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Existence and Stability Results for Random Impulsive Fractional Pantograph Equations

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Abstract. In this paper, we study the existence, uniqueness, stability through continuous dependence on initial conditions and Hyers-Ulam-Rassias stability results for random impulsive fractional pantograph differential systems by relaxing the linear growth conditions. Finally examples are given to illustrate the applications of the abstract results.

1. Introduction

Pantograph type equations have been studied extensively owing to the numerous applications in which these equations arise. The name pantograph originated from the work of Ockendon and Tayler [1] on the collection of current by the pantograph head of an electric locomotive. The pantograph equations are appeared in modeling of various problems in engineering and sciences such as biology, economy, control and electrodynamics. For some applications of this equation we refer [2–5].

Impulsive differential equations are suitable mathematical model to simulate the evolution of large classes of real processes. These processes are subjected to short temporary perturbations. The duration of these perturbations are negligible compared to the duration of whole process. These perturbations occurs in the form of impulses (see the monographs [6, 7]). When the impulses are exists at random it affect the nature of the differential system. There are few results that have been discussed. Iwankievicz and Nielsen [8], investigated the dynamic response of non- linear systems to Poisson distributed random impulses. A. Anguraj and A.Vinodkumar studied the existence, uniqueness and stability results of random impulsive semilinear differential systems [9]. Sanz - Serna and Stuart [10] first brought dissipative differential equations with random impulses and used Markov chains to simulate such systems. On further read on random impulsive type differential equations refer [11–17] and the references therein.

Recently, fractional differential / difference equations (FDEs / FDFEs) and impulsive fractional differential equations (IFDEs) have proved to be valuable tool in the modeling of many phenomena in various fields of science and engineering. Similarly, the stabilities like continuous dependence, Hyers-Ulam stability, Hyers-

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Ulam- Rassias stability, local stability and Mittag -Leffler stability for FDEs and IFDEs have attracted the attention of many mathematicians (see [18–24] and the references therein). In [25], the authors have given the Ulam's type stability and data dependence for FDEs. JinRong Wang et al. [26] studied stability of FDEs using fixed point theorem in a generalized complete metric space. In [27], JinRong Wang et al.studied Ulam's stability for the nonlinear IFDEs. Michal Fečkan et al. proved on the concept and existence of solution for IFDEs [28]. For more details on FDE and its stability concepts see [29–40] and for difference equations and its behaviors see [41–43].

Motivated by the above mentioned works and its importance in many applied fields, it is interesting to study the fractional model of the pantograph equations with random impulses. We relaxed the Lipschitz condition on the impulsive term and under our assumption it is enough to be bounded. To best of our knowledge there is no paper which study the random impulsive fractional pantograph equations.

The paper is organized as follows: In section 2, we recall briefly the notations, definitions, lemmas and preliminaries which are used throughout this paper. In section 3, we investigate the existence results and uniqueness of solutions of random impulsive fractional pantograph equations by relaxing the linear growth condition. In section 4, we study the stability through continuous dependence on initial conditions of random impulsive fractional pantograph equations. The Hyers Ulam stability and Hyers Ulam-Rassias stability of the solutions of random impulsive fractional pantograph equations are investigated in section 5 and finally in section 6, examples are given to illustrate our theoretical results.

2. Preliminaries

Let $\|\cdot\|$ denote Euclidean norm in $\mathfrak{R}^n$. Let $\mathfrak{R}^n$ be the $n-$dimensional Euclidean space and Ω a nonempty set. Assume that τ_k is a random variable defined from Ω to $D_k \overset{def.}{=} (0, d_k)$ for all $k = 1, 2, \cdots$, where $0 < d_k < +\infty$. Furthermore, assume that τ_k follow Erlang distribution, where $k = 1, 2, \ldots$ and let τ_i and τ_j are independent with each other as $i \neq j$ for $i, j = 1, 2, \cdots$. Let $\tau, T \in \mathfrak{R}$ be two constants satisfying $\tau < T$. For the sake of simplicity, we denote $\mathfrak{R}_\tau = [\tau, T]$.

Consider the fractional pantograph differential system with random impulses of the form

$$ {}^cD_t^\alpha x(t) = f(t, x(t), x(\lambda t)), \quad t \neq \xi_k, \quad t \geq \tau \tag{1} $$

$$ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \quad k = 1, 2, ... \tag{2} $$

$$ x_{t_0} = x_0, \tag{3} $$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$; $0 < \lambda < 1$; the function $f : \mathfrak{R}_\tau \times \mathfrak{R}^n \to \mathfrak{R}^n$; $b_k : D_k \to \mathfrak{R}^{n\times n}$ is a matrix-valued function for each $k = 1, 2, \cdots$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \cdots$, here $t_0 \in \mathfrak{R}_\tau$ is arbitrary real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots$; $x(\xi_k^-) = \lim_{t \uparrow \xi_k} x(t)$ according to their paths with the norm $\|x\| = \sup_{\tau \leq s \leq t} |x(s)|$ for each t satisfying $\tau \leq t \leq T$.

Let us denote $\{B_t, t \geq 0\}$ be the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote $\mathcal{F}_t$, the σ-algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. $E(.)$ is the expectation with respect to the measure P. Let $\mathcal{B}$ be the Banach space with the norm defined for any $\psi \in \mathcal{B}$,

$$ \|\psi\|^p = (\sup_{t\in[\tau,T]} E\|\psi(t)\|^p), $$

where $\psi(t)$, for any given $t \in [\tau, T]$.

Definition 2.1. *The fractional order integral of the function $h \in L^1([a, b], \mathfrak{R}^n)$ of order $\alpha \in \mathfrak{R}^+$ is defined by*

$$ I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(s)}{(t-s)^{1-\alpha}} ds, $$

where $\Gamma(.)$ is the gamma function.

Definition 2.2. *The Caputo derivative of order α for a function h on the given interval $[a, b]$ is defined by*

$$({}^{c}D^{\alpha}_{a,t}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{h^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds,$$

here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Theorem 2.3. *Let B be a convex subset of a Banach space E and assume $0 \in B$. Let $F : B \to B$ be a completely continuous operator and let*

$$U(F) \quad = \quad \{x \in B : x = \mathcal{V}Fx \text{ for some } 0 < \mathcal{V} < 1\}$$

then either $U(F)$ is unbounded or F has a fixed point.

Definition 2.4. *For a given $T \in (\tau, +\infty)$, a stochastic process $\{x(t) \in \mathcal{B}, \tau \leq t \leq T\}$ is called a solution to the equations $(1) - (3)$ in $(\Omega, P, \{\mathcal{F}_t\})$, if*
(i) $x(t) \in \mathcal{B}$ is $\mathcal{F}_t$ - adapted.
(ii)

$$x(t) = \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s, x(s), x(\lambda s))ds \tag{4}$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s, x(s), x(\lambda s))ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t), \quad t \in [\tau, T].$$

where $\prod_{j=m}^{n}(\cdot) = 1$ *as* $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1})\cdots b_i(\tau_i)$, *and* $I_A(\cdot)$ *is the index function, i.e.,*

$$I_A(t) = \begin{cases} 1, & \text{if } \ t \in A, \\ 0, & \text{if } \ t \notin A. \end{cases}$$

3. Existence Results

In this section, we give the main results on the existence of solutions of the system $(1) - (3)$.

(H_1) : There exists a constant $q_1 \in (0, \alpha)$ such that real valued function $m(t) \in L^{1/q_1}$ and there exists a L^p integrable and nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$E\|f(t, \vartheta_1, \vartheta_2)\|^p \leq m(t)W\Big(E\|\vartheta_1\|^p + E\|\vartheta_2\|^p\Big),$$

(H_2) The condition $E\Big(\max_{i,k}\Big\{\prod_{j=1}^{k}\|b_j(\tau_j)\|\Big\}\Big)$ is uniformly bounded if there is a constant $C > 0$ such that $E\Big(\max_{i,k}\Big\{\prod_{j=1}^{k}\|b_j(\tau_j)\|\Big\}\Big) \leq C$ for all $\tau_j \in D_j$, $j = 1, 2, \ldots$

Theorem 3.1. *If the hypotheses $(H_1) - (H_2)$ hold, then the system $(1) - (3)$ has a solution $x(t)$, defined on $[t_0, T]$ provided the following inequality satisfies*

$$M_1\int_{t_0}^{T} m(s)ds \quad < \quad \int_{c_1}^{\infty}\frac{ds}{W(2s)}, \tag{5}$$

where $M_1 = 2^{p-1}\max\Big\{1, C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}$, $c_1 = 2^{p-1}C^pE\|x_0\|^p$ *and* $2C \geq 2^{\frac{1}{p}}$.

Proof. Let T be an arbitrary number $t_0 < T < +\infty$ satisfying (5). We transform the problem (1) − (3) into a fixed point problem. We consider the operator $\mathcal{S} : \mathcal{B} \to \mathcal{B}$ defined by

$$\begin{aligned}(\mathcal{S}x)(t) \;&=\; \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds \\ &+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t), \quad t\in[\tau,T].\end{aligned}$$

In order to use the transversality theorem, first we establish the priori estimates for the solutions of the integral equation and $\lambda \in (0,1)$,

$$\begin{aligned}x(t) = \;&\mathcal{V}\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds \\ &+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t), \quad t\in[\tau,T].\end{aligned}$$

Thus by $(H_1) - (H_2)$, we have

$$\begin{aligned}\|x(t)\|^p \;&\le\; \Big[\mathcal{V}\sum_{k=0}^{+\infty}\Big[\Big\|\prod_{i=1}^{k} b_i(\tau_i)\Big\|\|x_0\| + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\Big\|\prod_{j=1}^{k} b_j(\tau_j)\Big\|\Big\{\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\|f(s,x(s),x(\lambda s))\|ds\Big\} \\ &+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\|f(s,x(s),x(\lambda s))\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^p \\ &\le\; 2^{p-1}\Big[\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\Big\|b_i(\tau_i)\Big\|^p\|x_0\|^p I_{[\xi_k,\xi_{k+1})}(t)\Big] \\ &+\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\Big\|\prod_{j=1}^{k} b_j(\tau_j)\Big\|\Big\{\frac{1}{\Gamma(\alpha)}\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\|f(s,x(s),x(\lambda s))\|ds\Big\} \\ &+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\|f(s,x(s),x(\lambda s))\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^p\Big] \\ &\le\; 2^{p-1}\max_k\Big\{\prod_{i=1}^{k}\Big\|b_i(\tau_i)\Big\|^p\Big\}\|x_0\|^p + 2^{p-1}\Big[\max_{i,k}\Big\{1,\prod_{j=1}^{k}\Big\|b_i(\tau_i)\Big\|\Big\}\Big]^p \\ &\Big[\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,x(s),x(\lambda s))\|ds\Big]^p\end{aligned}$$

Noting that the last term of the right hand side of the above inequality increases in t and choose $2C \ge 2^{\frac{1}{p}}$, we obtain that

$$\begin{aligned}E\|x(t)\|^p \;&\le\; 2^{p-1}C^pE\|x_0\|^p \\ &+2^{p-1}\max\Big\{1,C^p\Big\}\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{p(\alpha-1)}ds\times\int_{t_0}^{t}E\|f(s,x(s),x(\lambda s))\|^p ds \\ &\le\; 2^{p-1}C^pE\|x_0\|^p + 2^{p-1}\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t}E\|f(s,x(s),x(\lambda s))\|^p ds \\ &\le\; 2^{p-1}C^pE\|x_0\|^p + 2^{p-1}\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t}m(s)W(2E\|x(s)\|^p)ds\end{aligned}$$

$$\sup_{t_0\le v\le t} E\|x(v)\|^p \le 2^{p-1}C^pE\|x_0\|^p + 2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t} m(s)W(2\sup_{t_0\le v\le s}E\|x(v)\|^p)ds)$$

We consider the function $\ell(t)$ defined by

$$\ell(t) = \sup_{t_0\le v\le t} E\left[\|x(v)\|^p\right], \qquad t\in[t_0,T].$$

Then, for any $t\in[t_0,T]$ it follows that

$$\ell(t) \le 2^{p-1}C^pE\|x_0\|^p + 2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t} m(s)W(2\ell(s))ds \tag{6}$$

Denoting by $u(t)$ the right hand side of the above inequality (6), we obtain that

$$\ell(t)\le u(t), \quad t\in[t_0,T],$$

$$u(t_0) = 2^{p-1}C^pE\|x_0\|^p = c_1$$

and

$$\begin{aligned} u'(t) &= 2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}m(t)W(2\ell(t)) \\ &\le 2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}m(t)W(2u(t)), \qquad t\in[t_0,T]. \end{aligned}$$

Then

$$\frac{u'(t)}{W(2u(t))} \le 2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}m(t), \qquad t\in[t_0,T]. \tag{7}$$

Integrating (7) from t_0 to t and by making use of the change of variable, we obtain

$$\begin{aligned} \int_{u(t_0)}^{u(t)}\frac{ds}{W(2s)} &\le 2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t} m(s)ds \\ &\le 2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{T} m(s)ds \\ &< \int_{u(t_0)}^{\infty}\frac{ds}{W(2s)}, \quad t\in[t_0,T], \end{aligned} \tag{8}$$

where the last inequality is obtained by (5). From (8) and by mean value theorem, there is a constant η_1 such that $u(t)\le\eta_1$ and hence $\ell(t)\le\eta_1$. Since $\sup_{t_0\le v\le t}E\|x(v)\|^p = \ell(t)$ holds for every $t\in[t_0,T]$, we have $\sup_{t_0\le v\le T}E\|x(v)\|^p\le\eta_1$, where η_1 only depends on T, the functions m and W, and consequently

$$E\|x\|^p_{\mathcal{B}} = \sup_{t_0\le v\le T}E\|x(v)\|^p\le\eta_1.$$

Step :1 We prove that $\mathcal{S}$ is continuous.

Let $\{x_n\}$ be a convergent sequence of elements of x in $\mathcal{B}$. Then for each $t \in [t_0, T]$, we have

$$(\mathcal{S}x_n)(t) = \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x_n(s),x_n(\lambda s))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x_n(s),x_n(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t), \quad t \in [\tau, T].$$

Thus,

$$\|(\mathcal{S}x_n)(t) - (\mathcal{S}x)(t)\|^p \le \Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=1}^{k}\|b_j(\tau_j)\|\frac{1}{\Gamma(\alpha)}\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}$$
$$\|f(s,x_n(s),x_n(\lambda s)) - f(s,x(s),x(\lambda s))\|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\|f(s,x_n(s),x_n(\lambda s)) - f(s,x(s),x(\lambda s))\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^p$$
$$\le \Big[\max_{i,k}\Big\{1, \prod_{j=1}^{k}\|b_j(\tau_j)\|\Big\}\Big]^p$$
$$\left(\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,x_n(s),x_n(\lambda s)) - f(s,x(s),x(\lambda s))\|dsI_{[\xi_k,\xi_{k+1})}(t)\right)^p$$

$$E\|(\mathcal{S}x_n)(t) - (\mathcal{S}x)(t)\|^p \le 2^{p-1}\max\{1, C^p\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t}E\|f(s,x_n(s),x_n(\lambda s)) - f(s,x(s),x(\lambda s))\|^p ds$$
$$\longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus $\mathcal{S}$ is clearly continuous.
Step 2. We prove that $\mathcal{S}$ is completely continuous operator.

Denote

$$B_k = \Big\{x \in \mathcal{B}\Big| \|x\|_{\mathcal{B}}^p \le k\Big\}$$

for some $k \ge 0$.
Step 2.1 We show that $\mathcal{S}$ maps B_k into an equicontinuous family.

Let $x \in B_k$ and $t_1, t_2 \in [t_0, T]$. If $t_0 < t_1 < t_2 < T$, then by using hypotheses $(H_1) - (H_2)$ and condition (5), we have

$$[(\mathcal{S}x)(t_2) - (\mathcal{S}x)(t_1)]$$
$$\le \sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\Big\{\int_{\xi_{i-1}}^{\xi_i}(t_2-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big\}$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big]\Big(I_{[\xi_k,\xi_{k+1})}(t_2) - I_{[\xi_k,\xi_{k+1})}(t_1)\Big)$$
$$+\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\Big\{\int_{\xi_{i-1}}^{\xi_i}\Big[(t_2-s)^{\alpha-1} - (t_1-s)]^{\alpha-1}\Big]f(s,x(s),x(\lambda s))ds\Big\}$$
$$+\frac{1}{\Gamma(\alpha)}\Big[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds$$
$$+\int_{\xi_k}^{t_1}\Big[(t_2-s)^{\alpha-1} - (t_1-s)]^{\alpha-1}\Big]f(s,x(s),x(\lambda s))ds\Big]\Big]I_{[\xi_k,\xi_{k+1})}(t_2)$$

Then

$$E\|(\mathcal{S}x)(t_2)-(\mathcal{S}x)(t_1)\|^p \le 2^{p-1}E\|I_1\|^p + 2^{p-1}E\|I_2\|^p \tag{9}$$

where

$$\begin{aligned} I_1 &= \sum_{k=0}^{+\infty}\Bigg[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\Big\{\int_{\xi_{i-1}}^{\xi_i}(t_2-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big\} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Bigg]\Big(I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\Big) \end{aligned}$$

and

$$\begin{aligned} I_2 &= \sum_{k=0}^{+\infty}\Bigg[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\Big\{\int_{\xi_{i-1}}^{\xi_i}\Big[(t_2-s)^{\alpha-1}-(t_1-s)]^{\alpha-1}\Big]f(s,x(s),x(\lambda s))ds\Big\} \\ &\quad + \frac{1}{\Gamma(\alpha)}\Big[\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds \\ &\quad + \int_{\xi_k}^{t_1}\Big[(t_2-s)^{\alpha-1}-(t_1-s)]^{\alpha-1}\Big]f(s,x(s),x(\lambda s))ds\Big]\Bigg]I_{[\xi_k,\xi_{k+1})}(t_2) \end{aligned}$$

Furthermore,

$$\begin{aligned} E\|I_1\|^p &\le 2^{p-1}\max\{1,C^p\}\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_2-s)^{p(\alpha-1)}E\|f(s,x(s),x(\lambda s))\|^p ds \\ &\quad \times E\Big(I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\Big) \\ &\le 2^{p-1}\max\{1,C^p\}\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_2-s)^{p(\alpha-1)}m(s)W(2k)ds \\ &\quad \times E\Big(I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\Big) \\ &\le 2^{p-1}\max\{1,C^p\}\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t_2-s)^{p(\alpha-1)}m^*W(2k)ds \\ &\quad \times E\Big(I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\Big) \to 0 \text{ as } \quad t_2\to t_1 \end{aligned} \tag{10}$$

where $m^* = \sup\{m(t) : t \in [t_0, T]\}$, and
and

$$\begin{aligned} E\|I_2\|^p &\le 2^{p-1}\max\{1,C^p\}\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}\Big\{(t_1-s)^{\alpha-1}-(t_2-s)]^{\alpha-1}\Big\}^p E\|f(s,x(s),x(\lambda s))\|^p ds \\ &\quad +2^{p-1}\max\{1,C^p\}\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{p(\alpha-1)}E\|f(s,x(s),x(\lambda s))\|^p ds \\ &\le 2^{p-1}\max\{1,C^p\}\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2}\Big\{(t_1-s)^{\alpha-1}-(t_2-s)]^{\alpha-1}\Big\}^p m^*W(2k)ds \\ &\quad +2^{p-1}\max\{1,C^p\}\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{p(\alpha-1)}m^*W(2k)ds \\ &\quad \to 0 \text{ as } \quad t_2\to t_1 \end{aligned} \tag{11}$$

The right hand side of (10) and (11) is independent of $x \in B_k$. It follows that the right hand side of (9) tends to zero as $t_2 \to t_1$. Thus, $\mathcal{S}$ maps B_k into an equicontinuous family of functions.
Step 2.2 We show that $\mathcal{S}B_k$ is uniformly bounded.

From (5), $\|x\|^2_{\mathcal{B}} \leq k$ and by $(H_1)-(H_2)$ it yields that

$$\begin{aligned}\|(\mathcal{S}x)(t)\|^p &= \|\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\|^p\end{aligned}$$

Thus,

$$\begin{aligned}E\|\mathcal{S}x(t)\|^p &\leq 2^{p-1}C^pE\|x_0\|^p + 2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t} m(s)W(2E\|x(s)\|^p)ds \\ E\|\mathcal{S}x(t)\|^p &\leq 2^{p-1}C^pE\|x_0\|^p + 2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+2}}{(p(\alpha-1)+1)\Gamma(\alpha)}W(\|x\|_{L^p})\|m\|_{L^{1/q_1}}.\end{aligned}$$

This yields that the set $\{(\mathcal{S}x)(t),\ \|x\|^2_{\mathcal{B}} \leq k\}$ is uniformly bounded, so $\{\mathcal{S}B_k\}$ is uniformly bounded. We have already shown that $\{\mathcal{S}B_k\}$ is equicontinuous collection. Now it is sufficient, by the Arzela - Ascoli theorem, to show that $\mathcal{S}$ maps B_k into a precompact set in $\mathcal{B}$.
Step 2.3 We show that $\{\mathcal{S}B_k\}$ is compact. Let $t_0 < t \leq T$ be fixed and ϵ a real number satisfying $\epsilon \in (0, t-t_0)$, for $x \in B_k$. We define

$$\begin{aligned}(\mathcal{S}_\epsilon x)(t) &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t-\epsilon}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t),\ t \in (t_0, t-\epsilon). \qquad (12)\end{aligned}$$

The set

$$H_\epsilon(t) = \{(\mathcal{S}_\epsilon x)(t) :\ x \in B_k\}$$

is precompact in $\mathcal{B}$ for every $\epsilon \in (0, t-t_0)$.

By using $(H_1)-(H_2)$, (5) and $E\|x\|^p_{\mathcal{B}} \leq k$, we obtain

$$E\|(\mathcal{S}x) - (\mathcal{S}_\epsilon x)\|^p \leq 2^{p-1}\max\left\{1,C^p\right\}\frac{1}{\Gamma(\alpha)}\int_{t-\epsilon}^{t} m^*W(2k)ds.$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\mathcal{S}x)(t) :\ x \in B_k\}$. Hence the set $\{(\mathcal{S}x)(t) :\ x \in B_k\}$ is precompact in $\mathcal{B}$. Therefore, $\mathcal{S}$ is a completely continuous operator.

Moreover, the set $U(\mathcal{S}) = \{x \in \mathcal{B} : x = \lambda\mathcal{S}x,$ for some $0 < \lambda < 1\}$ is bounded. Consequently, by Theorem 2.1, the operator $\mathcal{S}$ has a fixed point in $\mathcal{B}$. Therefore, the system (1) – (3) has a solution. Thus, the proof is completed. □

Now, we give another existence result for the system (1) – (3) by means of Banach contraction principle. We make the following assumption:

(H_1'): The function f satisfies the Lipschitz condition. That is, for $\vartheta, \hat{\vartheta} \in \mathfrak{R}^n$ and $\tau \leq t \leq T$, there exists a constant $L > 0$ such that

$$E\|f(t,\vartheta_1,\vartheta_2) - f(t,\hat{\vartheta}_1,\hat{\vartheta}_2)\|^p \leq L\Big[E\|\vartheta_1-\hat{\vartheta}_1\|^p + E\|\vartheta_2-\hat{\vartheta}_2\|^p\Big]$$

Theorem 3.2. *Let the hypotheses (H_1'), (H_2) be hold. Then there exists a local unique continuous solution to* (1) – (3) *for any initial value (t_0, x_0) with $t_0 \geq 0$ and $x_0 \in \mathcal{B}$.*

Proof. Let T be an arbitrary number $\tau < T < +\infty$. Define the nonlinear operator $\mathcal{S} : \mathcal{B} \to \mathcal{B}$ as follows.

$$(\mathcal{S}x)(t) = \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t), \quad t\in[\tau,T].$$

Now we have to show that , $\mathcal{S}$ is a contraction mapping.

$$\begin{aligned}
&\|(\mathcal{S}x)(t)-(\mathcal{S}y)(t)\|^p\\
\le\ & \Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k}\prod_{j=1}^{k}\|b_j(\tau_j)\|\frac{1}{\Gamma(\alpha)}\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\|f(s,x(s),x(\lambda s))-f(s,y(s),y(\lambda s))\|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\|f(s,x(s),x(\lambda s))-f(s,y(s),y(\lambda s))\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big]^p\\
\le\ & \Big[\max_{i,k}\Big\{1,\prod_{j=1}^{k}\|b_j(\tau_j)\|\Big\}\Big]^p\\
&\left(\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,x(s),x(\lambda s))-f(s,y(s),y(\lambda s))\|dsI_{[\xi_k,\xi_{k+1})}(t)\right)^p
\end{aligned}$$

$$\begin{aligned}
E\|(\mathcal{S}x)(t)-(\mathcal{S}y)(t)\|^p \ \le\ & \max\{1,C^p\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\\
&\times L\int_{t_0}^{t}E\Big[\|x(s)-y(s)\|^p+\|x(\lambda s)-y(\lambda s)\|^p\Big]ds.
\end{aligned}$$

Taking supremum over t, we get,

$$\|(\mathcal{S}x)-(\mathcal{S}y)\|^p \le \max\{1,C^p\}\frac{2(T-\tau)^{p(\alpha-1)+2}}{(p(\alpha-1)+1)\Gamma(\alpha)}L\|x-y\|^p.$$

Thus,

$$\|\mathcal{S}x-\mathcal{S}y\|^p \le \Lambda(T)\|x-y\|^p,$$

with $\Lambda(T) = \max\{1,C^p\}\frac{2(T-\tau)^{p(\alpha-1)+2}}{(p(\alpha-1)+1)\Gamma(\alpha)}L$.

Then we can take a suitable $0 < T_1 < T$ sufficient small such that $\Lambda(T_1) < 1$, and hence $\mathcal{S}$ is a contraction on $\mathcal{B}_{T_1}$ ($\mathcal{B}_{T_1}$ denotes $\mathcal{B}$ with T substituted by T_1). Thus, by the well-known Banach fixed point theorem we obtain a unique fixed point $x \in \mathcal{B}_{T_1}$ for operator $\mathcal{S}$, and hence $\mathcal{S}x = x$ is a unique solution of (1) – (3). This procedure can be repeated to extend the solution to the entire interval $[\tau, T]$ in finitely many similar steps, thereby completing the proof for the existence and uniqueness of solutions on the whole interval $[\tau, T]$. □

4. Stability

In this section, we study the stability of the system (1) – (3) through the continuous dependence of solutions on initial condition.

Theorem 4.1. *Let $x(t)$ and $\bar{x}(t)$ be solutions of the system* (1) – (3) *with initial values x_0 and $\bar{x_0} \in \mathcal{R}^n$ respectively. If the assumptions of Theorem* 3.2 *are satisfied, then the solution of the system* (1) – (3) *is stable in the p^{th} mean.*

Proof. By the assumptions, x and $\bar{x}$ are the two solutions of the system $(1)-(3)$ for $t \in [\tau, T]$. Then,

$$x(t) - \bar{x}(t) = \sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} b_i(\tau_i)[x_0 - \bar{x_0}] + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1}[f(s, x(s), x(\lambda s) - f(s, \bar{x}(s), \bar{x}(\lambda s)]ds + \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1}[f(s, x(s), x(\lambda s)) - f(s, \bar{x}(s), \bar{x}(\lambda s))]ds \Bigg] I_{[\xi_k, \xi_{k+1})}(t), \quad t \in [\tau, T].$$

By using the hypotheses(H_1'), (H_2), we get

$$\sup_{t\in[\tau,T]} E\|x(t) - \bar{x}(t)\|^p \le 2^{p-1}C^p E\|x_0 - \bar{x_0}\|^p + 2^{p-1} \max\left\{1, C^p\right\} \frac{2(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)} L \int_{t_0}^{t} \sup_{s\in[\tau,T]} E\|x(s) - \bar{x}(s)\|^p ds$$

Therefore,

$$\|x - \bar{x}\|^p \le 2^{p-1}C^p E\|x_0 - \bar{x_0}\|^p + 2^{p-1} \max\left\{1, C^p\right\} \frac{2(T-\tau)^{p(\alpha-1)+2}}{(p(\alpha-1)+1)\Gamma(\alpha)} L \|x - \bar{x}\|^p.$$

$$\|x - \bar{x}\|^p \le \hat{\zeta} E\|x_0 - \bar{x_0}\|^p$$

where, $\hat{\zeta} = \frac{2^{p-1}C^p}{1 - 2^{p-1}\max\left\{1, C^p\right\} \frac{2(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)} L}$.
Now given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{\hat{\zeta}}$ such that $E\|x_0 - \bar{x_0}\|^p \le \delta$. Then

$$\|x - \bar{x}\|^p \le \epsilon.$$

Thus, it is apparent that the difference between the solution $x(t)$ and $\bar{x(t)}$ in the interval $[\tau, T]$ is small provided the change in the initial point (t_0, x_0) as well as in the function $f(s, x(s), x(\lambda s)$ do not exceed prescribed amounts. This completes the proof. □

5. Ulam-Hyers Stability Results

In this section, we study the Ulam- Hyers stability of random impulsive fractional pantograph differential equation $(1)-(3)$. Let $\epsilon > 0$ and $\phi : [\tau, T] \to \Re^+$ be a continuous function. We consider the following inequalities

$$\begin{cases} E\|{}^cD_t^\alpha x(t) - f(t, x(t), (x(\lambda t))\|^p \le \epsilon, & t \ne \xi_k, \quad t \ge \tau. \\ E\|x(\xi_k) - b_k(\tau_k)x(\xi_k^-)\|^p \le \epsilon, & k = 1, 2, \ldots.. \end{cases} \tag{13}$$

$$\begin{cases} E\|{}^cD_t^\alpha x(t) - f(t, x(t), x(\lambda t))\|^p \le \phi(t), & t \ne \xi_k, \quad t \ge \tau. \\ E\|x(\xi_k) - b_k(\tau_k)x(\xi_k^-)\|^p \le \mu, & k = 1, 2, \ldots.. \end{cases} \tag{14}$$

$$\begin{cases} E\|{}^cD_t^\alpha x(t) - f(t, x(t), x(\lambda t))\|^p \le \epsilon\phi(t), & t \ne \xi_k, \quad t \ge \tau. \\ E\|x(\xi_k) - b_k(\tau_k)x(\xi_k^-)\|^p \le \epsilon\mu, & k = 1, 2, \ldots.. \end{cases} \tag{15}$$

Definition 5.1. *The system* $(1)-(3)$ *is Ulam- Hyers stable in the* p^{th} *mean if there exists a real number* $\kappa > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $x \in \mathcal{B}$ *of the inequality* (13) *there exists a solution* $y \in \mathcal{B}$ *of the system* $(1)-(3)$ *with*

$$E\|x(t) - y(t)\|^p \le \kappa\epsilon, \qquad t \in [\tau, T].$$

Definition 5.2. *The system* (1) − (3) *is generalized Ulam- Hyers stable in the* p^{th} *mean if there exists a real number* $\eta \in \mathcal{B}, \eta(0) = 0$ *such that for each solution* $x \in \mathcal{B}$ *of the inequality* (13) *there exists a solution* $y \in \mathcal{B}$ *of the system* (1) − (3) *with*

$$E\|x(t) - y(t)\|^p \leq \eta(\epsilon), \qquad t \in [\tau, T].$$

Definition 5.3. *The system* (1) − (3) *is Ulam- Hyers- Rassias stable in the* p^{th} *mean with respect to* (ϕ, μ) *if there exists a real number* $\zeta > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $x \in \mathcal{B}$ *of the inequality* (15) *there exists a solution* $y \in \mathcal{B}$ *of the system* (1) − (3) *with*

$$E\|x(t) - y(t)\|^p \leq \zeta\epsilon(\phi(t) + \mu), \qquad t \in [\tau, T].$$

Definition 5.4. *The system* (1) − (3) *is generalized Ulam- Hyers- Rassias stable in the* p^{th} *mean with respect to* (ϕ, μ) *if there exists a real number* $\zeta > 0$ *such that for each solution* $x \in \mathcal{B}$ *of the inequality* (14) *there exists a solution* $y \in \mathcal{B}$ *of the system* (1) − (3) *with*

$$E\|x(t) - y(t)\|^p \leq \zeta(\phi(t) + \mu), \qquad t \in [\tau, T].$$

Remark 5.5. *It is clear that*

1. *Definition* (5.1) $\Rightarrow$ *Definition* (5.2)
2. *Definition* (5.3) $\Rightarrow$ *Definition* (5.4)
3. *Definition* (5.3) *for* $\phi(t) = \mu = 1 \Rightarrow$ *Definition* (5.1).

Remark 5.6. *A function* $x \in \mathcal{B}$ *is a solution of the inequality* (15) *if and only if there exists a function* $h \in \mathcal{B}$ *and the sequence* $h_k, k = 1, 2, \ldots$ *(which depend on x) such that*
(i): $E\|h(t)\|^p \leq \epsilon\phi(t), t \in [\tau, T]$ *and* $E\|h_k\|^p \leq \epsilon\mu, k = 1, 2, \ldots;$
(ii): ${}^cD_t^\alpha x(t) = f(t, x(t), x(\lambda t)) + h(t), \quad t \neq \xi_k, \quad t \geq \tau;$
(iii): $x(\xi_k) = b_k(\tau_k)x(\xi_k^-) + h_k, \quad k = 1, 2, ...$
One can have similar remarks for the inequalities (13) *and* (14).

Remark 5.7. *Let* $0 < \alpha < 1$, *if* $x \in \mathcal{B}$ *is a solution of the inequality* (15) *then x is a solution of the following integral inequality*

$$E\Bigg\|x(t) - \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s, x(s), x(\lambda s))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s, x(s), x(\lambda s))ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg\|^p$$
$$\leq 2^{p-1}\epsilon\Bigg\{C^p\mu + \max\big\{1, C^p\big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t}\phi(s)ds\Bigg\}, \ \ t \in [\tau, T].$$

From the Remark 5.6 we have

$$\begin{cases} {}^cD_t^\alpha x(t) = f(t, x(t), x(\lambda t)) + h(t), \quad t \neq \xi_k, \quad t \geq \tau. \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-) + h_k, \quad k = 1, 2, ... \end{cases} \tag{16}$$

Then

$$x(t) = \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \prod_{i=1}^{k} b_i(\tau_i)h_i + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s, x(s), x(\lambda s))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s, x(s), x(\lambda s))ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}h(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}h(s)ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t), \quad t \in [\tau, T].$$

Therefore,

$$
\begin{aligned}
& E\Big\|x(t)-\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}b_i(\tau_i)x_0+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds \\
& \qquad +\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \\
= \; & E\Big\|\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}b_i(\tau_i)h_i+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}h(s)ds \\
& \qquad +\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}h(s)ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \\
\leq \; & 2^{p-1}\max_k\Big\{\prod_{i=1}^{k}\Big\|b_i(\tau_i)\Big\|^p\Big\}E\|h_i\|^p \\
& \qquad +2^{p-1}\Big[\max_{i,k}\Big\{1,\prod_{j=1}^{k}\Big\|b_i(\tau_i)\Big\|\Big\}\Big]^p\Big[\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}E\|h(s)\|dsI_{[\xi_k,\xi_{k+1})}(t)\Big]^p \\
\leq \; & 2^{p-1}C^p\epsilon\mu+2^{p-1}\max\big\{1,C^p\big\}\Big[\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{p(\alpha-1)}ds\times\int_{t_0}^{t}E\|h(s)\|^pds\Big] \\
\leq \; & 2^{p-1}\epsilon\Big\{C^p\mu+\max\big\{1,C^p\big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t}\phi(s)ds\Big\}
\end{aligned}
$$

We have similar remarks for the solutions of the inequalities (13) and (14). Now, we give the main results, generalized Ulam-Hyers-Rassias results, in this section.

Theorem 5.8. *Asssumption* (H_1) *and* (H_2) *hold. Suppose there exists* $\lambda > 0$ *such that*

$$\int_{t_0}^{t}\phi(s)ds \leq \lambda\,\phi(t), \quad \text{for each } \; t\in[\tau,T],$$

where $\phi:[\tau,T]\to\mathfrak{R}^+$ *is a continuous nondecreasing function. Then the system* $(1)-(3)$ *is Ulam - Hyers- Rassias stable in the* p^{th} *mean square.*

Proof. Let $x\in\mathcal{B}$ be a solution of the inequality (15). By Theorem 3.2 there exist a unique solution y of the random impulsive fractional pantograph differential system

$$
\begin{aligned}
{}^cD_t^{\alpha}y(t) &= f(t,y(t),y(\lambda t)), \quad t\neq\xi_k, \quad t\geq\tau \\
y(\xi_k) &= b_k(\tau_k)y(\xi_k^-), \quad k=1,2,..... \\
y_{t_0} &= x_0.
\end{aligned} \tag{17}
$$

Then we have

$$
\begin{aligned}
y(t) = & \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}b_i(\tau_i)x_0+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,y(s),y(\lambda s))ds \\
& +\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,y(s),y(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t), \quad t\in[\tau,T].
\end{aligned}
$$

By differential inequality (15), we have

$$
\begin{aligned}
& E\Big\|x(t)-\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}b_i(\tau_i)x_0+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s))f(s,x(s),x(\lambda s))ds \\
&\qquad +\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \\
\leq\;& 2^{p-1}C^p\epsilon\mu+2^{p-1}\max\Big\{1,C^p\Big\}\Big[\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{p(\alpha-1)}ds\times\int_{t_0}^{t}E\|h(s)\|^p ds\Big] \\
\leq\;& 2^{p-1}\epsilon\Big\{C^p\mu+\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\int_{t_0}^{t}\phi(s)ds\Big\} \\
\leq\;& 2^{p-1}\epsilon\Big\{C^p\mu+\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\lambda\phi(t)\Big\}\;\; t\in[\tau,T].
\end{aligned}
$$

Hence for each $t\in[\tau,T]$, we have

$$
\begin{aligned}
& E\|x(t)-y(t)\|^p \\
\leq\;& 2^{p-1}E\Big\|x(t)-\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}b_i(\tau_i)x_0+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds \\
&\qquad +\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x(s),x(\lambda s))ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \\
& +2^{p-1}E\Big\|\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\Big[\prod_{j=1}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\Big\{f(s,x(s),x(\lambda s))-f(s,y(s),y(\lambda s))\Big\}ds \\
&\qquad +\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\Big\{f(s,x(s),x(\lambda s))-f(s,y(s),y(\lambda s))\Big\}\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \\
\leq\;& 4^{p-1}\epsilon\Big\{C^p\mu+\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\lambda\phi(t)\Big\} \\
& +2^{p-1}\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}L\int_{t_0}^{t}E\|x(s)-y(s)\|^p ds.
\end{aligned}
$$

Taking supremum over $t\in[\tau,T]$, we get

$$
\begin{aligned}
\sup_{t\in[\tau,T]}E\|x(t)-y(t)\|^p \;\leq\;& 4^{p-1}\epsilon\Big\{C^p\mu+\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\lambda\phi(t)\Big\} \\
& +2^{p-1}\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}L\int_{t_0}^{t}\sup_{s\in[\tau,t]}E\|x(s)-y(s)\|^p ds \\
\leq\;& 4^{p-1}\epsilon\Big\{C^p\mu+\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\lambda\phi(t)\Big\} \\
& +2^{p-1}\max\Big\{1,C^p\Big\}\frac{(T-\tau)^{p(\alpha-1)+2}}{(p(\alpha-1)+1)\Gamma(\alpha)}L\sup_{t\in[\tau,T]}E\|x(t)-y(t)\|^p.
\end{aligned}
$$

There exists a constant $\hat{h} = \frac{1}{1-2^{p-1}\max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+2}}{(p(\alpha-1)+1)\Gamma(\alpha)}L} > 0$ independent of $\lambda\phi(t)$ such that

$$\sup_{t\in[\tau,T]} E\|x(t)-y(t)\|^p \leq \hat{h}\,4^{p-1}\epsilon\left\{C^p\mu + \max\left\{1,C^p\right\}\frac{(T-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)\Gamma(\alpha)}\lambda\phi(t)\right\}, \ t\in[\tau,T].$$

Thus, the system (1) − (3) is Ulam - Hyers-Rassias stable in the p^{th} mean. Hence the proof. □

Remark 5.9. 1. *Under the assumption of Theorem* 5.8*, we consider the system* (1) − (3) *and the inequality* (13)*. One can repeat the same process to verify that the system* (1) − (3) *is Ulam - Hyers stable in the p^{th} mean.*

2. *Under the assumption of Theorem* 5.8*, we consider the system* (1) − (3) *and the inequality* (14)*. One can repeat the same process to verify that the system* (1) − (3) *is generalized Ulam - Hyers-Rassias stable in the p^{th} mean.*

6. Example

Consider the impulsive fractional pantograph equation of the form

$$\begin{aligned} &{}^cD_t^\alpha x(t) = ax(t) + f(t,x(\lambda t)), \quad t\neq\xi_k, \quad t\geq\tau \\ &x(\xi_k) = b_k(\tau_k)x(\xi_k^-), \quad k=1,2,..... \\ &x(0)=x_0, \end{aligned} \tag{18}$$

where $x_0\in R$, $\lambda\in(0,1)$, $\alpha\in(0,1)$ and $f:[0,T]\times\Re\to\Re$ is a scalar continuous function. The integral representation of equation (18) is

$$\begin{aligned} x(t) = \sum_{k=0}^{+\infty}\Bigg[&\prod_{i=1}^{k} b_i(\tau_i)x_0E_\alpha(at^\alpha) + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}E_{\alpha,\alpha}(a(t-s)^\alpha)f(s,x(\lambda s))ds \\ &+\int_{\xi_k}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(a(t-s)^\alpha)f(s,x(\lambda s))ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t), \quad t\in[\tau,T], \end{aligned} \tag{19}$$

where $E_\alpha(at^\alpha) = \sum_{k=0}^{\infty}\frac{a^kt^{k\alpha}}{\Gamma(1+\alpha k)}$, $E_{\alpha,\alpha}(a(t-s)^\alpha) = \sum_{k=0}^{\infty}\frac{a^k(t-s)^{k\alpha}}{\Gamma((1+k)\alpha)}$.

Case 6.1. *When $a=0$ and $f(t,x(\lambda t)) = cx(\lambda t)$, then the equation* (18) *becomes,*

$$\begin{aligned} &{}^cD_t^\alpha x(t) = cx(\lambda t), \quad t\neq\xi_k, \quad t\geq\tau \\ &x(\xi_k) = b(k)\tau_kx(\xi_k^-), \quad k=1,2,..... \\ &x_{t_0}=x_0, \end{aligned} \tag{20}$$

where $x_0\in R$, $c\neq 0$, $\lambda\in(0,1)$.

Let τ_k be a random variable defined on $D_k\equiv(0,d_k)$ for $k=1,2,\ldots$, where $0<d_k<+\infty$. Furthermore, assume that τ_k follow Erlang distribution, where $k=1,2,\ldots$ and τ_i and τ_j are independent with each other as $i\neq j$ for $i,j=1,2,\ldots$; b is a function of k; $\xi_0=t_0$; $\xi_k=\xi_{k-1}+\tau_k$ for $k=1,2,\ldots$.

We assume the following condition hold:

(i) $E\Big[\max_{i,k}\Big\{\prod_{j=i}^{k}\left\|b(j)(\tau_j)\right\|^2\Big\}\Big]<\infty.$

(ii) $E\|f(t,x(\lambda t))-f(t,y(\lambda t))\|^2 \leq L_1E\|x(t)-y(t)\|^2$*, for some $L_1>0$ and $t\in[\tau,T]$.*

Assuming that conditions (*i*) *and* (*ii*) *are verified, then the problem* (20) *can be modeled as the abstract random impulsive fractional pantograph equation* (1) − (3) *by defining*

$$f(t,x(t),x(\lambda t)) = cx(\lambda t), \quad b_k(\tau_k) = q(k)\tau_k \quad \textit{and} \ \ p=2.$$

Proposition 6.2. *Assume that the hypotheses* (H_1')-(H_2) *hold, then the system* (20) *has a unique solution* $x(t)$.

Proof. *Condition* (ii) *implies that* (H_1') *holds with* $L_1 > 0$ *and* (H_2) *follow from conditions (i).* □

The next results are consequences of Theorem 4.1 *and* 5.8 *respectively.*

Proposition 6.3. *Let the hypothesis* (H_1'), (H_2) *be hold. Then the trivial solution of* (20) *is stable in the mean square.*

Proof. *In the equations* (20), *integrating, we get*

$$x(t) = \sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}cx(\lambda s))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}cx(\lambda s))ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t), \quad t\in[\tau,T].$$

Then we have, $\|x-\bar{x}\|^2 \leq \hat{\zeta}E\|x_0-\bar{x_0}\|^2$ *where* $\hat{\zeta} = \frac{2c^2}{1-2\max\{1,c^2\}\frac{(T-\tau)2\alpha}{(2\alpha-1)\Gamma(\alpha)}L_1}$
Now given $\epsilon > 0$, *choose* $\delta = \frac{\epsilon}{\zeta}$ *such that* $E\|x_0-\bar{x_0}\|^2 \leq \delta$.
Then $\|x-\bar{x}\|^2 \leq \epsilon$. *Thus solution of system* (20) *is mean square stable.* □

Proposition 6.4. *Let the hypotheses* (H_1') *and* (H_2) *be hold. Suppose there exists* $\lambda > 0$ *such that*

$$\int_{t_0}^{t}\phi(s)ds \leq \lambda\,\phi(t), \;\; \text{for each} \;\; t\in[t_0,T],$$

where $\phi : [t_0,T] \to \Re^+$ *is a continuous nondecreasing function. Then the system* (20) *is generalized Ulam - Hyers-Rassias stable in the mean square.*

Remark 6.5. *The pure pantograph equation of the form,*

$$x'(t) = cx(\lambda t), t\geq 0, x(0) = x_0, (c\neq 0). \tag{21}$$

G.R. Morris et al [44] (also [5]) showed that the trivial solution of (21) *pantograph equation is unstable.*

Remark 6.6. *If* $\alpha = 1$, *the propositions* 6.1, 6.2, 6.3, *shows that the random impulsive perturbations can make the unstable pantograph system* (21) *as mean square stable.*

Case 6.7. *If* $a \neq 0$ *and* $f(t,x(\lambda t)) = cx(\lambda t)$, *the solution of the equation* (18) *is given as* (19). *The right hand side of* (19) *satisfies the Lipschitz condition of the form*

$$cT^\alpha\sum_{k=0}^{+\infty}\frac{a^kT^{k\alpha}}{((1+k)\alpha)\Gamma(1+k)\alpha}\|x(t)-y(t)\|^2.$$

By choosing a, c, and T in such a way that the hypotheses of Theorem 3.2, Theorem 4.1 and Theorem 5.8 are satisfied.

References

[1] J. R. Ockendon and A. B. Taylor, *The dynamics of a current collection system for an electric locomotive*, Proc R Soc London, Ser.A, 322(1971) 447 –468.
[2] A. Iserles *On the generalized pantograph functional-differential equation*, Eur J Appl Math, 4 (1993) 1–38.
[3] A.Iserles, *Exact and discretized stability of the pantograph equation*, Appl Numer Math, 24 (1997) 295–308.
[4] A.Iserles and Y. Liu, *On pantograph integro-differential equations*, J Integral Equations Appl, 6(1994) 213–237.
[5] K. Guan, Z. Luo, *Stability results for impulsive pantograph equations*, Appl. Math. Letters, 26(2013) 1169– 1174.
[6] V. Lakshmikantham, D.D. Bainov, and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[7] A.M. Samoilenko, N.A Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.
[8] R. Iwankievicz, S.R.K. Nielsen, *Dynamic response of non-linear systems to Poisson distributed random impulses*, J.Sound Vibration 156(1992) 407– 423.
[9] A Anguraj, A. Vinodkumar *Existence, uniqueness and stability results of random impulsive semilinear differential systems*, Nonlinear Analysis: Hybrid Systems 4(2010) 475–483.
[10] J.M. Sanz-Serna, A.M. Stuart, *Ergodicity of dissipative differential equations subject to random impulses*, J. Differential Equations 155(1999) 262–284.
[11] K. Tatsuyuki, K. Takashi, and S Satoshi, *Drift motion of granules in chara cells induced by random impulses due to the myosin-actin interaction*, Physica A 248 (1998) 21–27.
[12] S.J. Wu, X.L. Guo, and S. Q. Lin, *Existence and uniqueness of solutions to random impulsive differential systems*, Acta Math. Appl. Sin. 22(4)(2006) 595–600.
[13] S.J. Wu, X.L. Guo, and Y. Zhou, *p−moment stability of functional differential equations with random impulses*, Comput. Math. Appl. 52(2006) 1683–1694.
[14] S.J. Wu, *The Euler scheme for random impulsive differential equations*, Appl. Math. Comput. 191(2007) 164 –175.
[15] A. Anguraj, S. Wu, and A.Vinodkumar, *Existence and exponential stability of semilinear functional differential equations with random impulses under non-uniqueness*, Nonlinear Anal. TMA 74(2011) 331–342.
[16] M. Gowrisankar, P.Mohankumar, and A. Vinodkumar, *Stability results of random impulsive semilinear differential equations*, Acta Mathematica Scientia, 34 B(4) (2014) 1055–1071.
[17] S.Zhang, J. Sun, *Stability analysis of second-order differential systems with Erlang distribution random impulses*, Advances in Difference Equations 2013 (2013):4, 403–415.
[18] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., 2006.
[19] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[20] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
[21] Y. Li, Y. Chen, I. Podlubny, *Mittag-Leffler stability of fractional order nonlinear dynamic systems*, Automatica 45 (2009) 1965–1969.
[22] Y. Li, Y. Chen, I. Podlubny, *Stability of fractional- order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability*, Comput. Math. Appl. 59 (2010) 18 10–21.
[23] I. A. Rus , *Ulam stability of ordinary differential equations,* Studia Univ Babeş Bolyai Math. 54 (2009)1 25–33.
[24] W. Deng, *Smoothness and stability of the solutions for nonlinear fractional differential equations*, Nonlinear Anal. 72 (2010)17 68–77.
[25] J. Wang, L. Lv, Y. Zhou, *Ulam stability and data dependence for fractional differential equations with Caputo derivative*, Electron. J. Qual. Theory Differ. Equ. 63 (2011) 1–10.
[26] J. Wang, L. Lv, Y. Zhou, *New concepts and results in stability of fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 2530–2538.
[27] J. Wang, Y. Zhou, M. Fečkan, *Nonlinear impulsive problems for fractional differential equations and Ulam stability*, Comput. Math. Appl. 64 (2012) 3389–3405.
[28] Michal Fečkan, Yong Zhou, JinRong Wang, *On the concept and existence of solution for impulsive fractional differential equations*, Commun. Nonlinear Sci. Numer. Simulat. 17(2011) 3050–3060.
[29] K. Balachandran, S. Kiruthika and J. J. Trujiio, *Existence of solutions of nonlinear fractional pantograph equations*, Acta Mathematica Scientia 2013,33B(3) 712-720.
[30] X. Li, J. Wang, *Ulam-Hyers-Rassias stability of semilinear differential equations with impulses*, Electron. J. Diff. Equ., Vol. 2013 (2013), No. 172, pp. 1–8.
[31] F.B.Adda, J. Cresson, *Fractional differential equations and the schrödinger equation*, Appl. Math. Comput. 161 (2005)3 23–45.
[32] R.P. Agarwal, M. Benchohra, B.A. Slimani, *Existence results for differential equations with fractional order and impulses*, Memoirs Differ. Equ. Math. Phys. 44(2008) 1–21.
[33] M. Benchohra, BA. Slimani, *Existence and uniqueness of solutions to impulsive fractional differential equations*, Electron. J. Differ. Equ. 10 (2009) 1–11.
[34] G.M. Mophou, *Existence and uniqueness of mild solutions to impulsive fractional differential equations,* Nonlinear Anal. 72 (2010) 1604–1605.
[35] G.M. N′ Guérékata, *A Cauchy problem for some fractional abstract differential equation with non local conditions*, Nonlinear Anal. 70(2009) 1873–1876.
[36] S.M. Ulam, *A collection of mathematical problems,* New York, Interscience Publishers, 1968.
[37] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. 27(1941) 222–224.
[38] Th. M. Rassias, *On the stability of linear mappings in Banach spaces*, Proc. Amer. Soc. 72(1978) 297–300.
[39] Cădariu L, *Stabilitatea Ulam-Hyers-Bourgin pentru ecuatii functionale*, Ed. Univ. Vest Timisoara, Timisara 2007.
[40] C.P. Li, F.R. Zhang, *A survey on the stability of fractional differential equations*, Eur. Phys. J Special Topics 193(2011) 27–47.
[41] E. M. Elsayed, *Behavior and Expression of the Solutions of Some Rational Difference Equations*, J. Comput. Anal.Appl., 15 (1) (2013) 73–81.
[42] E. M. Elsayed, *On the solutions and periodic nature of some systems of difference equations*, I.J. Biomathematics, 7 (6) (2014) 1450067.
[43] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, *Equilibrium points, stability and numerical solutions of fractional-order predatorprey and rabies models*, J. Math. Anal. Appli., 1(325) (2007) 542- 553.
[44] G.R. Morris, A. Feldstein, E.W. Bowen, *The Phragmen-Lindelof principle and a class of functional differential equations, in: Ordinary differential equations*, Academic press, San Diego, (1972) 513 –540.