}_2]]c_3 \\ &\quad + [\bar{\beta}_1, [\bar{\alpha}_1, \bar{\beta}_2]]c_4 + [\bar{\beta}_2, [\bar{\alpha}_1, \bar{\beta}_2]]c_5\Big) \\ &= [\hat{\beta}_1, \hat{\beta}_2]a_1 + [\hat{\beta}_1, [\hat{\beta}_1, \hat{\beta}_2]]a_2 + [\hat{\beta}_2, [\hat{\beta}_1, \hat{\beta}_2]]a_3 \\ &\quad + [\hat{\alpha}_1, \hat{\alpha}_2]b_1 + [\hat{\alpha}_1, [\hat{\alpha}_1, \hat{\alpha}_2]]b_2 + [\hat{\alpha}_2, [\hat{\alpha}_1, \hat{\alpha}_2]]b_3 \\ &\quad + [\hat{\alpha}_1, \hat{\beta}_2]c_1 + [\hat{\alpha}_1, [\hat{\alpha}_1, \hat{\beta}_2]]c_2 + [\hat{\alpha}_2, [\hat{\alpha}_1, \hat{\beta}_2]]c_3 \\ &\quad + [\hat{\beta}_1, [\hat{\alpha}_1, \hat{\beta}_2]]c_4 + [\hat{\beta}_2, [\hat{\alpha}_1, \hat{\beta}_2]]c_5. \end{aligned}$$

We now compute $(\varphi \vee 1)Q_1$:

$$\begin{aligned} &(\varphi \vee 1)Q_1 \\ &= [J_{12}\varphi s, \hat{\beta}_3]a_1 + [J_{12}\varphi s, [J_{12}\varphi s, \hat{\beta}_3]]a_2 + [\hat{\beta}_3, [J_{12}\varphi s, \hat{\beta}_3]]a_3 \\ &\quad + [J_{12}\varphi r, \hat{\alpha}_3]b_1 + [J_{12}\varphi r, [J_{12}\varphi r, \hat{\alpha}_3]]b_2 + [\hat{\alpha}_3, [J_{12}\varphi r, \hat{\alpha}_3]]b_3 \\ &\quad + [J_{12}\varphi r, \hat{\beta}_3]c_1 + [J_{12}\varphi r, [J_{12}\varphi r, \hat{\beta}_3]]c_2 + [\hat{\alpha}_3, [J_{12}\varphi r, \hat{\beta}_3]]c_3 \\ &\quad + [J_{12}\varphi s, [J_{12}\varphi r, \hat{\beta}_3]]c_4 + [\hat{\beta}_3, [J_{12}\varphi r, \hat{\beta}_3]]c_5 \\ &= [\hat{\beta}_1 + \hat{\beta}_2, \hat{\beta}_3]a_1 + [\hat{\beta}_1 + \hat{\beta}_2, [\hat{\beta}_1 + \hat{\beta}_2, \hat{\beta}_3]]a_2 \\ &\quad + [\hat{\beta}_3, [\hat{\beta}_1 + \hat{\beta}_2, \hat{\beta}_3]]a_3 + [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\alpha}_3]b_1 \\ &\quad + [\hat{\alpha}_1 + \hat{\alpha}_2, [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\alpha}_3]]b_2 + [\hat{\alpha}_3, [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\alpha}_3]]b_3 \\ &\quad + [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\beta}_3]c_1 + [\hat{\alpha}_1 + \hat{\alpha}_2, [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\beta}_3]]c_2 \\ &\quad + [\hat{\alpha}_3, [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\beta}_3]]c_3 + [\hat{\beta}_1 + \hat{\beta}_2, [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\beta}_3]]c_4 \\ &\quad + [\hat{\beta}_3, [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\beta}_3]]c_5 \\ &= \Big([\hat{\beta}_1, \hat{\beta}_3] + [\hat{\beta}_2, \hat{\beta}_3]\Big)a_1 + \Big([\hat{\beta}_1, [\hat{\beta}_1, \hat{\beta}_3]] + [\hat{\beta}_1, [\hat{\beta}_2, \hat{\beta}_3]] \\ &\quad + [\hat{\beta}_2, [\hat{\beta}_1, \hat{\beta}_3]] + [\hat{\beta}_2, [\hat{\beta}_2, \hat{\beta}_3]]\Big)a_2 + \Big([\hat{\beta}_3, [\hat{\beta}_1, \hat{\beta}_3]] \\ &\quad + [\hat{\beta}_3, [\hat{\beta}_2, \hat{\beta}_3]]\Big)a_3 + \Big([\hat{\alpha}_1, \hat{\alpha}_3] + [\hat{\alpha}_2, \hat{\alpha}_3]\Big)b_1 \\ &\quad \Big([\hat{\alpha}_1, [\hat{\alpha}_1, \hat{\alpha}_3]] + [\hat{\alpha}_1, [\hat{\alpha}_2, \hat{\alpha}_3]] + [\hat{\alpha}_2, [\hat{\alpha}_1, \hat{\alpha}_3]] \end{aligned}$$

$$+[\hat{\alpha}_2,[\hat{\alpha}_2,\hat{\alpha}_3]]\Big)b_2+\Big([\hat{\alpha}_3,[\hat{\alpha}_1,\hat{\alpha}_3]]+[\hat{\alpha}_3,[\hat{\alpha}_2,\hat{\alpha}_3]]\Big)b_3$$
$$\Big([\hat{\alpha}_1,\hat{\beta}_3]+[\hat{\alpha}_2,\hat{\beta}_3]\Big)c_1+\Big([\hat{\alpha}_1,[\hat{\alpha}_1,\hat{\beta}_3]]+[\hat{\alpha}_1,[\hat{\alpha}_2,\hat{\beta}_3]]$$
$$+[\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\beta}_3]]+[\hat{\alpha}_2,[\hat{\alpha}_2,\hat{\beta}_3]]\Big)c_2+\Big([\hat{\alpha}_3,[\hat{\alpha}_1,\hat{\beta}_3]]$$
$$+[\hat{\alpha}_3,[\hat{\alpha}_2,\hat{\beta}_3]]\Big)c_3+\Big([\hat{\beta}_1,[\hat{\alpha}_1,\hat{\beta}_3]]+[\hat{\beta}_1,[\hat{\alpha}_2,\hat{\beta}_3]]$$
$$+[\hat{\beta}_2,[\hat{\alpha}_1,\hat{\beta}_3]]+[\hat{\beta}_2,[\hat{\alpha}_2,\hat{\beta}_3]]\Big)c_4+\Big([\hat{\beta}_3,[\hat{\alpha}_1,\hat{\beta}_3]]$$
$$+[\hat{\beta}_3,[\hat{\alpha}_2,\hat{\beta}_3]]\Big)c_5.$$

Since the Whitehead products such as $[\hat{\beta}_1,[\hat{\beta}_2,\hat{\beta}_3]],[\hat{\alpha}_1,[\hat{\alpha}_2,\hat{\alpha}_3]],[\hat{\alpha}_1,[\hat{\alpha}_2,\hat{\beta}_3]]$ and $[\hat{\beta}_1,[\hat{\alpha}_2,\hat{\beta}_3]]$ in the above equations are not basic products, we need to change them into the basic products by using the Jacobi identity [17, p. 478] as follows:

$$\begin{aligned}0 &= (-1)^{m(m-1)}[\hat{\beta}_1,[\hat{\beta}_2,\hat{\beta}_3]]+(-1)^{m(m-1)+m^2}[\hat{\beta}_2,[\hat{\beta}_1,\hat{\beta}_3]]\\ &\quad+(-1)^{m(m-1)}[\hat{\beta}_3,[\hat{\beta}_1,\hat{\beta}_2]],\end{aligned}$$

and

$$\begin{aligned}0 &= (-1)^{n(m-1)}[\hat{\alpha}_1,[\hat{\alpha}_2,\hat{\beta}_3]]+(-1)^{n(n-1)+mn}[\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\beta}_3]]\\ &\quad+(-1)^{m(n-1)}[\hat{\beta}_3,[\hat{\alpha}_1,\hat{\alpha}_2]].\end{aligned}$$

Similarly, for $[\hat{\alpha}_1,[\hat{\alpha}_2,\hat{\alpha}_3]]$ and $[\hat{\beta}_1,[\hat{\alpha}_2,\hat{\beta}_3]]$. More precisely,

$$[\hat{\beta}_1,[\hat{\beta}_2,\hat{\beta}_3]]=\begin{cases}+[\hat{\beta}_2,[\hat{\beta}_1,\hat{\beta}_3]]-[\hat{\beta}_3,[\hat{\beta}_1,\hat{\beta}_2]] & \text{for m odd,}\\ -[\hat{\beta}_2,[\hat{\beta}_1,\hat{\beta}_3]]-[\hat{\beta}_3,[\hat{\beta}_1,\hat{\beta}_2]] & \text{for m even,}\end{cases}$$

$$[\hat{\alpha}_1,[\hat{\alpha}_2,\hat{\alpha}_3]]=\begin{cases}+[\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\alpha}_3]]-[\hat{\alpha}_3,[\hat{\alpha}_1,\hat{\alpha}_2]] & \text{for n odd,}\\ -[\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\alpha}_3]]-[\hat{\alpha}_3,[\hat{\alpha}_1,\hat{\alpha}_2]] & \text{for n even,}\end{cases}$$

$$[\hat{\alpha}_1,[\hat{\alpha}_2,\hat{\beta}_3]]=\begin{cases}-[\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\beta}_3]]-[\hat{\beta}_3,[\hat{\alpha}_1,\hat{\alpha}_2]] & \text{for m, n even,}\\ +[\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\beta}_3]]+[\hat{\beta}_3,[\hat{\alpha}_1,\hat{\alpha}_2]] & \text{for m even, n odd,}\\ -[\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\beta}_3]]+[\hat{\beta}_3,[\hat{\alpha}_1,\hat{\alpha}_2]] & \text{for m odd, n even,}\\ +[\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\beta}_3]]-[\hat{\beta}_3,[\hat{\alpha}_1,\hat{\alpha}_2]] & \text{for m, n odd,}\end{cases}$$

and

$$[\hat{\beta}_1,[\hat{\alpha}_2,\hat{\beta}_3]]=\begin{cases}-[\hat{\alpha}_2,[\hat{\beta}_1,\hat{\beta}_3]]-[\hat{\beta}_3,[\hat{\beta}_1,\hat{\alpha}_2]] & \text{for m, n even,}\\ +[\hat{\alpha}_2,[\hat{\beta}_1,\hat{\beta}_3]]-[\hat{\beta}_3,[\hat{\beta}_1,\hat{\alpha}_2]] & \text{for m even, n odd,}\\ +[\hat{\alpha}_2,[\hat{\beta}_1,\hat{\beta}_3]]+[\hat{\beta}_3,[\hat{\beta}_1,\hat{\alpha}_2]] & \text{for m odd, n even,}\\ +[\hat{\alpha}_2,[\hat{\beta}_1,\hat{\beta}_3]]-[\hat{\beta}_3,[\hat{\beta}_1,\hat{\alpha}_2]] & \text{for m, n odd.}\end{cases}$$

We now compute $J_{23}Q_1$ and $(1\vee\varphi)Q_1$:

$$\begin{aligned}J_{23}Q_1 &= [\hat{\beta}_2,\hat{\beta}_3]a_1+[\hat{\beta}_2,[\hat{\beta}_2,\hat{\beta}_3]]a_2+[\hat{\beta}_3,[\hat{\beta}_2,\hat{\beta}_3]]a_3\\ &\quad+[\hat{\alpha}_2,\hat{\alpha}_3]b_1+[\hat{\alpha}_2,[\hat{\alpha}_2,\hat{\alpha}_3]]b_2+[\hat{\alpha}_3,[\hat{\alpha}_2,\hat{\alpha}_3]]b_3\\ &\quad+[\hat{\alpha}_2,\hat{\beta}_3]c_1+[\hat{\alpha}_2,[\hat{\alpha}_2,\hat{\beta}_3]]c_2+[\hat{\alpha}_3,[\hat{\alpha}_2,\hat{\beta}_3]]c_3\\ &\quad+[\hat{\beta}_2,[\hat{\alpha}_2,\hat{\beta}_3]]c_4+[\hat{\beta}_3,[\hat{\alpha}_2,\hat{\beta}_3]]c_5\end{aligned}$$

and

$$\begin{aligned}&(1\vee\varphi)Q_1\\ &= [\hat{\beta}_1,J_{23}\varphi s]a_1+[\hat{\beta}_1,[\hat{\beta}_1,J_{23}\varphi s]]a_2+[J_{23}\varphi s,[\hat{\beta}_1,J_{23}\varphi s]]a_3\\ &\quad+[\hat{\alpha}_1,J_{23}\varphi r]b_1+[\hat{\alpha}_1,[\hat{\alpha}_1,J_{23}\varphi r]]b_2+[J_{23}\varphi r,[\hat{\alpha}_1,J_{23}\varphi r]]b_3\\ &\quad+[\hat{\alpha}_1,J_{23}\varphi s]c_1+[\hat{\alpha}_1,[\hat{\alpha}_1,J_{23}\varphi s]]c_2+[J_{23}\varphi r,[\hat{\alpha}_1,J_{23}\varphi s]]c_3\\ &\quad+[\hat{\beta}_1,[\hat{\alpha}_1,J_{23}\varphi s]]c_4+[J_{23}\varphi s,[\hat{\alpha}_1,J_{23}\varphi s]]c_5\end{aligned}$$

$$
\begin{aligned}
&= [\hat\beta_1, \hat\beta_2 + \hat\beta_3]a_1 + [\hat\beta_1, [\hat\beta_1, [\hat\beta_2 + \hat\beta_3]]a_2 \\
&\quad + [\hat\beta_2 + \hat\beta_3, [\hat\beta_1, \hat\beta_2 + \hat\beta_3]]a_3 + [\hat\alpha_1, \hat\alpha_2 + \hat\alpha_3]b_1 \\
&\quad + [\hat\alpha_1, [\hat\alpha_1, \hat\alpha_2 + \hat\alpha_3]]b_2 + [\hat\alpha_2 + \hat\alpha_3, [\hat\alpha_1, \hat\alpha_2 + \hat\alpha_3]]b_3 \\
&\quad + [\hat\alpha_1, \hat\beta_2 + \hat\beta_3]c_1 + [\hat\alpha_1, [\hat\alpha_1, \hat\beta_2 + \hat\beta_3]]c_2 \\
&\quad + [\hat\alpha_2 + \hat\alpha_3, [\hat\alpha_1, \hat\beta_2 + \hat\beta_3]]c_3 + [\hat\beta_1, [\hat\alpha_1, \hat\beta_2 + \hat\beta_3]]c_4 \\
&\quad + [\hat\beta_2 + \hat\beta_3, [\hat\alpha_1, \hat\beta_2 + \hat\beta_3]]c_5 \\
&= \Big([\hat\beta_1, \hat\beta_2] + [\hat\beta_1, \hat\beta_3]\Big)a_1 + \Big([\hat\beta_1, [\hat\beta_1, \hat\beta_2]] + [\hat\beta_1, [\hat\beta_1, \hat\beta_3]]\Big)a_2 \\
&\quad + \Big([\hat\beta_2, [\hat\beta_1, \hat\beta_2]] + [\hat\beta_2, [\hat\beta_1, \hat\beta_3]] + [\hat\beta_3, [\hat\beta_1, \hat\beta_2]] + [\hat\beta_3, [\hat\beta_1, \hat\beta_3]]\Big)a_3 \\
&\quad + \Big([\hat\alpha_1, \hat\alpha_2] + [\hat\alpha_1, \hat\alpha_3]\Big)b_1 + \Big([\hat\alpha_1, [\hat\alpha_1, \hat\alpha_2]] + [\hat\alpha_1, [\hat\alpha_1, \hat\alpha_3]]\Big)b_2 \\
&\quad + \Big([\hat\alpha_2, [\hat\alpha_1, \hat\alpha_2]] + [\hat\alpha_2, [\hat\alpha_1, \hat\alpha_3]] + [\hat\alpha_3, [\hat\alpha_1, \hat\alpha_2]] + [\hat\alpha_3, [\hat\alpha_1, \hat\alpha_3]]\Big)b_3 \\
&\quad + \Big([\hat\alpha_1, \hat\beta_2] + [\hat\alpha_1, \hat\beta_3]\Big)c_1 + \Big([\hat\alpha_1, [\hat\alpha_1, \hat\beta_2]] + [\hat\alpha_1, [\hat\alpha_1, \hat\beta_3]]\Big)c_2 \\
&\quad + \Big([\hat\alpha_2, [\hat\alpha_1, \hat\beta_2]] + [\hat\alpha_2, [\hat\alpha_1, \hat\beta_3]] + [\hat\alpha_3, [\hat\alpha_1, \hat\beta_2]] + [\hat\alpha_3, [\hat\alpha_1, \hat\beta_3]]\Big)c_3 \\
&\quad + \Big([\hat\beta_1, [\hat\alpha_1, \hat\beta_2]] + [\hat\beta_1, [\hat\alpha_1, \hat\beta_3]]\Big)c_4 \\
&\quad + \Big([\hat\beta_2, [\hat\alpha_1, \hat\beta_2]] + [\hat\beta_2, [\hat\alpha_1, \hat\beta_3]] + [\hat\beta_3, [\hat\alpha_1, \hat\beta_2]] + [\hat\beta_3, [\hat\alpha_1, \hat\beta_3]]\Big)c_5.
\end{aligned}
$$

We now prove the following two cases since the other two cases are similar to the first two cases. We note that $(-1)x = -x + [1, 1]H_1^1(x) = -x$ by the Hilton's formula, where 1 is the identity map of a sphere. We also observe that the following Whitehead products in (1) and (2) are all the basic products.

(1) If m and n are even, then the equality $J_{12}Q_1 + (\varphi \vee 1)Q_1 = J_{23}Q_1 + (1 \vee \varphi)Q_1$ holds if and only if

$$
\begin{aligned}
0 &= (-[\hat\beta_3, [\hat\beta_1, \hat\beta_2]])a_2 + (-[\hat\alpha_3, [\hat\alpha_1, \hat\alpha_2]])b_2 + (-[\hat\beta_3, [\hat\alpha_1, \hat\alpha_2]])c_2 \\
&\quad + (-[\hat\alpha_2, [\hat\beta_1, \hat\beta_3]])c_4 + [\hat\beta_2, [\hat\alpha_1, \hat\beta_3]]c_4 + (-[\hat\beta_3, [\hat\beta_1, \hat\alpha_2]])c_4 \\
&\quad - ([\hat\beta_2, [\hat\beta_1, \hat\beta_3]] + [\hat\beta_3, [\hat\beta_1, \hat\beta_2]])a_3 \\
&\quad - ([\hat\alpha_2, [\hat\alpha_1, \hat\alpha_3]] + [\hat\alpha_3, [\hat\alpha_1, \hat\alpha_2]])b_3 \\
&\quad - ([\hat\alpha_2, [\hat\alpha_1, \hat\beta_3]] + [\hat\alpha_3, [\hat\alpha_1, \hat\beta_2]])c_3 \\
&\quad - ([\hat\beta_2, [\hat\alpha_1, \hat\beta_3]] + [\hat\beta_3, [\hat\alpha_1, \hat\beta_2]])c_5 \\
&= [\hat\beta_3, [\hat\beta_1, \hat\beta_2]](-a_2 - a_3) + [\hat\beta_2, [\hat\beta_1, \hat\beta_3]](-a_3) \\
&\quad + [\hat\alpha_3, [\hat\alpha_1, \hat\alpha_2]](-b_2 - b_3) + [\hat\alpha_2, [\hat\alpha_1, \hat\alpha_3]](-b_3) \\
&\quad + [\hat\beta_3, [\hat\alpha_1, \hat\alpha_2](-c_2) + [\hat\alpha_2, [\hat\alpha_1, \hat\beta_3]](-c_3) + [\hat\alpha_3, [\hat\alpha_1, \hat\beta_2]](-c_3) \\
&\quad + [\hat\alpha_2, [\hat\beta_1, \hat\beta_3]](-c_4) + [\hat\beta_3, [\hat\beta_1, \hat\alpha_2]](-c_4) \\
&\quad + [\hat\beta_2, [\hat\alpha_1, \hat\beta_3]](c_4 - c_5) + [\hat\beta_3, [\hat\alpha_1, \hat\beta_2](-c_5).
\end{aligned}
$$

Thus φ is associative if and only if $a_2 = a_3 = b_2 = b_3 = c_2 = c_3 = c_4 = c_5 = 0$.

(2) If m is even and n is odd, then the associativity of $\varphi = \varphi_{P,Q_1}$ holds if and only if

$$
\begin{aligned}
0 &= (-[\hat\beta_3, [\hat\beta_1, \hat\beta_2]])a_2 + 2[\hat\alpha_2, [\hat\alpha_1, \hat\alpha_3]]b_2 + (-[\hat\alpha_3, [\hat\alpha_1, \hat\alpha_2]])b_2 \\
&\quad + 2[\hat\alpha_2, [\hat\alpha_1, \hat\beta_3]]c_2 + [\hat\beta_3, [\hat\alpha_1, \hat\alpha_2]]c_2 + [\hat\alpha_2, [\hat\beta_1, \hat\beta_3]]c_4 \\
&\quad + (-[\hat\beta_3, [\hat\beta_1, \hat\alpha_2]])c_4 + [\hat\beta_2, [\hat\alpha_1, \hat\beta_3]]c_4 \\
&\quad - ([\hat\beta_2, [\hat\beta_1, \hat\beta_3]] + [\hat\beta_3, [\hat\beta_1, \hat\beta_2]])a_3 \\
&\quad - ([\hat\alpha_2, [\hat\alpha_1, \hat\alpha_3]] + [\hat\alpha_3, [\hat\alpha_1, \hat\alpha_2]])b_3 \\
&\quad - ([\hat\alpha_2, [\hat\alpha_1, \hat\beta_3]] + [\hat\alpha_3, [\hat\alpha_1, \hat\beta_2]])c_3 \\
&\quad - ([\hat\beta_2, [\hat\alpha_1, \hat\beta_3]] + [\hat\beta_3, [\hat\alpha_1, \hat\beta_2]])c_5 \\
&= [\hat\beta_3, [\hat\beta_1, \hat\beta_2]](-a_2 - a_3) + [\hat\beta_2, [\hat\beta_1, \hat\beta_3]](-a_3) \\
&\quad + [\hat\alpha_3, [\hat\alpha_1, \hat\alpha_2]](-b_2 - b_3) + [\hat\alpha_2, [\hat\alpha_1, \hat\alpha_3]](2b_2 - b_3) \\
&\quad + [\hat\alpha_2, [\hat\alpha_1, \hat\beta_3]](2c_2 - c_3) + [\hat\beta_3, [\hat\alpha_1, \hat\alpha_2]c_2 \\
&\quad + [\hat\alpha_3, [\hat\alpha_1, \hat\beta_2]](-c_3) + [\hat\alpha_2, [\hat\beta_1, \hat\beta_3]]c_4 \\
&\quad + [\hat\beta_3, [\hat\beta_1, \hat\alpha_2]](-c_4) + [\hat\beta_2, [\hat\alpha_1, \hat\beta_3]](c_4 - c_5) \\
&\quad + [\hat\beta_3, [\hat\alpha_1, \hat\beta_2](-c_5).
\end{aligned}
$$

Thus φ is associative if and only if $a_2 = a_3 = c_2 = c_3 = c_4 = c_5 = 0$, $b_2 = -b_3$ and $3b_2 = 0$. □

Remark 3.4. We note that if $p \leq 4n-4$, then the conclusion of Proposition 3.3 holds because x and y are suspensions by the Freudenthal suspension theorem [17] and thus the Hopf-Hilton invariants are trivial.

Definition 3.5. Let Q_2 be the sum of the maps

$$S^p \xrightarrow{c_6} S^{m+2n-2} \xrightarrow{[\alpha_2,[\alpha_1,\beta_1]]} X \vee X \xrightarrow{A\vee A} Y \vee Y$$

and

$$S^p \xrightarrow{c_7} S^{2m+n-2} \xrightarrow{[\beta_2,[\alpha_1,\beta_1]]} X \vee X \xrightarrow{A\vee A} Y \vee Y$$

for some c_6 and c_7; that is,

$$Q_2 = [\bar{\alpha}_2,[\bar{\alpha}_1,\bar{\beta}_1]]c_6 + [\bar{\beta}_2,[\bar{\alpha}_1,\bar{\beta}_1]]c_7,$$

and define $\varphi = \varphi_{P,Q_2} : Y \to Y \vee Y$ by Definition 3.1.

Proposition 3.6. *Let $m \leq 2n-2$ and $n < m < p \leq 8n-8$. If the Hopf-Hilton invariants $H_1^1(c_6)$ and $H_1^1(c_7)$ are trivial, then the comultiplication $\varphi = \varphi_{P,Q_2}$ is associative if and only if $c_6 = c_7 = 0$.*

Proof. We compute $J_{12}Q_2$ and $J_{23}Q_2$:

$$\begin{aligned} J_{12}Q_2 &= J_{12}([\bar{\alpha}_2,[\bar{\alpha}_1,\bar{\beta}_1]]c_6 + [\bar{\beta}_2,[\bar{\alpha}_1,\bar{\beta}_1]]c_7) \\ &= [\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\beta}_1]]c_6 + [\hat{\beta}_2,[\hat{\alpha}_1,\hat{\beta}_1]]c_7 \end{aligned}$$

and

$$J_{23}Q_2 = [\hat{\alpha}_3,[\hat{\alpha}_2,\hat{\beta}_2]]c_6 + [\hat{\beta}_3,[\hat{\alpha}_2,\hat{\beta}_2]]c_7.$$

We now compute $(\varphi \vee 1)Q_2$:

$$\begin{aligned} (\varphi \vee 1)Q_2 &= [\hat{\alpha}_3,[J_{12}\varphi r, J_{12}\varphi s]]c_6 + [\hat{\beta}_3,[J_{12}\varphi r, J_{12}\varphi s]]c_7 \\ &= [\hat{\alpha}_3,[J_{12}(\bar{\alpha}_1+\bar{\alpha}_2), J_{12}(\bar{\beta}_1+\bar{\beta}_2)]]c_6 \\ &\quad +[\hat{\beta}_3,[J_{12}(\bar{\alpha}_1+\bar{\alpha}_2), J_{12}(\bar{\beta}_1+\bar{\beta}_2)]]c_7 \\ &= [\hat{\alpha}_3,[\hat{\alpha}_1+\hat{\alpha}_2,\hat{\beta}_1+\hat{\beta}_2]]c_6 + [\hat{\beta}_3,[\hat{\alpha}_1+\hat{\alpha}_2,\hat{\beta}_1+\hat{\beta}_2]]c_7 \\ &= \Big([\hat{\alpha}_3,[\hat{\alpha}_1,\hat{\beta}_1]] + [\hat{\alpha}_3,[\hat{\alpha}_1,\hat{\beta}_2]] + (-1)^{mn}[\hat{\alpha}_3,[\hat{\beta}_1,\hat{\alpha}_2]] \\ &\quad +[\hat{\alpha}_3,[\hat{\alpha}_2,\hat{\beta}_2]]\Big)c_6 + \Big([\hat{\beta}_3,[\hat{\alpha}_1,\hat{\beta}_1]] + [\hat{\beta}_3,[\hat{\alpha}_1,\hat{\beta}_2]] \\ &\quad +(-1)^{mn}[\hat{\beta}_3,[\hat{\beta}_1,\hat{\alpha}_2]] + [\hat{\beta}_3,[\hat{\alpha}_2,\hat{\beta}_2]]\Big)c_7. \end{aligned}$$

Similarly,

$$\begin{aligned} (1 \vee \varphi)Q_2 &= [\hat{\alpha}_2+\hat{\alpha}_3,[\hat{\alpha}_1,\hat{\beta}_1]]c_6 + [\hat{\beta}_2+\hat{\beta}_3,[\hat{\alpha}_1,\hat{\beta}_1]]c_7 \\ &= \Big([\hat{\alpha}_2,[\hat{\alpha}_1,\hat{\beta}_1]] + [\hat{\alpha}_3,[\hat{\alpha}_1,\hat{\beta}_1]]\Big)c_6 \\ &\quad \Big([\hat{\beta}_2,[\hat{\alpha}_1,\hat{\beta}_1]] + [\hat{\beta}_3,[\hat{\alpha}_1,\hat{\beta}_1]]\Big)c_7. \end{aligned}$$

Thus φ is associative if and only if

$$\begin{aligned} 0 &= [\hat{\alpha}_3,[\hat{\alpha}_1,\hat{\beta}_2]]c_6 + (-1)^{mn}[\hat{\alpha}_3,[\hat{\beta}_1,\hat{\alpha}_2]]c_6 \\ &\quad +[\hat{\beta}_3,[\hat{\alpha}_1,\hat{\beta}_2]]c_7 + (-1)^{mn}[\hat{\beta}_3,[\hat{\beta}_1,\hat{\alpha}_2]]c_7. \end{aligned}$$

We note that the above Whitehead products are all basic products. From the Hilton's theorem, we conclude that the associativity of φ_{P,Q_2} holds if and only if $c_6 = c_7 = 0$. □

Definition 3.7. Let Q_3 be the sum of the maps

$$S^p \xrightarrow{c_8} S^{m+2n-2} \xrightarrow{[\beta_1,[\alpha_1,\alpha_2]]} X \vee X \xrightarrow{A\vee A} Y \vee Y$$

and

$$S^p \xrightarrow{c_9} S^{m+2n-2} \xrightarrow{[\beta_2,[\alpha_1,\alpha_2]]} X \vee X \xrightarrow{A \vee A} Y \vee Y$$

for some c_8 and c_9; that is,

$$Q_3 = [\bar{\beta}_1, [\bar{\alpha}_1, \bar{\alpha}_2]]c_8 + [\bar{\beta}_2, [\bar{\alpha}_1, \bar{\alpha}_2]]c_9,$$

and define $\varphi = \varphi_{P,Q_3} : Y \to Y \vee Y$ by Definition 3.1.

Proposition 3.8. *Let $m \leq 2n-2$ and $n < m < p \leq 8n-8$. If the Hopf-Hilton invariants $H_1^1(c_8)$ and $H_1^1(c_9)$ are trivial, then the comultiplication $\varphi = \varphi_{P,Q_3}$ is associative if and only if $c_8 = c_9 = 0$.*

Proof. We compute $J_{12}Q_3$:

$$\begin{aligned} J_{12}Q_3 &= J_{12}([\bar{\beta}_1, [\bar{\alpha}_1, \bar{\alpha}_2]]c_8 + [\bar{\beta}_2, [\bar{\alpha}_1, \bar{\alpha}_2]]c_9) \\ &= [\hat{\beta}_1, [\hat{\alpha}_1, \hat{\alpha}_2]]c_8 + [\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_2]]c_9. \end{aligned}$$

Similarly,

$$J_{23}Q_3 = [\hat{\beta}_2, [\hat{\alpha}_2, \hat{\alpha}_3]]c_8 + [\hat{\beta}_3, [\hat{\alpha}_2, \hat{\alpha}_3]]c_9.$$

We now compute $(\varphi \vee 1)Q_3$:

$$\begin{aligned} (\varphi \vee 1)Q_3 &= [J_{12}\varphi s, [J_{12}\varphi r, \hat{\alpha}_3]]c_8 + [\hat{\beta}_3, [J_{12}\varphi r, \hat{\alpha}_3]]c_9 \\ &= [J_{12}(\bar{\beta}_1 + \bar{\beta}_2), [J_{12}(\bar{\alpha}_1 + \bar{\alpha}_2), \hat{\alpha}_3]]c_8 \\ &\quad + [\hat{\beta}_3, [J_{12}(\bar{\alpha}_1 + \bar{\alpha}_2), \hat{\alpha}_3]]c_9 \\ &= [\hat{\beta}_1 + \hat{\beta}_2, [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\alpha}_3]]c_8 + [\hat{\beta}_3, [\hat{\alpha}_1 + \hat{\alpha}_2, \hat{\alpha}_3]]c_9 \\ &= \Big([\hat{\beta}_1, [\hat{\alpha}_1, \hat{\alpha}_3]] + [\hat{\beta}_1, [\hat{\alpha}_2, \hat{\alpha}_3]] + [\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_3]] \\ &\quad + [\hat{\beta}_2, [\hat{\alpha}_2, \hat{\alpha}_3]]\Big)c_8 + \Big([\hat{\beta}_3, [\hat{\alpha}_1, \hat{\alpha}_3]] + [\hat{\beta}_3, [\hat{\alpha}_2, \hat{\alpha}_3]]\Big)c_9. \end{aligned}$$

Similarly,

$$\begin{aligned} (1 \vee \varphi)Q_3 &= [\hat{\beta}_1, [\hat{\alpha}_1, \hat{\alpha}_2 + \hat{\alpha}_3]]c_8 + [\hat{\beta}_2 + \hat{\beta}_3, [\hat{\alpha}_1, \hat{\alpha}_2 + \hat{\alpha}_3]]c_9 \\ &= \Big([\hat{\beta}_1, [\hat{\alpha}_1, \hat{\alpha}_2]] + [\hat{\beta}_1, [\hat{\alpha}_1, \hat{\alpha}_3]]\Big)c_8 + \Big([\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_2]] \\ &\quad + [\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_3]] + [\hat{\beta}_3, [\hat{\alpha}_1, \hat{\alpha}_2]] + [\hat{\beta}_3, [\hat{\alpha}_1, \hat{\alpha}_3]]\Big)c_9. \end{aligned}$$

Since $[\hat{\beta}_1, [\hat{\alpha}_2, \hat{\alpha}_3]]$ is not a basic product, by using the Jacobi identity, we write it as a sum of basic products as follows:

$$[\hat{\beta}_1, [\hat{\alpha}_2, \hat{\alpha}_3]] = \begin{cases} -[\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_3]] - [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2]] & \text{for m, n even,} \\ +[\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_3]] - [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2]] & \text{for m even, n odd,} \\ +[\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_3]] + [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2]] & \text{for m odd, n even,} \\ +[\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_3]] - [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2]] & \text{for m, n odd.} \end{cases}$$

(1) If m and n are even, then φ_{P,Q_3} is associative if and only if

$$\begin{aligned} 0 &= \Big(-[\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_3]] - [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2]] + [\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_3]]\Big)c_8 \\ &\quad - \Big([\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_3]] + [\hat{\beta}_3, [\hat{\alpha}_1, \hat{\alpha}_2]]\Big)c_9 \\ &= [\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_3]](-c_8) + [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2]](-c_8) + [\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_3]](c_8 - c_9) \\ &\quad + [\hat{\beta}_3[\hat{\alpha}_1, \hat{\alpha}_2]](-c_9). \end{aligned}$$

(2) If m is even and n is odd, then the associativity holds if and only if

$$\begin{aligned} 0 &= \Big([\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_3]] - [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2]] + [\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_3]]\Big)c_8 \\ &\quad - \Big([\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_3]] + [\hat{\beta}_3, [\hat{\alpha}_1, \hat{\alpha}_2]]\Big)c_9 \\ &= [\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_3]](c_8) + [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2]](-c_8) + [\hat{\beta}_2, [\hat{\alpha}_1, \hat{\alpha}_3]](c_8 - c_9) \\ &\quad + [\hat{\beta}_3[\hat{\alpha}_1, \hat{\alpha}_2]](-c_9). \end{aligned}$$

Since the above Whitehead products in (1) and (2) are all basic products, φ is associative if and only if $c_8 = c_9 = 0$. Similarly, the other cases could be treated by using the same method. □

Definition 3.9. Let Q_4 be the map of compositions

$$S^p \xrightarrow{c_{10}} S^{2m+n-2} \xrightarrow{[\alpha_2,[\beta_1,\beta_2]]} X \vee X \xrightarrow{A\vee A} Y \vee Y$$

for some c_{10}; that is,

$$Q_4 = [\bar{\alpha}_2, [\bar{\beta}_1, \bar{\beta}_2]]c_{10},$$

and define $\varphi = \varphi_{P,Q_4} : Y \to Y \vee Y$ by Definition 3.1.

Proposition 3.10. *Let $m \leq 2n-2$ and $n < m < p \leq 8n-8$. If the Hopf-Hilton invariant $H_1^1(c_{10})$ is trivial, then the comultiplication $\varphi = \varphi_{P,Q_4}$ is associative if and only if $c_{10} = 0$.*

Proof. We compute $J_{12}Q_4$ and $(\varphi \vee 1)Q_4$:

$$J_{12}Q_4 = J_{12}[\bar{\alpha}_2, [\bar{\beta}_1, \bar{\beta}_2]]c_{10} = [\hat{\alpha}_2, [\hat{\beta}_1, \hat{\beta}_2]]c_{10}$$

and

$$\begin{aligned}(\varphi \vee 1)Q_4 &= (\varphi \vee 1)[\bar{\alpha}_2, [\bar{\beta}_1, \bar{\beta}_2]]c_{10} = [\hat{\alpha}_3, [J_{12}\varphi s, \hat{\beta}_3]]c_{10} \\ &= [\hat{\alpha}_3, [J_{12}(\bar{\beta}_1 + \bar{\beta}_2), \hat{\beta}_3]]c_{10} = [\hat{\alpha}_3, [\hat{\beta}_1 + \hat{\beta}_2, \hat{\beta}_3]]c_{10} \\ &= [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\beta}_3]]c_{10} + [\hat{\alpha}_3, [\hat{\beta}_2, \hat{\beta}_3]]c_{10}.\end{aligned}$$

Similarly, we have

$$J_{23}Q_4 = J_{23}[\bar{\alpha}_2, [\bar{\beta}_1, \bar{\beta}_2]]c_{10} = [\hat{\alpha}_3, [\hat{\beta}_2, \hat{\beta}_3]]c_{10}$$

and

$$\begin{aligned}(1 \vee \varphi)Q_4 &= (1 \vee \varphi)[\bar{\alpha}_2, [\bar{\beta}_1, \bar{\beta}_2]]c_{10} = [J_{23}\varphi r, [\hat{\beta}_1, J_{23}\varphi s]]c_{10} \\ &= [J_{23}(\bar{\alpha}_1 + \bar{\alpha}_2), [\hat{\beta}_1, J_{23}(\bar{\beta}_1 + \bar{\beta}_2)]]c_{10} \\ &= [\hat{\alpha}_2 + \hat{\alpha}_3, [\hat{\beta}_1, \hat{\beta}_2 + \hat{\beta}_3]]c_{10} \\ &= [\hat{\alpha}_2, [\hat{\beta}_1, \hat{\beta}_2]]c_{10} + [\hat{\alpha}_2, [\hat{\beta}_1, \hat{\beta}_3]]c_{10} \\ &\quad + [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\beta}_2]]c_{10} + [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\beta}_3]]c_{10}.\end{aligned}$$

Thus the associativity holds if and only if

$$0 = [\hat{\alpha}_2, [\hat{\beta}_1, \hat{\beta}_3]](-c_{10}) + [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\beta}_2]](-c_{10}).$$

We note that the above Whitehead products are basic products. Therefore, φ is associative if and only if $c_{10} = 0$. $\square$

Definition 3.11. Let Q_5 be the sum of the maps

$$S^p \xrightarrow{d_1} S^{m+n-1} \xrightarrow{[\beta_1,\alpha_2]} X \vee X \xrightarrow{A\vee A} Y \vee Y\,,$$

$$S^p \xrightarrow{d_2} S^{2m+n-2} \xrightarrow{[\beta_1,[\beta_1,\alpha_2]]} X \vee X \xrightarrow{A\vee A} Y \vee Y\,,$$

$$S^p \xrightarrow{d_3} S^{m+2n-2} \xrightarrow{[\alpha_2,[\beta_1,\alpha_2]]} X \vee X \xrightarrow{A\vee A} Y \vee Y$$

and

$$S^p \xrightarrow{d_4} S^{2m+n-2} \xrightarrow{[\beta_2,[\beta_1,\alpha_2]]} X \vee X \xrightarrow{A\vee A} Y \vee Y$$

for some d_1, d_2, d_3 and d_4; that is,

$$\begin{aligned}Q_5 &= [\bar{\beta}_1, \bar{\alpha}_2]d_1 + [\bar{\beta}_1, [\bar{\beta}_1, \bar{\alpha}_2]]d_2 \\ &\quad + [\bar{\alpha}_2, [\bar{\beta}_1, \bar{\alpha}_2]]d_3 + [\bar{\beta}_2, [\bar{\beta}_1, \bar{\alpha}_2]]d_4,\end{aligned}$$

and define $\varphi = \varphi_{P,Q_5} : Y \to Y \vee Y$ by Definition 3.1.

Proposition 3.12. *Let $m \leq 2n-2$ and $n < m < p \leq 8n-8$. If the Hopf-Hilton invariants $H_1^1(d_1), H_1^2(d_1), H_1^1(d_2), H_1^1(d_3)$ and $H_1^1(d_4)$ are trivial, then the comultiplication $\varphi = \varphi_{P,Q_5}$ is associative if and only if $d_2 = d_3 = d_4 = 0$.*

Proof. We compute $J_{12}Q_5$:

$$\begin{aligned} J_{12}Q_5 &= J_{12}\Big([\bar{\beta}_1, \bar{\alpha}_2]d_1 + [\bar{\beta}_1, [\bar{\beta}_1, \bar{\alpha}_2]]d_2 \\ &\quad +[\bar{\alpha}_2, [\bar{\beta}_1, \bar{\alpha}_2]]d_3 + [\bar{\beta}_2, [\bar{\beta}_1, \bar{\alpha}_2]]d_4\Big) \\ &= [\hat{\beta}_1, \hat{\alpha}_2]d_1 + [\hat{\beta}_1, [\hat{\beta}_1, \hat{\alpha}_2]]d_2 + [\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_2]]d_3 \\ &\quad +[\hat{\beta}_2, [\hat{\beta}_1, \hat{\alpha}_2]]d_4. \end{aligned}$$

Similarly,

$$\begin{aligned} J_{23}Q_5 &= [\hat{\beta}_2, \hat{\alpha}_3]d_1 + [\hat{\beta}_2, [\hat{\beta}_2, \hat{\alpha}_3]]d_2 + [\hat{\alpha}_3, [\hat{\beta}_2, \hat{\alpha}_3]]d_3 \\ &\quad +[\hat{\beta}_3, [\hat{\beta}_2, \hat{\alpha}_3]]d_4. \end{aligned}$$

We now compute $(\varphi \vee 1)Q_5$:

$$\begin{aligned} (\varphi \vee 1)Q_5 &= [J_{12}\varphi s, \hat{\alpha}_3]d_1 + [J_{12}\varphi s, [J_{12}\varphi s, \hat{\alpha}_3]]d_2 \\ &\quad +[\hat{\alpha}_3, [J_{12}\varphi s, \hat{\alpha}_3]]d_3 + [\hat{\beta}_3, [J_{12}\varphi s, \hat{\alpha}_3]]d_4 \\ &= [J_{12}(\bar{\beta}_1 + \bar{\beta}_2), \hat{\alpha}_3]d_1 \\ &\quad +[J_{12}(\bar{\beta}_1 + \bar{\beta}_2), [J_{12}(\bar{\beta}_1 + \bar{\beta}_2), \hat{\alpha}_3]]d_2 \\ &\quad +[\hat{\alpha}_3, [J_{12}(\bar{\beta}_1 + \bar{\beta}_2), \hat{\alpha}_3]]d_3 + [\hat{\beta}_3, [J_{12}(\bar{\beta}_1 + \bar{\beta}_2), \hat{\alpha}_3]]d_4 \\ &= [\hat{\beta}_1 + \hat{\beta}_2, \hat{\alpha}_3]d_1 + [\hat{\beta}_1 + \hat{\beta}_2, [\hat{\beta}_1 + \hat{\beta}_2, \hat{\alpha}_3]]d_2 \\ &\quad +[\hat{\alpha}_3, [\hat{\beta}_1 + \hat{\beta}_2, \hat{\alpha}_3]]d_3 + [\hat{\beta}_3, [\hat{\beta}_1 + \hat{\beta}_2, \hat{\alpha}_3]]d_4 \\ &= \Big([\hat{\beta}_1, \hat{\alpha}_3] + [\hat{\beta}_2, \hat{\alpha}_3]\Big)d_1 + \Big([\hat{\beta}_1, [\hat{\beta}_1, \hat{\alpha}_3]] \\ &\quad +[\hat{\beta}_1, [\hat{\beta}_2, \hat{\alpha}_3]] + [\hat{\beta}_2, [\hat{\beta}_1, \hat{\alpha}_3]] + [\hat{\beta}_2, [\hat{\beta}_2, \hat{\alpha}_3]]\Big)d_2 \\ &\quad +\Big([\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_3]] + [\hat{\alpha}_3, [\hat{\beta}_2, \hat{\alpha}_3]]\Big)d_3 \\ &\quad +\Big([\hat{\beta}_3, [\hat{\beta}_1, \hat{\alpha}_3]] + [\hat{\beta}_3, [\hat{\beta}_2, \hat{\alpha}_3]]\Big)d_4. \end{aligned}$$

We need to change the non-basic product $[\hat{\beta}_1, [\hat{\beta}_2, \hat{\alpha}_3]]$ into basic products by using the Jacobi identity as follows:

$$\begin{aligned} (-1)^{m(n-1)}[\hat{\beta}_1, [\hat{\beta}_2, \hat{\alpha}_3]] &= -(-1)^{m(m-1)+mn}[\hat{\beta}_2, [\hat{\beta}_1, \hat{\alpha}_3]] \\ &\quad -(-1)^{n(m-1)}[\hat{\alpha}_3, [\hat{\beta}_1, \hat{\beta}_2]]. \end{aligned}$$

We note that the Whitehead products on the right-hand side in the above equation are both basic products. Similarly,

$$\begin{aligned} (1 \vee \varphi)Q_5 &= [\hat{\beta}_1, J_{23}\varphi r]d_1 + [\hat{\beta}_1, [\hat{\beta}_1, J_{23}\varphi r]]d_2 \\ &\quad +[J_{23}\varphi r, [\hat{\beta}_1, J_{23}\varphi r]]d_3 + [J_{23}\varphi s, [\hat{\beta}_1, J_{23}\varphi r]]d_4 \\ &= [\hat{\beta}_1, J_{23}(\bar{\alpha}_1 + \bar{\alpha}_2)]d_1 + [\hat{\beta}_1, [\hat{\beta}_1, J_{23}(\bar{\alpha}_1 + \bar{\alpha}_2)]]d_2 \\ &\quad +[J_{23}(\bar{\alpha}_1 + \bar{\alpha}_2), [\hat{\beta}_1, J_{23}(\bar{\alpha}_1 + \bar{\alpha}_2)]]d_3 \\ &\quad +[J_{23}(\bar{\beta}_1 + \bar{\beta}_2), [\hat{\beta}_1, J_{23}(\bar{\alpha}_1 + \bar{\alpha}_2)]]d_4 \\ &= [\hat{\beta}_1, \hat{\alpha}_2 + \hat{\alpha}_3]d_1 + [\hat{\beta}_1, [\hat{\beta}_1, \hat{\alpha}_2 + \hat{\alpha}_3]]d_2 \\ &\quad +[\hat{\alpha}_2 + \hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2 + \hat{\alpha}_3]]d_3 \\ &\quad +[\hat{\beta}_2 + \hat{\beta}_3, [\hat{\beta}_1, \hat{\alpha}_2 + \hat{\alpha}_3]]d_4 \\ &= \Big([\hat{\beta}_1, \hat{\alpha}_2] + [\hat{\beta}_1, \hat{\alpha}_3]\Big)d_1 \\ &\quad +\Big([\hat{\beta}_1, [\hat{\beta}_1, \hat{\alpha}_2]] + [\hat{\beta}_1, [\hat{\beta}_1, \hat{\alpha}_3]]\Big)d_2 \\ &\quad +\Big([\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_2]] + [\hat{\alpha}_2, [\hat{\beta}_1, \hat{\alpha}_3]] + [\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_2]] \\ &\quad +[\hat{\alpha}_3, [\hat{\beta}_1, \hat{\alpha}_3]]\Big)d_3 + \Big([\hat{\beta}_2, [\hat{\beta}_1, \hat{\alpha}_2]] + [\hat{\beta}_2, [\hat{\beta}_1, \hat{\alpha}_3]] \\ &\quad +[\hat{\beta}_3, [\hat{\beta}_1, \hat{\alpha}_2]] + [\hat{\beta}_3, [\hat{\beta}_1, \hat{\alpha}_3]]\Big)d_4. \end{aligned}$$

We now find the conditions for the equation

$$J_{12}Q_5 + (\varphi \vee 1)Q_5 = J_{23}Q_5 + (1 \vee \varphi)Q_5$$

to be satisfied. We also note that the following Whitehead products in (1) and (2) are all basic products.

(1) If m is odd and n is even, then φ_{P,Q_5} is associative if and only if

$$\begin{aligned} 0 &= [\hat\beta_2,[\hat\beta_1,\hat\alpha_3]]d_2 + [\hat\alpha_3,[\hat\beta_1,\hat\beta_2]]d_2 + [\hat\beta_2,[\hat\beta_1,\hat\alpha_3]]d_2 \\ &\quad -\Big([\hat\alpha_2,[\hat\beta_1,\hat\alpha_3]] + [\hat\alpha_3,[\hat\beta_1,\hat\alpha_2]]\Big)d_3 \\ &\quad -\Big([\hat\beta_2,[\hat\beta_1,\hat\alpha_3]] + [\hat\beta_3,[\hat\beta_1,\hat\alpha_2]]\Big)d_4 \\ &= [\hat\alpha_3,[\hat\beta_1,\hat\beta_2]]d_2 + [\hat\alpha_2,[\hat\beta_1,\hat\alpha_3]](-d_3) + [\hat\alpha_3,[\hat\beta_1,\hat\alpha_2]](-d_3) \\ &\quad [\hat\beta_2,[\hat\beta_1,\hat\alpha_3]](2d_2 - d_4) + [\hat\beta_3,[\hat\beta_1,\hat\alpha_2]](-d_4). \end{aligned}$$

(2) If m and n are odd, then the associativity holds if and only if

$$\begin{aligned} 0 &= [\hat\beta_2,[\hat\beta_1,\hat\alpha_3]]d_2 + (-[\hat\alpha_3,[\hat\beta_1,\hat\beta_2]])d_2 + [\hat\beta_2,[\hat\beta_1,\hat\alpha_3]]d_2 \\ &\quad -\Big([\hat\alpha_2,[\hat\beta_1,\hat\alpha_3]] + [\hat\alpha_3,[\hat\beta_1,\hat\alpha_2]]\Big)d_3 \\ &\quad -\Big([\hat\beta_2,[\hat\beta_1,\hat\alpha_3]] + [\hat\beta_3,[\hat\beta_1,\hat\alpha_2]]\Big)d_4 \\ &= [\hat\alpha_3,[\hat\beta_1,\hat\beta_2]](-d_2) + [\hat\alpha_2,[\hat\beta_1,\hat\alpha_3]](-d_3) + [\hat\alpha_3,[\hat\beta_1,\hat\alpha_2]](-d_3) \\ &\quad [\hat\beta_2,[\hat\beta_1,\hat\alpha_3]](2d_2 - d_4) + [\hat\beta_3,[\hat\beta_1,\hat\alpha_2]](-d_4). \end{aligned}$$

Thus φ is associative if and only if $d_2 = d_3 = d_4 = 0$. Similarly, the other cases could be proved by the same way. □

We now consider the sum of the perturbations Q_j, $j = 1, 2, \ldots, 5$ defined above in order to investigate all the comultiplications on a wedge of three spheres completely in a certain range as follows:

Definition 3.13. Let Q be the sum of the perturbations $Q_1, Q_2, \ldots, Q_5$; that is,

$$\begin{aligned} Q = \textstyle\sum_{i=1}^5 Q_i &= [\bar\beta_1,\bar\beta_2]a_1 + [\bar\beta_1,[\bar\beta_1,\bar\beta_2]]a_2 + [\bar\beta_2,[\bar\beta_1,\bar\beta_2]]a_3 \\ &\quad +[\bar\alpha_1,\bar\alpha_2]b_1 + [\bar\alpha_1,[\bar\alpha_1,\bar\alpha_2]]b_2 + [\bar\alpha_2,[\bar\alpha_1,\bar\alpha_2]]b_3 \\ &\quad +[\bar\alpha_1,\bar\beta_2]c_1 + [\bar\alpha_1,[\bar\alpha_1,\bar\beta_2]]c_2 + [\bar\alpha_2,[\bar\alpha_1,\bar\beta_2]]c_3 \\ &\quad +[\bar\beta_1,[\bar\alpha_1,\bar\beta_2]]c_4 + [\bar\beta_2,[\bar\alpha_1,\bar\beta_2]]c_5 + [\bar\alpha_2,[\bar\alpha_1,\bar\beta_1]]c_6 \\ &\quad\quad +[\bar\beta_2,[\bar\alpha_1,\bar\beta_1]]c_7 + [\bar\beta_1,[\bar\alpha_1,\bar\alpha_2]]c_8 + [\bar\beta_2,[\bar\alpha_1,\bar\alpha_2]]c_9 \\ &\quad\quad +[\bar\alpha_2,[\bar\beta_1,\bar\beta_2]]c_{10} + [\bar\beta_1,\bar\alpha_2]d_1 + [\bar\beta_1,[\bar\beta_1,\bar\alpha_2]]d_2 \\ &\quad\quad +[\bar\alpha_2,[\bar\beta_1,\bar\alpha_2]]d_3 + [\bar\beta_2,[\bar\beta_1,\bar\alpha_2]]d_4, \end{aligned}$$

and define $\varphi = \varphi_{P,Q} : Y \to Y \vee Y$ by Definition 3.1.

The following is one of the technical results in this paper.

Theorem 3.14. *Let $m \le 2n-2, n < m < p \le 8n-8$ and let the Hopf-Hilton invariants $H_1^1(x)$ and $H_1^2(y)$ be trivial, where $x = a_i, b_i, c_j, d_k$ for $i = 1,2,3$, $j = 1,2,\ldots,10$, $k = 1,2,3,4$ and $y = a_1, b_1, c_1, d_1$. Then the comultiplication $\varphi = \varphi_{P,Q}$ is associative if and only if* (1) *m and n are even, $a_2 = a_3 = b_2 = b_3 = c_3 = c_6 = d_4 = 0$, $c_4 = c_5 = c_7 = -c_{10} = d_2$ and $-c_2 = c_8 = c_9 = -d_3$, or* (2) *m is even and n is odd, $a_2 = a_3 = d_4 = 0$, $c_4 = c_5 = c_7 = c_{10} = d_2$, $b_2 = -b_3, 3b_2 = 0$, $c_3 = c_6 = 2d_3$ and $c_2 = c_8 = c_9 = d_3$, or* (3) *m is odd and n is even, $b_2 = b_3 = c_3 = c_6 = 0$, $a_2 = -a_3, 3a_2 = 0$, $c_2 = c_8 = c_9 = d_3$, $c_4 = c_5 = c_7 = c_{10} = d_2$ and $d_4 = 2c_4$, or* (4) *m and n are odd, $a_2 = -a_3, b_2 = -b_3, 3a_2 = 0 = 3b_2$, $c_2 = -c_8 = -c_9 = -d_3$, $2c_2 = c_3 = c_6$, $c_4 = c_5 = c_7 = c_{10} = -d_2$ and $2d_2 = d_4$.*

Proof. To prove this theorem, we will make use of the previous results from Propositions 3.3, 3.6, 3.8, 3.10 and 3.12. We also note that following Whitehead products in (1), (2), (3) and (4) are all the basic products.

(1) If m and n are even, then the equality $J_{12}Q + (\varphi \vee 1)Q = J_{23}Q + (1 \vee \varphi)Q$ holds if and only if

$$\begin{aligned}0 &= [\hat\beta_3,[\hat\beta_1,\hat\beta_2]](-a_2-a_3) + [\hat\beta_2,[\hat\beta_1,\hat\beta_3]](-a_3)\\ &+[\hat\alpha_3,[\hat\alpha_1,\hat\alpha_2]](-b_2-b_3) + [\hat\alpha_2,[\hat\alpha_1,\hat\alpha_3]](-b_3)\\ &+[\hat\beta_3,[\hat\alpha_1,\hat\alpha_2]](-c_2-c_9) + [\hat\alpha_2,[\hat\alpha_1,\hat\beta_3]](-c_3)\\ &+[\hat\alpha_3,[\hat\alpha_1,\hat\beta_2]](-c_3+c_6) + [\hat\alpha_2,[\hat\beta_1,\hat\beta_3]](-c_4-c_{10})\\ &+[\hat\beta_3,[\hat\beta_1,\hat\alpha_2]](-c_4+c_7-d_4) + [\hat\beta_2,[\hat\alpha_1,\hat\beta_3]](c_4-c_5)\\ &+[\hat\beta_3,[\hat\alpha_1,\hat\beta_2]](-c_5+c_7) + [\hat\alpha_3,[\hat\beta_1,\hat\alpha_2]](c_6-c_8-d_3)\\ &+[\hat\alpha_2,[\hat\beta_1,\hat\alpha_3]](-c_8-d_3) + [\hat\beta_2,[\hat\alpha_1,\hat\alpha_3]](c_8-c_9)\\ &+[\hat\alpha_3,[\hat\beta_1,\hat\beta_2]](-d_2-c_{10}) + [\hat\beta_2,[\hat\beta_1,\hat\alpha_3]](-d_4).\end{aligned}$$

Thus φ is associative if and only if $a_2 = a_3 = b_2 = b_3 = c_3 = c_6 = d_4 = 0$, $c_4 = c_5 = c_7 = -c_{10} = d_2$ and $-c_2 = c_8 = c_9 = -d_3$.

(2) If m is even and n is odd, then the associativity of the comultiplication $\varphi = \varphi_{P,Q} : Y \to Y \vee Y$ holds if and only if

$$\begin{aligned}0 &= [\hat\beta_3,[\hat\beta_1,\hat\beta_2]](-a_2-a_3) + [\hat\beta_2,[\hat\beta_1,\hat\beta_3]](-a_3)\\ &+[\hat\alpha_3,[\hat\alpha_1,\hat\alpha_2]](-b_2-b_3) + [\hat\alpha_2,[\hat\alpha_1,\hat\alpha_3]](2b_2-b_3)\\ &+[\hat\alpha_2,[\hat\alpha_1,\hat\beta_3]](2c_2-c_3) + [\hat\beta_3,[\hat\alpha_1,\hat\alpha_2]](c_2-c_9)\\ &+[\hat\alpha_3,[\hat\alpha_1,\hat\beta_2]](-c_3+c_6) + [\hat\alpha_2,[\hat\beta_1,\hat\beta_3]](c_4-c_{10})\\ &+[\hat\beta_3,[\hat\beta_1,\hat\alpha_2]](-c_4+c_7-d_4) + [\hat\beta_2,[\hat\alpha_1,\hat\beta_3]](c_4-c_5)\\ &+[\hat\beta_3,[\hat\alpha_1,\hat\beta_2]](-c_5+c_7) + [\hat\alpha_3,[\hat\beta_1,\hat\alpha_2]](c_6-c_8-d_3)\\ &+[\hat\alpha_2,[\hat\beta_1,\hat\alpha_3]](c_8-d_3) + [\hat\beta_2,[\hat\alpha_1,\hat\alpha_3]](c_8-c_9)\\ &+[\hat\alpha_3,[\hat\beta_1,\hat\beta_2]](d_2-c_{10}) + [\hat\beta_2,[\hat\beta_1,\hat\alpha_3]](-d_4).\end{aligned}$$

Thus φ is associative if and only if $a_2 = a_3 = d_4 = 0$, $c_4 = c_5 = c_7 = c_{10} = d_2$, $b_2 = -b_3, 3b_2 = 0$, $c_3 = c_6 = 2d_3$ and $c_2 = c_8 = c_9 = d_3$.

(3) If m is odd and n is even, then the comultiplication $\varphi = \varphi_{P,Q}$ is associative if and only if

$$\begin{aligned}0 &= [\hat\beta_3,[\hat\beta_1,\hat\beta_2]](-a_2-a_3) + [\hat\beta_2,[\hat\beta_1,\hat\beta_3]](2a_2-a_3)\\ &+[\hat\alpha_3,[\hat\alpha_1,\hat\alpha_2]](-b_2-b_3) + [\hat\alpha_2,[\hat\alpha_1,\hat\alpha_3]](-b_3)\\ &+[\hat\alpha_2,[\hat\alpha_1,\hat\beta_3]](-c_3) + [\hat\beta_3,[\hat\alpha_1,\hat\alpha_2]](c_2-c_9)\\ &+[\hat\alpha_3,[\hat\alpha_1,\hat\beta_2]](-c_3+c_6) + [\hat\alpha_2,[\hat\beta_1,\hat\beta_3]](c_4-c_{10})\\ &+[\hat\beta_3,[\hat\beta_1,\hat\alpha_2]](c_4+c_7-d_4) + [\hat\beta_2,[\hat\alpha_1,\hat\beta_3]](c_4-c_5)\\ &+[\hat\beta_3,[\hat\alpha_1,\hat\beta_2]](-c_5+c_7) + [\hat\alpha_3,[\hat\beta_1,\hat\alpha_2]](c_6+c_8-d_3)\\ &+[\hat\alpha_2,[\hat\beta_1,\hat\alpha_3]](c_8-d_3) + [\hat\beta_2,[\hat\alpha_1,\hat\alpha_3]](c_8-c_9)\\ &+[\hat\beta_2,[\hat\beta_1,\hat\alpha_3]](2d_2-d_4) + [\hat\alpha_3,[\hat\beta_1,\hat\beta_2]](d_2-c_{10}).\end{aligned}$$

Thus φ is associative if and only if $b_2 = b_3 = c_3 = c_6 = 0$, $a_2 = -a_3, 3a_2 = 0$, $c_2 = c_8 = c_9 = d_3$, $c_4 = c_5 = c_7 = c_{10} = d_2$ and $d_4 = 2c_4$.

(4) If m and n are odd, then the given comultiplication is associative if and only if

$$\begin{aligned}0 &= [\hat\beta_3,[\hat\beta_1,\hat\beta_2]](-a_2-a_3) + [\hat\beta_2,[\hat\beta_1,\hat\beta_3]](2a_2-a_3)\\ &+[\hat\alpha_3,[\hat\alpha_1,\hat\alpha_2]](-b_2-b_3) + [\hat\alpha_2,[\hat\alpha_1,\hat\alpha_3]](2b_2-b_3)\\ &+[\hat\alpha_2,[\hat\alpha_1,\hat\beta_3]](2c_2-c_3) + [\hat\beta_3,[\hat\alpha_1,\hat\alpha_2]](-c_2-c_9)\\ &+[\hat\alpha_3,[\hat\alpha_1,\hat\beta_2]](-c_3+c_6) + [\hat\alpha_2,[\hat\beta_1,\hat\beta_3]](c_4-c_{10})\\ &+[\hat\beta_3,[\hat\beta_1,\hat\alpha_2]](-c_4-c_7-d_4) + [\hat\beta_2,[\hat\alpha_1,\hat\beta_3]](c_4-c_5)\\ &+[\hat\beta_3,[\hat\alpha_1,\hat\beta_2]](-c_5+c_7) + [\hat\alpha_3,[\hat\beta_1,\hat\alpha_2]](-c_6-c_8-d_3)\\ &+[\hat\alpha_2,[\hat\beta_1,\hat\alpha_3]](c_8-d_3) + [\hat\beta_2,[\hat\alpha_1,\hat\alpha_3]](c_8-c_9)\\ &+[\hat\beta_2,[\hat\beta_1,\hat\alpha_3]](2d_2-d_4) + [\hat\alpha_3,[\hat\beta_1,\hat\beta_2]](-d_2-c_{10}).\end{aligned}$$

Thus φ is associative if and only if $a_2 = -a_3, 3a_2 = 0, b_2 = -b_3, 3b_2 = 0$, $c_2 = -c_8 = -c_9 = -d_3$, $2c_2 = c_3 = c_6$, $c_4 = c_5 = c_7 = c_{10} = -d_2$ and $2d_2 = d_4$. □

Let $\mathcal{A}(Y)$ be the set of homotopy classes of associative comultiplications on a co-H-space Y and let $|S|$ be the number of elements in a set S. Then we have the one of the main theorems in this paper as follows:

Theorem 3.15. *Let $m \le 2n-2, n < m < p \le 4n-4$ and let $Y = S^n \vee S^m \vee S^p$.* (1) *If m and n are even, then*

$$\begin{aligned}|\mathcal{A}(Y)| &= |\pi_p(S^{2m-1})| \times |\pi_p(S^{2n-1})| \times |\pi_p(S^{m+n-1})|^2 \times |\pi_p(S^{2m+n-2})| \\ &\quad \times |\pi_p(S^{m+2n-2})|.\end{aligned}$$

(2) *If m is even and n is odd, then*

$$\begin{aligned}|\mathcal{A}(Y)| &= |\pi_p(S^{2m-1})| \times |\pi_p(S^{2n-1})| \times |\{b_2 \in \pi_p(S^{3n-2}) \,|\, 3b_2 = 0\}| \\ &\quad \times |\pi_p(S^{m+n-1})|^2 \times |\pi_p(S^{m+2n-2})| \times |\pi_p(S^{2m+n-2})|.\end{aligned}$$

(3) *If m is odd and n is even, then*

$$\begin{aligned}|\mathcal{A}(Y)| &= |\pi_p(S^{2m-1})| \times |\pi_p(S^{2n-1})| \times |\{a_2 \in \pi_p(S^{3m-2}) \,|\, 3a_2 = 0\}| \\ &\quad \times |\pi_p(S^{m+n-1})|^2 \times |\pi_p(S^{m+2n-2})| \times |\pi_p(S^{2m+n-2})|.\end{aligned}$$

(4) *If m and n are odd, then*

$$\begin{aligned}|\mathcal{A}(Y)| &= |\pi_p(S^{2m-1})| \times |\pi_p(S^{2n-1})| \times |\{a_2 \in \pi_p(S^{3m-2}) \,|\, 3a_2 = 0\}| \\ &\quad \times |\{b_2 \in \pi_p(S^{3n-2}) \,|\, 3b_2 = 0\}| \times |\pi_p(S^{m+n-1})|^2 \\ &\quad \times |\pi_p(S^{m+2n-2})| \times |\pi_p(S^{2m+n-2})|.\end{aligned}$$

Proof. Since every comultiplication has the form in Definition 3.13, by Theorem 3.14, the result follows. □

Example 3.16. The set $\mathcal{A}(S^4 \vee S^5 \vee S^{12})$ is infinite, and $|\mathcal{A}(S^8 \vee S^{12} \vee S^{27})| = 32$. Indeed, $|\mathcal{A}(S^8 \vee S^{12} \vee S^{27})| = |\pi_{27}(S^{23})| \times |\pi_{27}(S^{15})| \times |\pi_{27}(S^{19})|^2 \times |\pi_{27}(S^{30})| \times |\pi_{27}(S^{26})| = 1 \times 1 \times 16 \times 1 \times 2 = 32$, and similarly for the other case.

4. Commutativity

We recall that $\varphi = \varphi_{P,Q} : Y \to Y \vee Y$ in Definition 3.1 is commutative if and only if $TP = P$ and $TQ = Q$. We also note that $T\iota_1 = \iota_2$ and $T\iota_2 = \iota_1$, where $T : Y \vee Y \to Y \vee Y$ is the switching map.

Proposition 4.1. *Let $\varphi = \varphi_{P,Q_1} : Y \to Y \vee Y$ be the comultiplication in Definition 3.2. Then, under the same hypotheses of Proposition 3.3, φ is commutative if and only if* (1) *m and n are even, $a_2 = a_3, b_2 = b_3$ and $c_i = 0$ for $i = 1, 2, \dots, 5$, or* (2) *m is even and n is odd, $a_2 = a_3, b_2 = -b_3, 2b_1 = 0$ and $c_i = 0$ for $i = 1, 2, \dots, 5$, or* (3) *m is odd and n is even, $a_2 = -a_3, b_2 = b_3, 2a_1 = 0$ and $c_i = 0$ for $i = 1, 2, \dots, 5$, or* (4) *m and n are odd, $a_2 = -a_3, b_2 = -b_3, 2a_1 = 0 = 2b_1$ and $c_i = 0$ for $i = 1, 2, \dots, 5$.*

Proof. Since

$$\begin{aligned}Q_1 &= [\bar\beta_1, \bar\beta_2]a_1 + [\bar\beta_1, [\bar\beta_1, \bar\beta_2]]a_2 + [\bar\beta_2, [\bar\beta_1, \bar\beta_2]]a_3 \\ &\quad + [\bar\alpha_1, \bar\alpha_2]b_1 + [\bar\alpha_1, [\bar\alpha_1, \bar\alpha_2]]b_2 + [\bar\alpha_2, [\bar\alpha_1, \bar\alpha_2]]b_3 \\ &\quad + [\bar\alpha_1, \bar\beta_2]c_1 + [\bar\alpha_1, [\bar\alpha_1, \bar\beta_2]]c_2 + [\bar\alpha_2, [\bar\alpha_1, \bar\beta_2]]c_3 \\ &\quad + [\bar\beta_1, [\bar\alpha_1, \bar\beta_2]]c_4 + [\bar\beta_2, [\bar\alpha_1, \bar\beta_2]]c_5,\end{aligned}$$

we have

$$\begin{aligned}TQ_1 &= [\bar\beta_2, \bar\beta_1]a_1 + [\bar\beta_2, [\bar\beta_2, \bar\beta_1]]a_2 + [\bar\beta_1, [\bar\beta_2, \bar\beta_1]]a_3 \\ &\quad + [\bar\alpha_2, \bar\alpha_1]b_1 + [\bar\alpha_2, [\bar\alpha_2, \bar\alpha_1]]b_2 + [\bar\alpha_1, [\bar\alpha_2, \bar\alpha_1]]b_3 \\ &\quad + [\bar\alpha_2, \bar\beta_1]c_1 + [\bar\alpha_2, [\bar\alpha_2, \bar\beta_1]]c_2 + [\bar\alpha_1, [\bar\alpha_2, \bar\beta_1]]c_3 \\ &\quad + [\bar\beta_2, [\bar\alpha_2, \bar\beta_1]]c_4 + [\bar\beta_1, [\bar\alpha_2, \bar\beta_1]]c_5 \\ &= (-1)^{m^2}[\bar\beta_1, \bar\beta_2]a_1 + (-1)^{m^2}[\bar\beta_2, [\bar\beta_1, \bar\beta_2]]a_2 \\ &\quad + (-1)^{m^2}[\bar\beta_1, [\bar\beta_1, \bar\beta_2]]a_3 + (-1)^{n^2}[\bar\alpha_1, \bar\alpha_2]b_1 \\ &\quad + (-1)^{n^2}[\bar\alpha_2, [\bar\alpha_1, \bar\alpha_2]]b_2 + (-1)^{n^2}[\bar\alpha_1, [\bar\alpha_1, \bar\alpha_2]]b_3 \\ &\quad + (-1)^{mn}[\bar\beta_1, \bar\alpha_2]c_1 + (-1)^{mn}[\bar\alpha_2, [\bar\beta_1, \bar\alpha_2]]c_2 \\ &\quad + (-1)^{mn}[\bar\alpha_1, [\bar\beta_1, \bar\alpha_2]]c_3 + (-1)^{mn}[\bar\beta_2, [\bar\beta_1, \bar\alpha_2]]c_4 \\ &\quad + (-1)^{mn}[\bar\beta_1, [\bar\beta_1, \bar\alpha_2]]c_5.\end{aligned}$$

We note that the Whitehead product $[\bar{\alpha}_1, [\bar{\beta}_1, \bar{\alpha}_2]$ in the above equations is still not a basic product. We need to change it into the basic product by using the Jacobi identity as follows:

$$\begin{aligned}(-1)^{n(n-1)}[\bar{\alpha}_1, [\bar{\beta}_1, \bar{\alpha}_2] &= -(-1)^{m(n-1)+n^2}[\bar{\beta}_1, [\bar{\alpha}_1, \bar{\alpha}_2]] \\ &\quad -(-1)^{n(m-1)}[\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\beta}_1]].\end{aligned}$$

By using the basic products on the right-hand side of the above equation, we see that the following Whitehead products in (1) and (2) are all basic products. We recall that

$$(-[\bar{\beta}_1, \bar{\beta}_2])a_1 = [\bar{\beta}_1, \bar{\beta}_2](-1)a_1 = [\bar{\beta}_1, \bar{\beta}_2](-a_1 + [1,1]H_1^1(a_1))$$

by the Hilton's formula.

(1) If m is even and n is odd, then

$$\begin{aligned}Q_1 - TQ_1 &= [\bar{\beta}_1, [\bar{\beta}_1, \bar{\beta}_2]](a_2 - a_3) + [\bar{\beta}_2, [\bar{\beta}_1, \bar{\beta}_2]](a_3 - a_2) \\ &\quad +[\bar{\alpha}_1, \bar{\alpha}_2](2b_1 - [1,1]H_1^1(b_1)) \\ &\quad +[\bar{\alpha}_1, [\bar{\alpha}_1, \bar{\alpha}_2]](b_2 + b_3) + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\alpha}_2]](b_3 + b_2) \\ &\quad +[\bar{\alpha}_1, \bar{\beta}_2]c_1 + [\bar{\alpha}_1, [\bar{\alpha}_1, \bar{\beta}_2]]c_2 + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\beta}_2]]c_3 \\ &\quad +[\bar{\beta}_1, [\bar{\alpha}_1, \bar{\beta}_2]]c_4 + [\bar{\beta}_2, [\bar{\alpha}_1, \bar{\beta}_2]]c_5 \\ &\quad +[\bar{\beta}_1, \bar{\alpha}_2](-c_1) + [\bar{\alpha}_2, [\bar{\beta}_1, \bar{\alpha}_2]](-c_2) \\ &\quad +[\bar{\beta}_1, [\bar{\alpha}_1, \bar{\alpha}_2]](-c_3) + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\beta}_1]](-c_3) \\ &\quad +[\bar{\beta}_2, [\bar{\beta}_1, \bar{\alpha}_2]](-c_4) + [\bar{\beta}_1, [\bar{\beta}_1, \bar{\alpha}_2]](-c_5).\end{aligned}$$

Since the Hopf-Hilton invariant $H_1^1(b_1)$ is trivial, φ is commutative if and only if $a_2 = a_3, b_2 = -b_3, 2b_1 = 0$ and $c_i = 0$ for $i = 1, 2, \ldots, 5$.

(2) If m and n are odd, then

$$\begin{aligned}Q_1 - TQ_1 &= [\bar{\beta}_1, \bar{\beta}_2](2a_1 - [1,1]H_1^1(a_1)) + [\bar{\alpha}_1, \bar{\alpha}_2](2b_1 - [1,1]H_1^1(b_1)) \\ &\quad +[\bar{\beta}_1, [\bar{\beta}_1, \bar{\beta}_2]](a_2 + a_3) + [\bar{\beta}_2, [\bar{\beta}_1, \bar{\beta}_2]](a_3 + a_2) \\ &\quad +[\bar{\alpha}_1, [\bar{\alpha}_1, \bar{\alpha}_2]](b_2 + b_3) + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\alpha}_2]](b_3 + b_2) \\ &\quad +[\bar{\alpha}_1, \bar{\beta}_2]c_1 + [\bar{\alpha}_1, [\bar{\alpha}_1, \bar{\beta}_2]]c_2 + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\beta}_2]]c_3 \\ &\quad +[\bar{\beta}_1, [\bar{\alpha}_1, \bar{\beta}_2]]c_4 + [\bar{\beta}_2, [\bar{\alpha}_1, \bar{\beta}_2]]c_5 \\ &\quad +[\bar{\beta}_1, \bar{\alpha}_2]c_1 + [\bar{\alpha}_2, [\bar{\beta}_1, \bar{\alpha}_2]]c_2 \\ &\quad +[\bar{\beta}_1, [\bar{\alpha}_1, \bar{\alpha}_2]]c_3 + [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\beta}_1]](-c_3) \\ &\quad +[\bar{\beta}_2, [\bar{\beta}_1, \bar{\alpha}_2]]c_4 + [\bar{\beta}_1, [\bar{\beta}_1, \bar{\alpha}_2]]c_5.\end{aligned}$$

Since the Hopf-Hilton invariants $H_1^1(a_1)$ and $H_1^1(b_1)$ are trivial, φ is commutative if and only if $a_2 = -a_3, b_2 = -b_3, 2a_1 = 0, 2b_1 = 0$ and $c_i = 0$ for $i = 1, 2, \ldots, 5$. Similarly, the other cases could be proved by the same method. □

Proposition 4.2. *Let $\varphi = \varphi_{P,Q_2} : Y \to Y \vee Y$ be the comultiplication in Definition 3.5. Then, under the same hypotheses of Proposition 3.6, φ is commutative if and only if $c_6 = c_7 = 0$.*

Proof. From $Q_2 = [\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\beta}_1]]c_6 + [\bar{\beta}_2, [\bar{\alpha}_1, \bar{\beta}_1]]c_7$, we have

$$TQ_2 = [\bar{\alpha}_1, [\bar{\alpha}_2, \bar{\beta}_2]]c_6 + [\bar{\beta}_1, [\bar{\alpha}_2, \bar{\beta}_2]]c_7.$$

Since TQ_2 consists of two non-basic Whitehead products, we need to change them into the basic product by using the Jacobi identity as follows:

$$\begin{aligned}(-1)^{n(m-1)}[\bar{\alpha}_1, [\bar{\alpha}_2, \bar{\beta}_2]] &= -(-1)^{n(n-1)+mn}[\bar{\alpha}_2, [\bar{\alpha}_1, \bar{\beta}_2]] \\ &\quad -(-1)^{m(n-1)}[\bar{\beta}_2, [\bar{\alpha}_1, \bar{\alpha}_2]]\end{aligned}$$

and

$$\begin{aligned}(-1)^{m(m-1)}[\bar{\beta}_1, [\bar{\alpha}_2, \bar{\beta}_2]] &= -(-1)^{n(m-1)+m^2}[\bar{\alpha}_2, [\bar{\beta}_1, \bar{\beta}_2]] \\ &\quad -(-1)^{m(n-1)}[\bar{\beta}_2, [\bar{\beta}_1, \bar{\alpha}_2]].\end{aligned}$$

We recall that $(-1)c_j = -c_j + [1,1]H_1^1(c_j) = -c_j$, for $j = 6, 7$.

(1) If m and n are even, then

$$\begin{aligned} Q_2 - TQ_2 &= [\bar\alpha_2,[\bar\alpha_1,\bar\beta_1]]c_6 + [\bar\beta_2,[\bar\alpha_1,\bar\beta_1]]c_7 \\ &\quad +[\bar\alpha_2,[\bar\alpha_1,\bar\beta_2]]c_6 + [\bar\beta_2,[\bar\alpha_1,\bar\alpha_2]]c_6 \\ &\quad +[\bar\alpha_2,[\bar\beta_1,\bar\beta_2]]c_7 + [\bar\beta_2,[\bar\beta_1,\bar\alpha_2]]c_7. \end{aligned}$$

(2) If m is odd and n is even, then

$$\begin{aligned} Q_2 - TQ_2 &= [\bar\alpha_2,[\bar\alpha_1,\bar\beta_1]]c_6 + [\bar\beta_2,[\bar\alpha_1,\bar\beta_1]]c_7 \\ &\quad +[\bar\alpha_2,[\bar\alpha_1,\bar\beta_2]]c_6 + [\bar\beta_2,[\bar\alpha_1,\bar\alpha_2]](-c_6) \\ &\quad +[\bar\alpha_2,[\bar\beta_1,\bar\beta_2]](-c_7) + [\bar\beta_2,[\bar\beta_1,\bar\alpha_2]](-c_7). \end{aligned}$$

Since the above Whitehead products in (1) and (2) are all the basic products, φ is commutative if and only if $c_6 = c_7 = 0$ in any case. Similarly, we obtain the same result for the other cases. □

Proposition 4.3. *Let $\varphi = \varphi_{P,Q_3} : Y \to Y \vee Y$ be the comultiplication in Definition 3.7. Then, under the same hypotheses of Proposition 3.8, φ is commutative if and only if* (1) *n is even and $c_8 = c_9$, or* (2) *n is odd and $c_8 = -c_9$.*

Proof. Since $Q_3 = [\bar\beta_1,[\bar\alpha_1,\bar\alpha_2]]c_8 + [\bar\beta_2,[\bar\alpha_1,\bar\alpha_2]]c_9$, we have

$$\begin{aligned} TQ_3 &= [\bar\beta_2,[\bar\alpha_2,\bar\alpha_1]]c_8 + [\bar\beta_1,[\bar\alpha_2,\bar\alpha_1]]c_9 \\ &= (-1)^{n^2}[\bar\beta_2,[\bar\alpha_1,\bar\alpha_2]]c_8 + (-1)^{n^2}[\bar\beta_1,[\bar\alpha_1,\bar\alpha_2]]c_9. \end{aligned}$$

If n is even, then

$$Q_3 - TQ_3 = [\bar\beta_1,[\bar\alpha_1,\bar\alpha_2]](c_8 - c_9) + [\bar\beta_2,[\bar\alpha_1,\bar\alpha_2]](c_9 - c_8).$$

In case that n is odd, we obtain

$$Q_3 - TQ_3 = [\bar\beta_1,[\bar\alpha_1,\bar\alpha_2]](c_8 + c_9) + [\bar\beta_2,[\bar\alpha_1,\bar\alpha_2]](c_9 + c_8).$$

Since the right-hand side of the above equations are basic products, we complete the proof. □

Proposition 4.4. *Let $\varphi = \varphi_{P,Q_4} : Y \to Y \vee Y$ be the comultiplication in Definition 3.9. Then, under the same hypotheses of Proposition 3.10, φ is commutative if and only if $c_{10} = 0$.*

Proof. We note that $Q_4 = [\bar\alpha_2,[\bar\beta_1,\bar\beta_2]]c_{10}$ and

$$TQ_4 = [\bar\alpha_1,[\bar\beta_2,\bar\beta_1]]c_{10} = (-1)^{m^2}[\bar\alpha_1,[\bar\beta_1,\bar\beta_2]]c_{10}.$$

Since the last term is not a basic Whitehead product, we change it into the basic products as follows:

$$\begin{aligned} (-1)^{n(m-1)}[\bar\alpha_1,[\bar\beta_1,\bar\beta_2]] &= -(-1)^{m(n-1)+mn}[\bar\beta_1,[\bar\alpha_1,\bar\beta_2]] \\ &\quad -(-1)^{m(m-1)}[\bar\beta_2,[\bar\alpha_1,\bar\beta_1]]. \end{aligned}$$

Therefore, by considering the equation $Q_4 - TQ_4 = 0$, we complete the proof. □

Proposition 4.5. *Let $\varphi = \varphi_{P,Q_5} : Y \to Y \vee Y$ be the comultiplication in Definition 3.11. Then, under the same hypotheses of Proposition 3.12, φ is commutative if and only if $d_1 = d_2 = d_3 = d_4 = 0$.*

Proof. We compute Q_5 and TQ_5 as follows:

$$Q_5 = [\bar\beta_1,\bar\alpha_2]d_1 + [\bar\beta_1,[\bar\beta_1,\bar\alpha_2]]d_2 + [\bar\alpha_2,[\bar\beta_1,\bar\alpha_2]]d_3 + [\bar\beta_2,[\bar\beta_1,\bar\alpha_2]]d_4$$

and

$$\begin{aligned} TQ_5 &= [\bar\beta_2,\bar\alpha_1]d_1 + [\bar\beta_2,[\bar\beta_2,\bar\alpha_1]]d_2 + [\bar\alpha_1,[\bar\beta_2,\bar\alpha_1]]d_3 \\ &\quad +[\bar\beta_1,[\bar\beta_2,\bar\alpha_1]]d_4 \\ &= (-1)^{mn}[\bar\alpha_1,\bar\beta_2]d_1 + (-1)^{mn}[\bar\beta_2,[\bar\alpha_1,\bar\beta_2]]d_2 \\ &\quad +(-1)^{mn}[\bar\alpha_1,[\bar\alpha_1,\bar\beta_2]]d_3 + (-1)^{mn}[\bar\beta_1,[\bar\alpha_1,\bar\beta_2]]d_4. \end{aligned}$$

By considering the equation $Q_5 - TQ_5 = 0$ again whose terms consist of basic Whitehead products, we complete the proof. □

The following is the technical result in this section.

Theorem 4.6. *Let $\varphi = \varphi_{P,Q} : Y \to Y \vee Y$ be the comultiplication in Definition 3.13 and let m, n and p have the range as in Theorem 3.14 with the same hypotheses on Hopf-Hilton invariants. Then the comultiplication $\varphi = \varphi_{P,Q}$ is commutative if and only if* (1) *m and n are even, $a_2 = a_3, b_2 = b_3$, $c_3 = c_6 = 0$, $c_1 = d_1, c_2 = d_3, c_5 = d_2, c_7 = -c_{10}, c_8 = c_9$ and $d_4 = c_4 - c_7$, or* (2) *m is even and n is odd, $a_2 = a_3, b_2 = -b_3, 2b_1 = 0$, $c_3 = c_6 = 0$, $c_1 = d_1, c_2 = d_3, c_5 = d_2, c_7 = c_{10}, c_8 = -c_9$ and $d_4 = c_4 - c_7$, or* (3) *m is odd and n is even, $a_2 = -a_3, b_2 = b_3, 2a_1 = 0$, $c_3 = c_6 = 0$, $c_1 = d_1, c_2 = d_3, c_5 = d_2, c_8 = c_9, c_7 = c_{10}$ and $c_4 + c_7 = d_4$, or* (4) *m and n are odd, $a_2 = -a_3, b_2 = -b_3, 2a_1 = 0 = 2b_1$, $c_3 = c_6 = 0$, $c_1 = -d_1, c_2 = -d_3, c_5 = -d_2, c_8 = -c_9, c_7 = c_{10}$ and $-c_4 - c_7 = d_4$.*

Proof. By using the results of Propositions 4.1, 4.2, 4.3, 4.4 and 4.5, especially concentrating on the basic products, we prove the theorem. We can see that the following Whitehead products in (1), (2), (3) and (4) are all the basic products.

(1) If m and n are even, then $\varphi_{P,Q}$ is commutative if and only if

$$\begin{aligned}
0 \ &= [\bar\beta_1, [\bar\beta_1, \bar\beta_2]](a_2 - a_3) + [\bar\beta_2, [\bar\beta_1, \bar\beta_2]](a_3 - a_2)\\
&+[\bar\alpha_1, [\bar\alpha_1, \bar\alpha_2]](b_2 - b_3) + [\bar\alpha_2, [\bar\alpha_1, \bar\alpha_2]](b_3 - b_2)\\
&+[\bar\alpha_1, \bar\beta_2](c_1 - d_1) + [\bar\alpha_1, [\bar\alpha_1, \bar\beta_2]](c_2 - d_3) + [\bar\alpha_2, [\bar\alpha_1, \bar\beta_2]](c_3 + c_6)\\
&+[\bar\beta_1, [\bar\alpha_1, \bar\beta_2]](c_4 - d_4 + c_{10}) + [\bar\beta_2, [\bar\alpha_1, \bar\beta_2]](c_5 - d_2)\\
&+[\bar\beta_1, \bar\alpha_2](-c_1 + d_1) + [\bar\alpha_2, [\bar\beta_1, \bar\alpha_2]](-c_2 + d_3)\\
&+[\bar\beta_1, [\bar\alpha_1, \bar\alpha_2]](c_3 + c_8 - c_9) + [\bar\alpha_2, [\bar\alpha_1, \bar\beta_1]](c_3 + c_6)\\
&+[\bar\beta_2, [\bar\beta_1, \bar\alpha_2]](-c_4 + d_4 + c_7) + [\bar\beta_1, [\bar\beta_1, \bar\alpha_2]](-c_5 + d_2)\\
&+[\bar\beta_2, [\bar\alpha_1, \bar\beta_1]](c_7 + c_{10}) + [\bar\beta_2, [\bar\alpha_1, \bar\alpha_2]]c_6\\
&+[\bar\alpha_2, [\bar\beta_1, \bar\beta_2]](c_7 + c_{10}) + [\bar\beta_2, [\bar\alpha_1, \bar\alpha_2]](c_9 - c_8).
\end{aligned}$$

Thus φ is commutative if and only if $a_2 = a_3, b_2 = b_3$, $c_3 = c_6 = 0$, $c_1 = d_1, c_2 = d_3, c_5 = d_2, c_7 = -c_{10}, c_8 = c_9$, and $d_4 = c_4 - c_7$.

(2) If m is even and n is odd, then the equality $Q = TQ$ holds if and only if

$$\begin{aligned}
0 \ &= [\bar\beta_1, [\bar\beta_1, \bar\beta_2]](a_2 - a_3) + [\bar\beta_2, [\bar\beta_1, \bar\beta_2]](a_3 - a_2)\\
&+[\bar\alpha_1, \bar\alpha_2](2b_1 - [1, 1]H_1^1(b_1))\\
&+[\bar\alpha_1, [\bar\alpha_1, \bar\alpha_2]](b_2 + b_3) + [\bar\alpha_2, [\bar\alpha_1, \bar\alpha_2]](b_3 + b_2)\\
&+[\bar\alpha_1, \bar\beta_2](c_1 - d_1) + [\bar\alpha_1, [\bar\alpha_1, \bar\beta_2]](c_2 - d_3) + [\bar\alpha_2, [\bar\alpha_1, \bar\beta_2]](c_3 - c_6)\\
&+[\bar\beta_1, [\bar\alpha_1, \bar\beta_2]](c_4 - d_4 - c_{10}) + [\bar\beta_2, [\bar\alpha_1, \bar\beta_2]](c_5 - d_2)\\
&+[\bar\beta_1, \bar\alpha_2](-c_1 + d_1) + [\bar\alpha_2, [\bar\beta_1, \bar\alpha_2]](-c_2 + d_3)\\
&+[\bar\beta_1, [\bar\alpha_1, \bar\alpha_2]](-c_3 + c_8 + c_9) + [\bar\alpha_2, [\bar\alpha_1, \bar\beta_1]](-c_3 + c_6)\\
&+[\bar\beta_2, [\bar\beta_1, \bar\alpha_2]](-c_4 + d_4 + c_7) + [\bar\beta_1, [\bar\beta_1, \bar\alpha_2]](-c_5 + d_2)\\
&+[\bar\beta_2, [\bar\alpha_1, \bar\beta_1]](c_7 - c_{10}) + [\bar\beta_2, [\bar\alpha_1, \bar\alpha_2]](-c_6)\\
&+[\bar\alpha_2, [\bar\beta_1, \bar\beta_2]](-c_7 + c_{10}) + [\bar\beta_2, [\bar\alpha_1, \bar\alpha_2]](c_8 + c_9).
\end{aligned}$$

Thus φ is commutative if and only if $a_2 = a_3, b_2 = -b_3, 2b_1 = 0$, $c_3 = c_6 = 0$, $c_1 = d_1, c_2 = d_3, c_5 = d_2, c_7 = c_{10}, c_8 = -c_9$, and $d_4 = c_4 - c_7$.

(3) If m is odd and n is even, then the commutativity of $\varphi_{P,Q}$ holds if and only if

$$\begin{aligned}
0 \ &= [\bar\beta_1, \bar\beta_2](2a_1 - [1, 1]H_1^1(a_1))\\
&+[\bar\beta_1, [\bar\beta_1, \bar\beta_2]](a_2 + a_3) + [\bar\beta_2, [\bar\beta_1, \bar\beta_2]](a_3 + a_2)\\
&+[\bar\alpha_1, [\bar\alpha_1, \bar\alpha_2]](b_2 - b_3) + [\bar\alpha_2, [\bar\alpha_1, \bar\alpha_2]](b_3 - b_2)\\
&+[\bar\alpha_1, \bar\beta_2](c_1 - d_1) + [\bar\alpha_1, [\bar\alpha_1, \bar\beta_2]](c_2 - d_3) + [\bar\alpha_2, [\bar\alpha_1, \bar\beta_2]](c_3 + c_6)\\
&+[\bar\beta_1, [\bar\alpha_1, \bar\beta_2]](c_4 - d_4 + c_{10}) + [\bar\beta_2, [\bar\alpha_1, \bar\beta_2]](c_5 - d_2)\\
&+[\bar\beta_1, \bar\alpha_2](-c_1 + d_1) + [\bar\alpha_2, [\bar\beta_1, \bar\alpha_2]](-c_2 + d_3)\\
&+[\bar\beta_1, [\bar\alpha_1, \bar\alpha_2]](-c_3 + c_8 - c_9) + [\bar\alpha_2, [\bar\alpha_1, \bar\beta_1]](c_3 + c_6)\\
&+[\bar\beta_2, [\bar\beta_1, \bar\alpha_2]](-c_4 + d_4 - c_7) + [\bar\beta_1, [\bar\beta_1, \bar\alpha_2]](-c_5 + d_2)\\
&+[\bar\beta_2, [\bar\alpha_1, \bar\beta_1]](c_7 - c_{10}) + [\bar\beta_2, [\bar\alpha_1, \bar\alpha_2]](-c_6)\\
&+[\bar\alpha_2, [\bar\beta_1, \bar\beta_2]](-c_7 + c_{10}) + [\bar\beta_2, [\bar\alpha_1, \bar\alpha_2]](c_9 - c_8).
\end{aligned}$$

Thus φ is commutative if and only if $a_2 = -a_3, b_2 = b_3, 2a_1 = 0$, $c_3 = c_6 = 0$, $c_1 = d_1, c_2 = d_3, c_5 = d_2, c_8 = c_9, c_7 = c_{10}$, and $c_4 + c_7 = d_4$.

(4) If m and n are odd, then the given comultiplication $\varphi_{P,Q}$ is commutative if and only if

$$\begin{aligned} 0 &= [\bar\beta_1, \bar\beta_2](2a_1 - [1,1]H_1^1(a_1)) + [\bar\alpha_1, \bar\alpha_2](2b_1 - [1,1]H_1^1(b_1)) \\ &+[\bar\beta_1,[\bar\beta_1,\bar\beta_2]](a_2+a_3) + [\bar\beta_2,[\bar\beta_1,\bar\beta_2]](a_3+a_2) \\ &+[\bar\alpha_1,[\bar\alpha_1,\bar\alpha_2]](b_2+b_3) + [\bar\alpha_2,[\bar\alpha_1,\bar\alpha_2]](b_3+b_2) \\ &+[\bar\alpha_1,\bar\beta_2](c_1+d_1) + [\bar\alpha_1,[\bar\alpha_1,\bar\beta_2]](c_2+d_3) + [\bar\alpha_2,[\bar\alpha_1,\bar\beta_2]](c_3-c_6) \\ &+[\bar\beta_1,[\bar\alpha_1,\bar\beta_2]](c_4+d_4+c_{10}) + [\bar\beta_2,[\bar\alpha_1,\bar\beta_2]](c_5+d_2) \\ &+[\bar\beta_1,\bar\alpha_2](c_1+d_1) + [\bar\alpha_2,[\bar\beta_1,\bar\alpha_2]](c_2+d_3) \\ &+[\bar\beta_1,[\bar\alpha_1,\bar\alpha_2]](c_3+c_8+c_9) + [\bar\alpha_2,[\bar\alpha_1,\bar\beta_1]](-c_3+c_6) \\ &+[\bar\beta_2,[\bar\beta_1,\bar\alpha_2]](c_4+d_4+c_7) + [\bar\beta_1,[\bar\beta_1,\bar\alpha_2]](c_5+d_2) \\ &+[\bar\beta_2,[\bar\alpha_1,\bar\beta_1]](c_7-c_{10}) + [\bar\beta_2,[\bar\alpha_1,\bar\alpha_2]]c_6 \\ &+[\bar\alpha_2,[\bar\beta_1,\bar\beta_2]](-c_7+c_{10}) + [\bar\beta_2,[\bar\alpha_1,\bar\alpha_2]](c_9+c_8). \end{aligned}$$

Thus φ is commutative if and only if $a_2 = -a_3, b_2 = -b_3, 2a_1 = 0 = 2b_1$, $c_3 = c_6 = 0$, $c_1 = -d_1, c_2 = -d_3, c_5 = -d_2, c_8 = -c_9, c_7 = c_{10}$, and $-c_4 - c_7 = d_4$. □

Let $CO(Y)$ be the set of homotopy classes of commutative comultiplications on a co-H-space $Y = S^n \vee S^m \vee S^p$. Then we have the following main theorem in this section.

Theorem 4.7. *Let $m \le 2n-2$ and $n < m < p \le 4n-4$.* (1) *If m and n are even, then*

$$\begin{aligned} |CO(Y)| &= |\pi_p(S^{2m-1})| \times |\pi_p(S^{2n-1})| \times |\pi_p(S^{3m-2})| \times |\pi_p(S^{3n-2})| \\ &\times |\pi_p(S^{m+n-1})| \times |\pi_p(S^{2m+n-2})|^3 \times |\pi_p(S^{m+2n-2})|^2. \end{aligned}$$

(2) *If m is even and n is odd, then*

$$\begin{aligned} |CO(Y)| &= |\pi_p(S^{2m-1})| \times |\{b_1 \in \pi_p(S^{2n-1}) \,|\, 2b_1 = 0\}| \times |\pi_p(S^{3m-2})| \\ &\times |\pi_p(S^{3n-2})| \times |\pi_p(S^{m+n-1})| \times |\pi_p(S^{2m+n-2})|^3 \\ &\times |\pi_p(S^{m+2n-2})|^2. \end{aligned}$$

(3) *If m is odd and n is even, then*

$$\begin{aligned} |CO(Y)| &= |\{a_1 \in \pi_p(S^{2m-1}) \,|\, 2a_1 = 0\}| \times |\pi_p(S^{2n-1})| \times |\pi_p(S^{3m-2})| \\ &\times |\pi_p(S^{3n-2})| \times |\pi_p(S^{m+n-1})| \times |\pi_p(S^{2m+n-2})|^3 \\ &\times |\pi_p(S^{m+2n-2})|^2. \end{aligned}$$

(4) *If m and n are odd, then*

$$\begin{aligned} |CO(Y)| &= |\{a_1 \in \pi_p(S^{2m-1}) \,|\, 2a_1 = 0\}| \times |\{b_1 \in \pi_p(S^{2n-1}) \,|\, 2b_1 = 0\}| \\ &\times |\pi_p(S^{3m-2})| \times |\pi_p(S^{3n-2})| \times |\pi_p(S^{m+n-1})| \\ &\times |\pi_p(S^{2m+n-2})|^3 \times |\pi_p(S^{m+2n-2})|^2. \end{aligned}$$

Proof. Since every comultiplication has the form in Definition 3.13 in this range, by Theorem 4.6, we have the result. □

Example 4.8. $CO(S^4 \vee S^5 \vee S^{12})$ is an infinite set, and $|CO(S^8 \vee S^{12} \vee S^{27})| = 16$. Indeed, $|CO(S^8 \vee S^{12} \vee S^{27})| = |\pi_{27}(S^{23})| \times |\pi_{27}(S^{15})| \times |\pi_{27}(S^{34})| \times |\pi_{27}(S^{22})| \times |\pi_{27}(S^{19})| \times |\pi_{27}(S^{26})|^2 \times |\pi_{27}(S^{30})|^3 = 1 \times 1 \times 1 \times 1 \times 4 \times 4 \times 1 = 16$, and similarly for the other case.

Example 4.9. The set $\mathcal{A}CO(S^8 \vee S^{12} \vee S^{27})$ of associative and commutative comultiplications of the wedge of spheres $S^8 \vee S^{12} \vee S^{27}$ consists of 8-homotopy classes. Indeed, $|\mathcal{A}CO(S^8 \vee S^{12} \vee S^{27})| = |\pi_{27}(S^{23})| \times |\pi_{27}(S^{15})| \times |\pi_{27}(S^{19})| \times |\pi_{27}(S^{26})| \times |\pi_{27}(S^{30})| = 1 \times 1 \times 4 \times 2 \times 1 = 8$.

Remark 4.10. It is possible for us to weaken the inequality in this paper; that is, we can give a larger upper bound for n and require more Hopf-Hilton invariants to be zero. We still conclude that the above comultiplications $\varphi : Y \to Y \vee Y$ in this paper have the desirable properties. On the other hand, we are able to restrict our range to $m \leq 2n-2$ and $p \leq 4n-4$ so that we can handle the corresponding results more easily.

Acknowledgement The author is grateful to the anonymous referees for a careful reading and many helpful suggestions that improved the quality of the paper.

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