Periodic Orbits of Nonlinear First-Order
General Periodic Boundary Value Problem

Feng Wang[a,b], Fang Zhang[a], Hailong Zhu[c], Shengjun Li[d]

[a]*School of Mathematics and Physics, Changzhou University, Changzhou, Jiangsu 213164, China*
[b]*Department of Mathematics, College of Science, Nanjing University, Nanjing 210098, China*
[c]*Department of Mathematics, Anhui University of Finance & Economics, Bengbu, Anhui 233030, China*
[d]*Department of Mathematics, Hainan University, Haikou, Hainan 570228, China*

Abstract. In this paper, the existence and multiplicity of periodic orbits are obtained for first-order general periodic boundary value problem

$$x'(t) + a(t)x(t) = f(t, x), \quad t \in [0, T],$$
$$x(0) = \alpha x(T),$$

where $a : [0, T] \to [0, +\infty)$ and $f : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, $\alpha > 0$ and $T > 0$ with $\alpha e^{-\int_0^T a(s)ds} = 1$. The proofs are carried out by the use of topological degree theory. We also prove some nonexistence theorems. Our results extend and improve some recent work in the literature.

1. Introduction

The existence of solutions for first-order periodic boundary value problems has been considered by several authors; see [2]–[3], [5]–[7], [9]–[13] and references therein. Some theorems and methods of nonlinear functional analysis have been applied to research on this problem, such as the method of Mawhin coincidence degree theory [2], [6], [9], [12], the upper and lower solutions method and monotone iterative technique [11], [13] and fixed point theorems of cone mapping [3], [5], [7], [10]. In this paper, we consider the first order general periodic boundary value problem (GPBVP)

$$\begin{cases} x'(t) + a(t)x(t) = f(t, x(t)), & t \in [0, T], \\ x(0) = \alpha x(T), \end{cases} \tag{1.1}$$

where $a : [0, T] \to [0, +\infty)$ and $f : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, $\alpha > 0$ and $T > 0$ with

$$\alpha e^{-\int_0^T a(s)ds} = 1. \tag{1.2}$$

The GPBVP (1.1) was first considered by Zima [12]. For the special case of GPBVP (1.1) in which $\alpha = 1$ and $a(t) \equiv 0$ on $[0, T]$, Peng [7] studied the existence of positve solutions of the first order periodic boundary value problem, Peng's approach is to rewrite the original PBVP as an equivalent one, so that the fixed point theorem [1] can be applied, and by Leggett-Williams multiple fixed point theorem [4] and fixed point theorem of cone expansion and compression [1], Liu [3] established some multiplicity results for periodic boundary value problem.

In [12], the existence of positive solutions of the GPBVP (1.1) was studied. By using Leggett-Williams norm-type theorem for coincidence due to O'Regan and Zima [6], it was established the existence of at least one positive solution under some conditions.

Inspired by the above work, the aim of this paper is to consider the existence, multiplicity and nonexistence of positive solutions for the GPBVP (1.1). The main ingredient is an existence result for semilinear equations in cones due to Zhang and Wang [14].

2. An Abstract Existence Result

For the convenience of the readers, we present here the necessary definitions and an order-type existence theorem due to Zhang and Wang.

Definition 2.1. *Let X be a Banach space. A nonempty convex closed set $K \subset X$ is said to be a cone provided that*
(i) $ax \in K$ for any $x \in K$ and any $a \geq 0$ and
(ii) $x, -x \in K$ implies $x = \theta$.
Note that every cone $K \subset X$ induces an ordering in X given by

$$x \leq y \Leftrightarrow y - x \in K.$$

Definition 2.2. *Let X and Y be Banach spaces, D a linear subspace of X, $\{X_n\} \subset D$, and $\{Y_n\} \subset Y$ sequences of oriented finite dimensional subspaces such that $Q_n y \rightarrow y$ in Y for every y and $dist(x, X_n) \rightarrow 0$ for every $x \in D$ where $Q_n : Y \rightarrow Y_n$ and $P_n : X \rightarrow X_n$ are sequences of continuous linear projections. The projection scheme $\Gamma = \{X_n, Y_n, P_n, Q_n\}$ is then said to be admissible for maps from $D \subset X$ to Y.*

Definition 2.3. *A map $T : D \subset X \rightarrow Y$ is called approximation-proper (abbreviated A-proper) at a point $y \in Y$ with respect to Γ, if $T_n \equiv Q_n T|_{D \cap X_n}$ is continuous for each $n \in \mathbb{N}$ and whenever $\{x_{n_j} : x_{n_j} \in D \cap X_{n_j}\}$ is bounded with $T_{n_j} x_{n_j} \rightarrow y$, then there exists a subsequence $\{x_{n_{j_k}}\}$ such that $x_{n_{j_k}} \rightarrow x \in D$, and $Tx = y$. T is said to be A-proper on a set Ω if it is A-proper at all points of Ω.*

Let X and Y be Banach spaces, K be a cone in X, then $K_1 = (L + J^{-1}P)(\text{dom}L \cap K)$ is a cone in Y. Consider the semilinear equation

$$Lx = Nx, \tag{2.1}$$

where $L : \text{dom}L \subset X \rightarrow Y$ is a linear mapping, $\Omega \subset X$ is open, and $N : \overline{\Omega \cap K} \subset X \rightarrow Y$ is a bounded nonlinear mapping such that $L - \lambda N : \overline{\Omega \cap K} \subset X \rightarrow Y$ is A-proper for each $\lambda \in [0, 1]$ with respect to a suitable approximation scheme. We will assume that $L : \text{dom } L \subset X \rightarrow Y$ is a Fredholm operator of index zero, that is, Im L is closed and dim Ker L = codim Im $L < \infty$. As a consequence of this property X and Y may be expressed as direct sums; $X = X_0 \bigoplus X_1$, $Y = Y_0 \bigoplus Y_1$ with continuous linear projections $P : X \rightarrow$ Ker $L = X_0$ and $Q : Y \rightarrow Y_0$. The restriction of L to dom $L \cap X_1$, denoted L_1, is a bijection onto Im $L = Y_1$ with continuous inverse $L_1^{-1} : Y_1 \rightarrow$ dom $L \cap X_1$. Since X_0 and Y_0 have the same finite dimension, there exists a continuous bijection $J : Y_0 \rightarrow X_0$.

Concerning equation (2.1), by [14, Theorem 2.5], we have the following existence result which will be used later.

Lemma 2.1[14]. *If $L : \text{dom}L \rightarrow Y$ is Fredholm of index zero, and let $L - \lambda N$ be A-proper for $\lambda \in [0, 1]$. Assume that N is bounded and $P + JQN + L_1^{-1}(I - Q)N$ maps K to K. Suppose Ω_1 and Ω_2 are two bounded open sets in X such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $\Omega_2 \cap K \cap \text{dom}L \neq \emptyset$. If one of the following two conditions is satisfied:*
(C_1) $Nx \not\geq Lx$, for any $x \in \partial\Omega_1 \cap K$ and $Nx \not\leq Lx$, for any $x \in \partial\Omega_2 \cap K$,

(C_2) $Nx \nleq Lx$, for any $x \in \partial\Omega_1 \cap K$ and $Nx \ngeq Lx$, for any $x \in \partial\Omega_2 \cap K$, then equation (2.1) has at least one solution $x \in K \cap (\overline{\Omega_2 \setminus \Omega_1})$.

For convenience we adopt the notation, let

$$\varphi(t) = e^{\int_0^t a(s)ds}, \quad \psi(t) = \int_0^t \frac{ds}{\varphi(s)}, \quad t \in [0, T].$$

From (1.2) we get $\varphi(T) = \alpha$. We also define

$$G(t,s) = \frac{\varphi(s)}{\varphi(t)} \begin{cases} 1 + \frac{\psi(s)}{\psi(T)}, & 0 \le s \le t \le T, \\ \frac{\psi(s)}{\psi(T)}, & 0 \le t < s \le T, \end{cases}$$

and

$$H(t,s) = \frac{\varphi(s)}{\varphi(t) \int_0^T \varphi(\tau)d\tau} + G(t,s) - \frac{\int_0^T G(t,\tau)d\tau}{\int_0^T \varphi(\tau)d\tau} \varphi(s), \ t,s \in [0,T].$$

Set

$$\kappa = \min\left\{\frac{\int_0^T \varphi(s)ds}{\alpha\psi(T)}, \min_{t,s\in[0,T]} \frac{1}{\alpha\psi(T)H(t,s)}\right\}.$$

3. Existence and Multiplicity

In this section we use Lemma 2.1 to establish the existence and multiplicity of periodic orbits to the GPBVP (1.1).

Consider the Banach spaces

$$X = Y = C[0, T]$$

with the sup norm

$$\|x\| = \max_{t\in[0,T]} |x(t)|.$$

Let $K = \{x \in X : x(t) \ge 0, \ t \in [0, T]\}$, then K is a cone of X. We can write GPBVP (1.1) as an abstract equation

$$Lx = Nx, \ x \in \text{dom}L \cap K,$$

where

$$L : \text{dom}L \to Z, \ x \to x' + a(\cdot)x, \ \text{dom}L = \{x \in X : x' \in C[0,T], x(0) = \alpha x(T)\},$$

and

$$N : X \cap K \to Y, \ x \to f(\cdot, x(\cdot)).$$

We note that

$$\text{Ker}L = \{x \in \text{dom}L : x(t) \equiv \frac{c}{\varphi(t)}, \ c \in \mathbb{R}, \ \text{for any } t \in [0, T]\},$$
$$\text{Im}L = \left\{y \in Y : \int_0^T \varphi(s)y(s)ds = 0\right\},$$
$$\dim \text{Ker}L = \text{codim } \text{Im}L = 1.$$

Thus L is linear and Fredholm of index zero. We define

$$P : X \to X, \ x \to \frac{1}{\varphi(\cdot)\psi(T)} \int_0^T x(s)ds$$

and

$$Q : Y \to Y, \ y \to \frac{\int_0^T \varphi(s)y(s)ds}{\int_0^T \varphi(s)ds}.$$

For ImQ define

$$Jy(t) = \frac{y}{\varphi(t)}, \ t \in [0, T].$$

Clearly, J is an isomorphism from ImQ to Ker$L$. We see that, as in [4], we can obtain the inverse operator $L_1^{-1} : \text{Im}L \to \text{dom}L \cap \text{Ker}P$ of $L|_{\text{dom}L\cap\text{Ker}P} : \text{dom}L \cap \text{Ker}P \to \text{Im}L$ as $(L_1^{-1}y)(t) = \int_0^1 G(t, s)y(s)ds$.

Next we state our first result:

Theorem 3.1. *Assume that there exist two positive numbers $0 < a < b$ such that*
   (H_1) $f(t, b) < 0$, *for any* $t \in [0, T]$,
   (H_2) $f(t, x) > 0$, *for any* $(t, x) \in [0, T] \times [0, a]$,
   (H_3) $f(t, x) \geq -\kappa x$, *for any* $(t, x) \in [0, T] \times [0, b]$,
   (H_4) $H(t, s) \geq 0$ for $t, s \in [0, T]$. *Then the GPBVP(1.1) has at least one periodic orbit $x^* \in \text{dom}L \cap K$ satisfying* $a \leq \|x^*\| \leq b$.

Proof. First, since L_1^{-1} is compact by Arzelà-Ascoli theorem. Hence, by (a) of Lemma 2 in [8], the mapping $L - \lambda N$ is A-proper for each $\lambda \in [0, 1]$.

By conditions $(H_3), (H_4)$ we get for any $x \in K$

$$
\begin{aligned}
& (P + JQN + L_1^{-1}(I - Q)N)x(t) \\
= {}& \frac{1}{\varphi(t)\psi(T)} \int_0^T x(s)ds + \frac{1}{\varphi(t)\int_0^T \varphi(s)ds} \int_0^T f(s, x(s))\varphi(s)ds \\
& + \int_0^T G(t, s)\Big(f(s, x(s)) - \frac{\int_0^T \varphi(s)f(s, x(s))ds}{\int_0^T \varphi(s)ds}\Big)ds \\
= {}& \frac{1}{\varphi(t)\psi(T)} \int_0^T x(s)ds + \int_0^T H(t, s)f(s, x(s))ds \\
\geq {}& \int_0^T \Big(\frac{1}{\alpha\psi(T)} - \kappa H(t, s)\Big)x(s)ds \\
\geq {}& 0.
\end{aligned}
\tag{3.1}
$$

Thus $(P + JQN + L_1^{-1}(I - Q)N)(K) \subset K$.

Let

$$\Omega_1 = \{x \in X : \|x\| < a\}, \ \Omega_2 = \{x \in X : \|x\| < b\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2.$$

We now show that

$$Nx \ngeq Lx, \ for \ any \ x \in \partial\Omega_2 \cap K. \tag{3.2}$$

In fact, if there is $x_1 \in \partial\Omega_2 \cap K$ such that

$$Nx_1 \geq Lx_1.$$

Then

$$x_1'(t) + a(t)x_1(t) \leq f(t, x_1(t)), \ t \in [0, T].$$

Let $t_1 \in [0, T]$ be such that $x_1(t_1) = b$. Clearly, the function x_1 attains a maximum on $[0, T]$ at $t = t_1$. Therefore $x_1'(t_1) = 0$. As a consequence,

$$0 \leq ba(t_1) \leq f(t_1, b),$$

which is contraction to (H_1). Therefore (3.2) holds.

On the other hand, we claim that

$$Nx \nleq Lx, \quad for \; any \; x \in \partial\Omega_1 \cap K. \tag{3.3}$$

In fact, if not, there exists $x_2 \in \partial\Omega_1 \cap K$, such that $Nx_2 \leq Lx_2$, and therefore

$$x_2'(t) + a(t)x_2(t) \geq f(t, x_2(t)).$$

Multiplying the last inequality by $e^{\int_0^t a(s)ds}$ yields

$$(e^{\int_0^t a(s)ds} x_2(t))' \geq e^{\int_0^t a(s)ds} f(t, x_2(t)). \tag{3.4}$$

For any $x_2 \in \partial\Omega_1 \cap K$, we have $\|x_2\| = a$, then $0 \leq x_2(t) \leq a$ for $t \in [0, T]$. Integrating (3.4) from 0 to T and using the boundary conditions and condition (H_2), we have

$$\begin{aligned}
0 &= \alpha x_2(T) - x_2(0) \\
&\geq \int_0^T e^{\int_0^t a(s)ds} f(t, x_2(t))dt \\
&> 0,
\end{aligned}$$

which is a contradiction. As a result, (3.3) is verified.

Thus all the hypotheses of Lemma 2.1 have been verified, so the conclusion of Theorem 3.1 follow from Lemma 2.1. □

Remark 3.1. In [12], the following condition is required:
(H^*) there exist $a \in (0, b/\alpha), t_0 \in [0, T], \beta > 0, m \in (0, 1)$, and continuous functions $g : [0, T] \to [0, \infty)$, $h : (0, a] \to [0, \infty)$ such that $f(t, x) \geq g(t)h(x)$ for any $t \in [0, T]$ and $x \in (0, a]$, $h(x)/x^\beta$ is non-increasing on $(0, a]$ with

$$\frac{h(a)}{a} m^\beta \int_0^T G(t_0, s)g(s)ds \geq 1 - \frac{mT}{\varphi(t_0)\psi(T)}.$$

Obviously, our condition (H_2) is much weaker and less strict compared with (H^*). Moreover, (H_2) is easier to check than (H^*). So our result generalizes and improves [10, Theorem 3.1].

We shall use the following assumptions:
(H_5) there are positive solutions m_1, m_2 such that $m_1 \leq H(t, s) \leq m_2$ for $t, s \in [0, T]$,
(H_6) $f(t, x) \geq -\kappa x$, for any $t \in [0, T]$, $x \geq 0$, where κ satisfies

$$0 < \kappa < \frac{1}{m_2 \alpha \psi(T)}.$$

We now construct a particular closed convex set $\widetilde{K}$ in X, defined by

$$\widetilde{K} = \left\{ x \in X : x(t) \geq 0, \; x(t) \geq \sigma\|x\|, t \in [0, T] \right\},$$

where $\sigma = \frac{m_1}{m_2}$. Thus $0 < \sigma < 1$. It is easy to see that $\widetilde{K}$ is a cone in X. Now we show

$$(P + JQN + L_1^{-1}(I - Q)N)(\widetilde{K}) \subset \widetilde{K}.$$

For any $x \in \widetilde{K}$. It follows from conditions $(H_5) - (H_6)$ that, by using the same method to get (3.1), we can get

$$(P + JQN + L_1^{-1}(I - Q)N)x(t) \geq 0.$$

We claim that

$$(Px + JQN)x(t) + L_1^{-1}(I - Q)Nx(t) \geq \sigma\|(P + JQN)x + L_1^{-1}(I - Q)Nx\|.$$

By conditions $(H_1) - (H_2)$, we have

$$
\begin{aligned}
&(Px + JQN)x(t) + L_1^{-1}(I - Q)Nx(t) \\
&= \frac{1}{\varphi(t)\psi(T)} \int_0^T x(s)ds + \int_0^T H(t,s)f(s,x(s))ds \\
&= \int_0^T [\frac{1}{\varphi(t)\psi(T)} - \kappa H(t,s)]x(s)ds + \int_0^T H(t,s)[f(s,x(s)) + \kappa x(x)]ds \\
&\leq \int_0^T [\frac{1}{\varphi(t)\psi(T)} - \kappa H(t,s)]x(s)ds + m_2 \int_0^T [f(s,x(s)) + \kappa x(x)]ds.
\end{aligned}
$$

From the last inequality, we have from conditions $(H_1) - (H_2)$

$$
\begin{aligned}
&(Px + JQN)x(t) + L_1^{-1}(I - Q)Nx(t) \\
&= \int_0^T [\frac{1}{\varphi(t)\psi(T)} - \kappa H(t,s)]x(s)ds + \int_0^T H(t,s)[f(s,x(s)) + \kappa x(x)]ds \\
&\geq \int_0^T [\frac{1}{\varphi(t)\psi(T)} - \kappa H(t,s)]x(s)ds + m_1 \int_0^T [f(s,x(s)) + \kappa x(x)]ds \\
&= \sigma\Big[\frac{1}{\sigma} \int_0^T [\frac{1}{\varphi(t)\psi(T)} - \kappa H(t,s)]x(s)ds + \frac{m_1}{\sigma} \int_0^T [f(s,x(s)) + \kappa x(x)]ds\Big] \\
&\geq \sigma\Big[\int_0^T [\frac{1}{\varphi(t)\psi(T)} - \kappa H(t,s)]x(s)ds + m_2 \int_0^T [f(s,x(s)) + \kappa x(x)]ds\Big] \\
&\geq \sigma\|(P + JQN)x + L_1^{-1}(I - Q)Nx\|.
\end{aligned}
$$

Therefore, $(P + JQN)x + L_1^{-1}(I - Q)Nx \in \widetilde{K}$.

Theorem 3.2. *Under assumptions (H_5)–(H_6), if moreover there exist constants a, b and c, such that $0 < a < \sigma b < c$ and*

$(H_1)'$ $f(t, a) < 0$, $\forall\, t \in [0, T]$,
$(H_2)'$ $f(t, x) > 0$, $\forall\, (t, x) \in [0, T] \times [\sigma b, b]$,
$(H_3)'$ $f(t, c) < 0$, $\forall\, t \in [0, T]$,
then the GPBVP (1.1) has at least two periodic orbits $x^*, x^{**} \in \widetilde{K}$ with

$$
a \leq \|x^*\| < b < \|x^{**}\| \leq c.
$$

Proof. We construct the sets $\Omega_a = \{x \in X : \|x\| < a\}$, $\Omega_b = \{x \in X : \|x\| < b\}$ and $\Omega_c = \{x \in X : \|x\| < c\}$ in order to apply Lemma 2.1.

Let $x \in \widetilde{K}$ with $\|x\| = b$, we have $\sigma b = \sigma\|x\| \leq x(t) \leq b$. It follows from $(H_2)'$ that, by using the similar method to get (3.3), we can get

$$
Nx \nleq Lx, \; \text{for any } x \in \partial\Omega_b \cap \widetilde{K}. \tag{3.6}
$$

It follows from $(H_1)'$ and $(H_3)'$ that, by using the same method to get (3.2), we can get

$$
Nx \ngeq Lx, \; \text{for any } x \in \partial\Omega_a \cap \widetilde{K}, \tag{3.7}
$$

$$
Nx \ngeq Lx, \; \text{for any } x \in \partial\Omega_c \cap \widetilde{K}. \tag{3.8}
$$

Now, (3.6), (3.7) and the first part of Lemma 2.1 guarantee that there exists $x^* \in \widetilde{K} \cap (\overline{\Omega}_b \backslash \Omega_a)$ such that $Lx^* = Nx^*$. By (3.7), (3.8) and the second part of Lemma 2.1 guarantee that there exists $x^{**} \in \widetilde{K} \cap (\overline{\Omega}_c \backslash \Omega_b)$ such that $Lx^{**} = Nx^{**}$. Thus $a \leq \|x^*\| < b < \|x^{**}\| \leq c$. $\quad\square$

Theorem 3.3. *Under assumptions* (H_5)–(H_6), *if moreover there exist constants a, b and c, such that $0 < a < b < \sigma c < c$ and*

$(H_1)''$ $f(t, x) > 0, \ \forall \ (t, x) \in [0, T] \times [0, a]$,
$(H_2)''$ $f(t, b) < 0, \ \forall \ t \in [0, T]$,
$(H_3)''$ $f(t, x) > 0, \ \forall \ (t, x) \in [0, T] \times [\sigma c, c]$,

then the GPBVP (1.1) has at least two periodic orbits $x^*, x^{**} \in \widetilde{K}$ with

$$a \leq \|x^*\| < b < \|x^{**}\| \leq c.$$

Proof. We omit the details because they are much similar to that in the proof of Theorem 3.2. □

4. Nonexistence Results and Example

First, we give some sufficient conditions for the nonexistence of positive solutions.

Theorem 4.1. *If*

$$\inf_{x>0} \min_{t\in[0,T]} \frac{f(t, x)}{x} > 0, \tag{4.1}$$

then the GPBVP (1.1) has no positive solution.

Proof. Assume, to the contrary, that $x(t)$ is a positive solution of the GPBVP (1.1). Then $x \in K, x(t) > 0$ for $0 \leq t \leq T$, and

$$x'(t) + a(t)x(t) = f(t, x), \ \ 0 \leq t \leq T. \tag{4.2}$$

Multiply (4.2) by $\varphi(t)$ to obtain

$$\varphi(t)x'(t) + \varphi(t)a(t)x(t) = (\varphi(t)x(t))' = \varphi(t)f(t, x), \ \ 0 \leq t \leq T. \tag{4.3}$$

From assumption (4.1), there exists $\epsilon > 0$ such that

$$f(t, x(t)) \geq \epsilon x(t), \ \ t \in [0, T]. \tag{4.4}$$

Integrating Eq. (4.3) form 0 to T and using (4.4) and boundary conditions we get

$$0 = \int_0^T \varphi(t)f(t, x)dt \geq \epsilon \int_0^T \varphi(t)x(t)dt.$$

Since $\int_0^T \varphi(t)x(t)dt > 0$, we have reached a contradiction. Therefore the GPBVP (1.1) has no positive solution. □

In a similar fashion, we can prove the next theorem.

Theorem 4.2. *If* $\sup_{x>0} \max_{t\in[0,T]} \frac{f(t,x)}{x} < 0$, *then the GPBVP (1.1) has no positive solution.*

Remark 4.1. It is to be noted that for $T = 1, \alpha = 1$ and $a(t) \equiv 0$ on $[0, 1]$, the same conclusions of Theorems 3.1, 3.2, 4.1 and 4.2 hold for the periodic boundary value problem

$$\begin{cases} x'(t) = f(t, x), \ \ 0 \leq t \leq 1, \\ x(0) = x(1). \end{cases} \tag{4.5}$$

Especially, for PBVP (4.5), our conclusions are also true and new.

Example 4.1. Consider the following boundary value problem

$$\begin{cases} x'(t) + x(t) = f(t,x), & 0 \le t \le 1, \\ x(0) = ex(1). \end{cases} \tag{4.6}$$

We choose $a(t) \equiv 1$ and

$$f(t,x) = \begin{cases} \frac{3e(t+1)}{(e-1)^2}x, & 0 \le x \le \frac{1}{2}, \\ \frac{2e(t+1)}{3(e-1)^2}(x^2 - 6x + 5), & x \ge \frac{1}{2}. \end{cases}$$

A simple calculation shows that $\alpha = e, \kappa = \frac{e}{(e-1)^2}, \varphi(t) = e^t$ and $\psi(t) = 1 - e^{-t}$. Thus $f(t,x) \ge -\kappa x$, $x \ge 0$, $t \in [0,1]$. If we choose $a = \frac{1}{4}, b = 3$, then it is easy to verify that the BVP (4.6) satisfies Theorem 3.1 and hence it has at least one periodic orbit x^* on $[0,1]$ with $\frac{1}{4} \le \|x^*\| \le 3$.

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