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Multivalued Almost F-contractions on Complete Metric Spaces

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Abstract. In the present paper, considering a new concept of multivalued almost F-contraction, we give a general class of multivalued weakly Picard operators on complete metric spaces. Also, we give some illustrative examples showing that our results are proper generalizations of some previous theorems.

1. Introduction and Preliminaries

Given a metric space (X, d), by $P(X)$, $CB(X)$ and $K(X)$ we will denote the family of all nonempty subsets of X, the family of all nonempty closed and bounded subsets of X, and the family of all nonempty compact subsets of X, respectively. It is clear that, $K(X) \subseteq CB(X) \subseteq P(X)$. For $A, B \in CB(X)$, let

$$H(A, B) = \max\left\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\right\},$$

where $D(x, B) = \inf\{d(x, y) : y \in B\}$. Then H is a metric on $CB(X)$, which is called the Pompeiu-Hausdorff metric induced by d. We can find detailed information about the Pompeiu-Hausdorff metric in [9, 16]. Let $T : X \to CB(X)$ be a mapping, then T is called a multivalued contraction if for all $x, y \in X$ there exists $c \in [0, 1)$ such that

$$H(Tx, Ty) \leq cd(x, y).$$

In 1969, Nadler [20] proved that every multivalued contraction on a complete metric space has a fixed point.

Since then, a lot of generalizations of the result of Nadler were given (see, for example [1, 2, 10–15, 17, 18, 26, 28]). An interesting important generalization of it were given by Berinde and Berinde [8] where the authors introduced the concept of a multivalued weakly Picard operator as follows (for single-valued Picard and weakly Picard operators we refer to [6, 7, 23]):

Definition 1.1. *Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued operator. T is said to be a multivalued weakly Picard (MWP) operator if for each $x \in X$ and any $y \in Tx$, there exists a sequence $\{x_n\}$ in X such that*
(i) $x_0 = x$, $x_1 = y$,
(ii) $x_{n+1} \in Tx_n$,
(iii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of T.

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Then Berinde and Berinde [8] show that the type multivalued contractions on complete metric spaces considered by Nadler [20], Mizoguchi and Takahashi [19], Reich [24], Rus [27] and Petruşel [21], are MWP operators.

In the same paper, Berinde and Berinde [8] introduced the concepts of multivalued almost contraction (the original name was multivalued (δ, L)-weak contraction) and proved the following nice fixed point theorem:

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued almost contraction, that is, there exist two constants $\delta \in (0, 1)$ and $L \geq 0$ such that*

$$H(Tx, Ty) \leq \delta d(x, y) + LD(y, Tx) \tag{1}$$

for all $x, y \in X$. Then T is an MWP operator.

We can find some detailed information about the singe-valued case of (δ, L)-weak contraction and the nonlinear case of it in [4, 5, 22].

In the present paper we introduce the concept of multivalued almost F-contraction and, from it, we give a general class of MWP operators on complete metric spaces. Our results are based on the notion of an F-contraction which was introduced by Wardowski [29] for the case of single-valued maps on complete metric spaces. First, we recall this notion and some related results.

Let $F : (0, \infty) \to \mathbb{R}$ be a function. Consider the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,

(F2) For each sequence $\{\alpha_n\}$ of positive numbers

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty,$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$,

(F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

We denote by $\mathcal{F}$ and $\mathcal{F}_*$ be the set of all functions F satisfying (F1)-(F3) and (F1)-(F4), respectively. It is clear that $\mathcal{F}_* \subset \mathcal{F}$ and some examples of the functions belonging $\mathcal{F}_*$ are $F_1(\alpha) = \ln \alpha$, $F_2(\alpha) = \alpha + \ln \alpha$, $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln\left(\alpha^2 + \alpha\right)$. If we define $F_5(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F_5(\alpha) = \alpha$ for $\alpha > 1$, then $F_5 \in \mathcal{F} \backslash \mathcal{F}_*$.

Remark 1.3. *If F satisfies (F1), then it satisfies (F4) if and only if it is right continuous.*

Definition 1.4 ([29]). *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then, we say that T is an F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that*

$$\forall x, y \in X\ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))]. \tag{2}$$

Observe that if T is an F-contraction for F given by $F(\alpha) = \ln \alpha$, then the inequality (2) turns to

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, Tx \neq Ty. \tag{3}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds. Thus T is an ordinary contraction with contractive constant $c = e^{-\tau}$. Conversely, if T is an ordinary contraction with contractive constant c, then T is also an F-contraction for F given by $F(\alpha) = \ln \alpha$ and $\tau = -\ln c$. However, not every F-contraction is an ordinary contraction. Indeed, Wardowski gave in Example 2.5 of [29] an instance of a self-mapping T on a complete metric space which is an F-contraction for F given by $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$, but is not an ordinary contraction.

In addition, Wardowski showed that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

Thereby, Wardowski proved that every F-contraction on a complete metric space has a unique fixed point.

By combining the ideas of Wardowski and Nadler, Altun et al [3] introduced the concept of multivalued F-contraction and obtained some fixed point results for this type of mappings on complete metric spaces.

Definition 1.5 ([3]). *Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. Then we say that T is a multivalued F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that*

$$\forall x, y \in X\ [H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leq F(d(x, y))]. \tag{4}$$

By the considering $F(\alpha) = \ln \alpha$, then every multivalued contraction in the sense of Nadler is also multivalued F-contraction.

Theorem 1.6 ([3]). *Let (X, d) be a complete metric space and $T : X \to K(X)$ be a multivalued F-contraction, then T has a fixed point in X.*

Theorem 1.7 ([3]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued F-contraction. If $F \in \mathcal{F}_*$, then T has a fixed point in X.*

2. Main Result

We begin this section by introducing the following definition.

Definition 2.1. *Let (X, d) be a metric space and $T : X \to CB(X)$ be a mapping. We say that T is a multivalued almost F-contraction if $F \in \mathcal{F}$ and there exist two constants $\tau > 0$ and $\lambda \geq 0$ such that, for all $x, y \in X$ with $H(Tx, Ty) > 0$,*

$$\tau + F(H(Tx, Ty)) \leq F(d(x, y) + \lambda D(y, Tx)). \tag{5}$$

By considering $F(\alpha) = \ln \alpha$, we can say that every multivalued almost contraction (in the sense of (1)) is also a multivalued almost F-contraction.

Our main result is as follows:

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued almost F-contraction with $F \in \mathcal{F}_*$, then T is an MWP operator.*

Proof. Let $x_0 \in X$. As Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. In this case, we construct a sequence $\{x_n\}$ by $x_n = x_1$ for $n \geq 1$, then $x_{n+1} \in Tx_n$ and $\{x_n\}$ converges to a fixed point of T. Now, let $x_1 \notin Tx_1$. Then, as Tx_1 is closed, $D(x_1, Tx_1) > 0$. On the other hand, as $D(x_1, Tx_1) \leq H(Tx_0, Tx_1)$, from (F1) we have

$$F(D(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)).$$

From (5), we can write that

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq F(H(Tx_0, Tx_1)) \\ &\leq F(d(x_1, x_0) + \lambda D(x_1, Tx_0)) - \tau \\ &= F(d(x_1, x_0)) - \tau \end{aligned} \tag{6}$$

From (F4) we can write (note that $D(x_1, Tx_1) > 0$)

$$F(D(x_1, Tx_1)) = \inf_{y \in Tx_1} F(d(x_1, y)),$$

and so from (6) we have

$$\inf_{y \in Tx_1} F(d(x_1, y)) \leq F(d(x_1, x_0)) - \tau. \tag{7}$$

Then, from (7) there exists $x_2 \in Tx_1$ such that

$$F(d(x_1, x_2)) \leq F(d(x_1, x_0)) - \tau.$$

If $x_2 \in Tx_2$ we are finished. Otherwise, by the same way we can find $x_3 \in Tx_2$ such that

$$F(d(x_2, x_3)) \le F(d(x_2, x_1)) - \tau.$$

We continue recursively, then we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$F(d(x_n, x_{n+1})) \le F(d(x_n, x_{n-1})) - \tau \tag{8}$$

If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} \in Tx_{n_0}$, then x_{n_0} is a fixed point of T and so the proof is complete. Thus, suppose that for every $n \in \mathbb{N}$, $x_n \notin Tx_n$. Denote $a_n = d(x_n, x_{n+1})$, for $n = 0, 1, 2, \cdots$. Then $a_n > 0$ for all $n \in \mathbb{N}$ and, using (8), the following holds:

$$F(a_n) \le F(a_{n-1}) - \tau \le F(a_{n-2}) - 2\tau \le \cdots \le F(a_0) - n\tau. \tag{9}$$

From (9), we get $\lim_{n\to\infty} F(a_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} a_n = 0.$$

From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0.$$

By (9), the following holds for all $n \in \mathbb{N}$

$$a_n^k F(a_n) - a_n^k F(a_0) \le -a_n^k n\tau \le 0. \tag{10}$$

Letting $n \to \infty$ in (10), we obtain that

$$\lim_{n\to\infty} n a_n^k = 0. \tag{11}$$

From (11), there exits $n_1 \in \mathbb{N}$ such that $na_n^k \le 1$ for all $n \ge n_1$. So, we have, for all $n \ge n_1$

$$a_n \le \frac{1}{n^{1/k}}. \tag{12}$$

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \ge n_1$.Using the triangular inequality for the metric and from (12), we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= a_n + a_{n+1} + \cdots + a_{m-1} \\ &= \sum_{i=n}^{m-1} a_i \le \sum_{i=n}^{\infty} a_i \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$, passing to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

Now, from (5), for all $x, y \in X$ with $H(Tx, Ty) > 0$, we get

$$H(Tx, Ty) < d(x, y) + \lambda D(y, Tx)$$

and so

$$H(Tx, Ty) \le d(x, y) + \lambda D(y, Tx)$$

for all $x, y \in X$. Then

$$\begin{aligned} D(x_{n+1}, Tz) &\le H(Tx_n, Tz) \\ &\le d(x_n, z) + \lambda D(z, Tx_n) \\ &\le d(x_n, z) + \lambda d(z, x_{n+1}) \end{aligned}$$

Passing to limit $n \to \infty$, we obtain $D(z, Tz) = 0$. Thus, we get $z \in \overline{Tz} = Tz$. Therefore T is a MWP operator. □

3. Final Remarks and Examples

Remark 3.1. *In Theorem 1.7 (or Theorem 2.2), the condition (F4) on F cannot be removed. The following example shows this fact.*

Example 3.2. *Let* $X = [0, 1]$ *and*

$$d(x,y) = \begin{cases} 0 & , \quad x = y \\ 1 + |x - y| & , \quad x \neq y \end{cases},$$

then it is clear that (X, d) *is complete metric space, which is also bounded. Since* τ_d *is discrete topology, all subsets of* X *are closed. Therefore all subsets of* X *are closed and bounded. Define a mapping* $T : X \to CB(X)$ *as:*

$$Tx = \begin{cases} A & , \quad x \in B \\ B & , \quad x \in A \end{cases},$$

where A *is the set of all rational numbers in* X *and* B *is the set of all irrational numbers in* X. *Therefore* T *has no fixed point. Now, define* $F : (0, \infty) \to \mathbb{R}$ *by*

$$F(\alpha) = \begin{cases} \ln \alpha & , \quad \alpha \leq 1 \\ \alpha & , \quad \alpha > 1 \end{cases},$$

then we can see that $F \in \mathcal{F} \backslash \mathcal{F}_*$. *Now we show that*

$$\forall x, y \in X \; [H(Tx, Ty) > 0 \Rightarrow 1 + F(H(Tx, Ty)) \leq F(d(x, y))].$$

Note that $H(Tx, Ty) > 0 \Rightarrow \{x, y\} \cap A$ *is singleton. Therefore we have*

$$\begin{aligned} H(Tx, Ty) > 0 \quad &\Rightarrow \quad H(Tx, Ty) = H(A, B) = 1 < 1 + |x - y| = d(x, y) \\ &\Rightarrow \quad F(H(Tx, Ty)) = 0 < 1 + |x - y| = F(d(x, y)) \\ &\Rightarrow \quad 1 + F(H(Tx, Ty)) \leq F(d(x, y)). \end{aligned}$$

Consequently all conditions of Theorem 1.7 (or in Theorem 2.2) except for (F4) are satisfied, but T *has no fixed point.*

Remark 3.3. *If we take* $T : X \to K(X)$ *in Theorem 2.2, we can remove the condition (F4) on F. Indeed, let* $x_0 \in X$ *and* $x_1 \in Tx_0$. *If* $x_1 \in Tx_1$, *then the proof is complete. Let* $x_1 \notin Tx_1$. *Then, as* Tx_1 *is closed,* $D(x_1, Tx_1) > 0$. *On the other hand, as* $D(x_1, Tx_1) \leq H(Tx_0, Tx_1)$, *from (F1) we have*

$$F(D(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)).$$

From (5), we can write that

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq F(H(Tx_0, Tx_1)) \\ &\leq F(d(x_1, x_0) + \lambda D(x_1, Tx_0)) - \tau \\ &= F(d(x_1, x_0)) - \tau. \end{aligned} \tag{13}$$

Since Tx_1 *is compact, there exists* $x_2 \in Tx_1$ *such that* $d(x_1, x_2) = D(x_1, Tx_1)$. *Then from (13) we have*

$$F(d(x_1, x_2)) \leq F(d(x_1, x_0)) - \tau.$$

The rest of the proof can be completed as in the proof of Theorem 2.2.

Remark 3.4. *If there exist $\delta \in (0,1)$ and $L \geq 0$ satisfying (1), then (5) is satisfied with $F(\alpha) = \ln \alpha$, $\tau = -\ln \delta$ and $\lambda = \frac{L}{\delta}$. Therefore, Theorem 1.2 is a special case of Theorem 2.2.*

Remark 3.5. *If there exist $\tau > 0$ and $F \in \mathcal{F}_*$ satisfying (4), then (5) is satisfied with $\lambda = 0$. Therefore, Theorem 1.7 is a special case of Theorem 2.2.*

Now we give two examples to show that Theorem 2.2 is a real generalization of Theorem 1.2 and Theorem 1.7, respectively.

Example 3.6. *Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $d(x,y) = \left|x - y\right|$. Then (X,d) is a complete metric space. Define a mapping $T : X \to CB(X)$ by:*

$$Tx = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \\ \{x_1, x_2, \cdots, x_{n-1}\} & , \quad x = x_n \end{cases}.$$

Then, as shown in Example 3 of [3], T is multivalued almost F-contraction with respect to $F(\alpha) = \alpha + \ln \alpha$, $\tau = 1$ and $\lambda \geq 0$. Thus, by Theorem 2.2, T is an MWP operator.

On the other hand, since $D(x_1, Tx_n) = 0$ and

$$\lim_{n\to\infty} \frac{H(Tx_n, Tx_1)}{d(x_n, x_1)} = \lim_{n\to\infty} \frac{x_{n-1} - 1}{x_n - 1} = 1,$$

then we can not find $\delta \in (0,1)$ and $L \geq 0$ satisfying (1). Therefore, T is not a multivalued almost contraction. That is, Theorem 1.2 cannot be applied to this example.

Example 3.7. *Let $X = [0,1] \cup \{2,3\}$ and $d(x,y) = \left|x - y\right|$, then (X,d) is complete metric space. Define a mapping $T : X \to CB(X)$,*

$$Tx = \begin{cases} \left[\frac{1-x}{3}, \frac{1-x}{2}\right] & , \quad x \in [0,1] \\ \\ \{x\} & , \quad x \in \{2,3\} \end{cases}.$$

Since $H(T2, T3) = 1 = d(2,3)$, then for all $F \in \mathcal{F}$ and $\tau > 0$ we have

$$\tau + F(H(T2, T3)) > F(d(2,3)).$$

Therefore, T is not a multivalued F-contraction, and so Theorem 1.7 can not be applied to this example.

Now, let us consider the mapping F defined by $F(\alpha) = \ln \alpha$. Then T is multivalued almost F-contraction with $\tau = \ln 2$ and $\lambda = 10$. Note that if $H(Tx, Ty) > 0$, then $x \neq y$, and so

$$\forall x, y \in X[H(Tx, Ty) > 0) \Rightarrow \tau + F(H(Tx, Ty)) \leq F(d(x,y) + \lambda D(y, Tx))]$$

is equivalent to

$$\forall x, y \in X[x \neq y \Rightarrow H(Tx, Ty) \leq e^{-\tau} d(x,y) + \lambda e^{-\tau} D(y, Tx)]$$

and so

$$\forall x, y \in X[x \neq y \Rightarrow H(Tx, Ty) \leq \frac{1}{2} d(x,y) + 5D(y, Tx)]. \tag{14}$$

Now we consider the following cases:

Case 1. Let $x, y \in [0,1]$, then

$$H(Tx, Ty) = \frac{1}{2}\left|x - y\right| = \frac{1}{2} d(x,y).$$

It is clear that (14) is satisfied.

Case 2. Let $x, y \in \{2, 3\}$, *then*

$$H(Tx, Ty) = d(x, y) = D(y, Tx) = \left|x - y\right|.$$

It is clear that (14) is satisfied.

Case 3. Let $x \in [0, 1]$ *and* $y \in \{2, 3\}$, *then*

$$H(Tx, Ty) = \frac{3y + x - 1}{3}, d(x, y) = y - x \text{ and } D(y, Tx) = \frac{2y + x - 1}{2}.$$

Therefore

$$H(Tx, Ty) = \frac{3y + x - 1}{3}$$

and

$$\frac{1}{2}d(x, y) + 5d(y, Tx) = \frac{11y + 4x - 5}{2}.$$

Since $\frac{3y+x-1}{3} \leq \frac{11y+4x-5}{2}$, *(14) is satisfied.*

Case 4. Let $x \in \{2, 3\}$ *and* $y \in [0, 1]$, *then*

$$H(Tx, Ty) = \frac{3x + y - 1}{3}, d(x, y) = x - y \text{ and } D(y, Tx) = x - y.$$

Therefore

$$H(Tx, Ty) = \frac{3x + y - 1}{3}$$

and

$$\frac{1}{2}d(x, y) + 5d(y, Tx) = \frac{11}{2}(x - y).$$

Since $\frac{3x+y-1}{3} \leq \frac{11}{2}(x - y)$, *(14) is satisfied.*

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References

[1] R. P. Agarwal, D. O'Regan, N. S. Papageorgiou, Common fixed point theory for multivalued contractive maps of Reich type in uniform spaces, Appl. Anal. 83 (1) (2004), 37–47.
[2] R. P. Agarwal, D. O'Regan, N. Shahzad, Fixed point theorems for generalized contractive maps of Meir-Keeler type, Math. Nachr. 276 (2004), 3–12.
[3] I. Altun, G. Mınak, H. Dağ, Multivalued F-Contractions On Complete Metric Space, J. Nonlinear Convex Anal., 16 (4) (2015), 659-666.
[4] V. Berinde, Approximating fixed points of weak $\varphi-$contractions using the Picard iteration, Fixed Point Theory (2003), 131–147.
[5] V. Berinde, General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces, Carpathian J. Math. 24 (2) (2008), 10–19.
[6] V. Berinde, Iterative Approximation of Fixed Points, Springer-Verlag, Berlin Heidelberg, 2007.
[7] V. Berinde, On the approximation of fixed points of weak contractive mappings. Carpathian J. Math. 19 (1) (2003), 7–22.
[8] M. Berinde, V. Berinde, On a general class of multivalued weakly Picard mappings, J. Math. Anal. Appl. 326 (2007), 772–782.
[9] V. Berinde, M. Păcurar, The role of the Pompeiu-Hausdorff metric in fixed point theory, Creat. Math. Inform. 22 (2) (2013), 35–42.
[10] Lj. B. Ćirić, Multi-valued nonlinear contraction mappings, Nonlinear Anal. 71 (2009), 2716–2723.

[11] Lj. B. Ćirić, J. S. Ume, Common fixed point theorems for multi-valued nonself mappings, Publ. Math. Debrecen 60 (2002), 359–371.
[12] P. Z. Daffer, H. Kaneko, Fixed points of generalized contractive multivalued mappings, J. Math. Anal. Appl. 192 (1995), 655–666.
[13] W.-S. Du, On coincidence point and fixed point theorems for nonlinear multivalued maps, Topology Appl. 159 (1) (2012), 49–56.
[14] W.-S. Du, Some new results and generalizations in metric fixed point theory, Nonlinear Anal. 73 (5) (2010), 1439–1446.
[15] Y. Feng, S. Liu, Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings, J. Math. Anal. Appl. 317 (2006), 103–112.
[16] V. I. Istrăţescu, Fixed Point Theory an Introduction, Dordrecht D. Reidel Publishing Company 1981.
[17] T. Kamran, Q. Kiran, Fixed point theorems for multi-valued mappings obtained by altering distances, Math. Comput. Model. 54 (2011), 2772–2777.
[18] D. Klim, D. Wardowski, Fixed point theorems for set-valued contractions in complete metric spaces, J. Math. Anal. Appl. 334 (2007), 132–139.
[19] N. Mizoguchi, W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl. 141 (1989), 177–188.
[20] S.B. Nadler, Multi-valued contraction mappings, Pacific J. Math. 30 (1969), 475–488.
[21] A. Petruşel, On Frigon-Granas-type multifunctions, Nonlinear Anal. Forum 7 (2002), 113–121.
[22] M. Păcurar, Remark regarding two classes of almost contractions with unique fixed point, Creat. Math. Inform. 19 (2) (2010), 178–183.
[23] M. Păcurar, Sequences of almost contractions and fixed points, Carpathian J. Math. 24 (2) (2008), 101–109.
[24] S. Reich, Fixed points of contractive functions, Boll. Un. Mat. Ital. 4 (5) (1972), 26–42.
[25] S. Reich, Kannan's fixed point theorem, Boll. Un. Mat. Ital. 4 (4) (1971), 1–11.
[26] S. Reich, Some problems and results in fixed point theory, Topological methods in nonlinear functional analysis (Toronto, Ont., 1982), 179–187, Contemp. Math., 21, Amer. Math. Soc., Providence, RI, 1983.
[27] I. A. Rus, Basic problems of the metric fixed point theory revisited (II), Stud. Univ. Babeş-Bolyai Math. 36 (1991), 81–99.
[28] T. Suzuki, Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's, J. Math. Anal. Appl. 340 (2008), 752–755.
[29] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012, 2012:94, 6 pp.