Stability of a Generalized Quadratic Functional Equation in Intuitionistic Fuzzy 2-normed Space

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Abstract. In this paper, we recall the notion of intuitionistic fuzzy 2-normed space introduced by Mursaleen and Lohani [26] and using the direct method, we investigate the Hyers Ulam-Rassias stability of the following quadratic functional equation

$$f(ax + by) + f(ax - by) - \frac{a}{2}f(x + y) = \frac{a}{2}f(x - y) - (2a^2 - a)f(x) - (2b^2 - a)f(y)$$

in this space.

1. Introduction

In 1940 Ulam [30] proposed the famous Ulam stability problem: When is it true that a function which satisfies some functional equation approximately must be close to one satisfying the equation exactly?. If the answer is affirmative, we would say that the equation is stable. In 1941, Hyers [8] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In 1951, Bourgin [5] was the second author to treat the Ulam stability problem for additive mappings. Subsequently the result of Hyers was generalized by Rassias [28] for linear mapping by considering an unbounded Cauchy difference. The paper of Rassias [28] has provided a lot of influence in the development of what we call the Hyers-Ulam stability or the Hyers-Ulam-Rassias stability of functional equations.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [29] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space.

In 1984, Katrasas [10] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined a fuzzy norm on a linear space from various points of view [2, 24]. In particular, in 2003, Bag and Samanta [3], following Cheng and Mordeson [4], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek

type [12]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces. Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several various fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [1, 6, 11, 23, 31]

Quite recently, the stability results in the setting of intuitionistic fuzzy normed space were studied in [7, 18–20, 22]respectively, while the idea of intuitionistic fuzzy normed space was introduced in [25].

2. Preliminary Estimates

Definition 2.1. *Let X be a real linear space of dimension greater than one and let $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following condition:*

    1. $\|x, y\|=\|y, x\|$ for all $x, y \in X$
    2. $\|x, y\| = 0$ if and only if x, y are linearly dependent.
    3. $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathcal{R}$.
    4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$
    Then the function $\|.,.\|$ is called a 2-norm on X and pair $(X, \|.,.\|)$ is called a 2-normed linear space.

Definition 2.2. *A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:*

    1. $*$ is commutative and associative;
    2. $*$ is continuous;
    3 $a * 1 = a$ for all $a \in [0,1]$;
    4 $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Example 2.3. *The example of continuous $t-$norm is*

$$a * b = \min\{a, b\}$$

Definition 2.4. *A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-conorm if $\diamond$ satisfies the following conditions:*

    1. $\diamond$ is commutative and associative;
    2. $\diamond$ is continuous;
    3 $a \diamond 0 = a$ for all $a \in [0,1]$;
    4 $a \diamond b \leq c \diamond d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Example 2.5. *The example of continuous $t-$conorm is*

$$a \diamond b = \max\{a, b\}$$

Definition 2.6. *Let X be a linear space over the filed $\mathcal{F}$. A fuzzy subset μ of $X \times X \times \mathcal{R}$ is called a fuzzy 2-norm on X if and only if for $x, y, z \in X$, $t, s \in \mathcal{R}$ and $c \in \mathcal{F}$:*

    1. $\mu(x, y, t) = 0$ if $t \leq 0$.
    2. $\mu(x, y, t) = 1$ if and only if x, y are linearly dependent, for all $t > 0$.
    3. $\mu(x, y, t)$ is invariant under any permutation of x, y.
    4. $\mu(x, cy, t) = \mu(x, y, \frac{t}{|c|})$ for all $t > 0$ and $c \neq 0$.
    5. $\mu(x + z, y, t + s) \geq \mu(x, y, t) * \mu(z, y, s)$ for all $t, s > 0$.
    6. $\mu(x, y, .)$ is a non-decreasing function on $\mathcal{R}$ and

$$\lim_{t \to \infty} \mu(x, y, t) = 1$$

Then μ is said to be a fuzzy 2-norm on a linear space X, and the pair (X, μ) is called a fuzzy 2-normed linear space.

Example 2.7. *Let* $(X, \|.,.\|)$ *be a 2-normed linear space. Define*

$$\mu(x, y, t) = \begin{cases} \frac{t}{t+\|x,y\|} & \text{if } t > 0 \\ \\ 0 & \text{if } t \leq 0 \end{cases}$$

where $x, y \in X$ *and* $t \in \mathcal{R}$. *Then* (X, μ) *is a fuzzy 2-normed linear space.*

Definition 2.8. *Let* (X, μ) *be a fuzzy 2-normed linear space. Let* $\{x_n\}$ *be a sequence in* X *then* $\{x_n\}$ *is said to be convergent if there exists* $x \in X$ *such that*

$$\lim_{n \to \infty} \mu(x_n - x, y, t) = 1$$

for all $t > 0$.

Definition 2.9. *Let* (X, μ) *be a fuzzy 2-normed linear space. Let* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *is said to be Cauchy sequence if*

$$\lim_{n \to \infty} \mu(x_{n+p} - x_n, y, t) = 1$$

for all $t > 0$ *and* $p = 1, 2, 3, \dots$.

Let (X, μ) be a fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in X. If $\{x_n\}$ be a convergent in X then (X, μ) is said to be fuzzy 2-Banach space.

Definition 2.10. *Let* X *be a linear space over a real field* $\mathcal{F}$. *A fuzzy subset* ν *of* $X \times X \times \mathcal{R}$ *such that for all* $x, y, z \in X$, $t, s \in \mathcal{R}$ *and* $c \in \mathcal{F}$

1. $\nu(x, y, t) = 1$, *for all* $t \leq 0$.
2. $\nu(x, y, t) = 0$ *if and only if* x, y *are linearly dependent, for all* $t > 0$.
3. $\nu(x, y, t)$ *is invariant under any permutation of* x, y.
4. $\nu(x, cy, t) = \nu(x, y, \frac{t}{|c|})$ *for all* $t > 0$, $c \neq 0$.
5. $\nu(x, y + z, t + s) \leq \nu(x, y, t) \diamond \nu(x, z, s)$ *for all* $s, t > 0$
6. $\nu(x, y, .)$ *is a non-increasing function and*

$$\lim_{t \to \infty} \nu(x, y, t) = 0$$

Then ν *is said to be an anti fuzzy 2-norm on a linear space* X *and the pair* (X, ν) *is called a anti fuzzy 2-normed linear space.*

Definition 2.11. *Let* (X, ν) *be an anti fuzzy 2-normed linear space and* $\{x_n\}$ *be a sequence in X, then* $\{x_n\}$ *is said to be convergent if there exists* $x \in X$ *such that*

$$\lim_{n \to \infty} \nu(x_n - x, y, t) = 0$$

for all $t > 0$.

Definition 2.12. *Let* (X, ν) *be an anti fuzzy 2-normed linear space and* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *is said to be a Cauchy sequence if*

$$\lim_{n \to \infty} \nu(x_{n+p} - x_n, y, t) = 0$$

for all $t > 0$ *and* $p = 1, 2, 3, \dots$.

Let (X, ν) be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in X. If $\{x_n\}$ is convergent in X then (X, ν) is said to be anti fuzzy 2-Banach space.

Lemma 2.13. *Consider the set L^* and operation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1 , \; x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 2.14. *A continuous t-norm τ on $L = [0, 1]^2$ is said to be continuous t-representable if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L$*

$$\tau(x, y) = (x_1 * y_1, x_2 \diamond y_2)$$

The notion of intuitionistic fuzzy 2-normed space was introduced by by Mursaleen and Lohani [26]. We present here its slight modification as follows:

Definition 2.15. *The 3-tuple $(X, \rho_{\mu, \nu}, \tau)$ is said to be an intuitionistic fuzzy 2-normed space(for short IF2NS) if X is a linear space over the field F. μ and ν is a fuzzy 2-norm and anti fuzzy 2-norm respectively, such that $\nu(x, y, t) + \mu(x, y, t) \leq 1$. τ is a continuous t-representable, and*

$$\rho_{\mu, \nu} : X \times X \times \mathcal{R} \to L^*$$

$$\rho_{\mu, \nu}(x, y, t) = (\mu(x, y, t), \nu(x, y, t))$$

be a function satisfies on the following condition, for all $x, y, z \in X$, $t, s \in \mathcal{R}$ and $\alpha \in \mathcal{F}$

(1) $\rho_{\mu, \nu}(x, y, t) = (0, 1) = 0_{L^*}$ *for all $t \leq 0$.*

(2) $\rho_{\mu, \nu}(x, y, t) = (1, 0) = 1_{L^*}$ *if and only if x, y are linearly dependent, for all $t > 0$.*

(3) $\rho_{\mu, \nu}(\alpha x, y, t) = \rho_{\mu, nu}(x, y, \frac{t}{|\alpha|})$, *for all $t > 0$ and $\alpha \neq 0$*

(4) $\rho_{\mu, \nu}(x, y, t)$ *is invariant under any permutation of x, y.*

(5) $\rho_{\mu, \nu}(x + z, y, t + s) \geq_{L^*} \tau(\rho_{\mu, \nu}(x, y, t), \rho_{\mu, \nu}(z, y, s))$ *for all $t, s > 0$.*

(6) $\rho_{\mu, \nu}(x, y, .)$ *is a continuous and*

$$\lim_{t \to 0} \rho_{\mu, \nu}(x, y, t) = 0_{L^*} \quad and \quad \lim_{t \to \infty} \rho_{\mu, \nu}(x, y, t) = 1_{L^*}$$

Then $\rho_{\mu, \nu}$ is said to be an intuitionstic fuzzy 2-norm on a linear space X.

Example 2.16. *Let $(X, \|., .\|)$ be a 2-normed space,*

$$\tau(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$$

be a continuous t-representable for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be a fuzzy and anti fuzzy 2-norm respectively. We define

$$\rho_{\mu, \nu}(x, y, t) = \left(\frac{t}{t + m\|x, y\|}, \frac{\|x, y\|}{t + m\|x, y\|} \right)$$

for all $t \in \mathcal{R}^+$ in which $m > 1$. Then $(X, \rho_{\mu, \nu}, \tau)$ is an IF2NS.

Definition 2.17. *A sequence $\{x_n\}$ in an IF2NS $(X, \rho_{\mu, \nu}, \tau)$ is said to be convergent to a point $x \in X$, if*

$$\lim_{n \to \infty} \rho_{\mu, \nu}(x_n - x, y, t) = 1_{L^*}$$

for every $t > 0$.

Definition 2.18. *A sequence $\{x_n\}$ in an IF2NS $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be Cauchy sequence if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathcal{N}$ such that*

$$\rho_{\mu,\nu}(x_n - x_m, y, t) \geq_{L^*} (1 - \epsilon, \epsilon)$$

for all $n, m \geq n_0$.

Definition 2.19. *An IF2N space $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be complete if every Cauchy sequence in $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is convergent. A complete intuitionistic fuzzy 2-normed space is called an intuitionistic fuzzy 2-Banach space.*

3. Main Results

We investigate the Hyers Ulam-Rassias stability of the quadratic functional equation

$$f(ax + by) + f(ax - by) - \frac{a}{2}f(x + y) = \frac{a}{2}f(x - y) - (2a^2 - a)f(x) - (2b^2 - a)f(y)$$

in intuitionistic fuzzy 2-normed space.

For related problems, we refer to [7]–[25].

Definition 3.1. *Let $\mathcal{X}, \mathcal{Y}$ be linear spaces. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$, we define*

$$Df(x, y) := f(ax + by) + f(ax - by) - \frac{a}{2}f(x + y) - \frac{a}{2}f(x - y) - (2a^2 - a)f(x) - (2b^2 - a)f(y)$$

where $a, b \geq 1$, $a \neq 2b^2$ and $x, y \in \mathcal{X}$.

Theorem 3.2. *Let $\mathcal{X}$ be a linear space over the filed $\mathcal{F}$ and $(\mathcal{Y}, \rho'_{\mu,\nu}, \tau')$ be an intuitionistic fuzzy 2-Banach space. Let $(\mathcal{Z}, \rho_{\mu,\nu}, \tau)$ be an intuitionistic fuzzy 2-normed space and the mappings $\phi, \varphi : \mathcal{X} \to \mathcal{Z}$ be such that*

$$\rho_{\mu,\nu}(\phi(ax), \varphi(ax), t) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{1}$$

for all $x \in X$, $t > 0$. Let $\xi : \mathcal{X} \to \mathcal{Y}$ be a function such that $\xi(ax) = \frac{1}{\alpha}\xi(x)$, for all $x \in X$ and $a, \alpha \in \mathcal{F}$, such that $0 < \alpha < a^2$, and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping such that

$$\rho'_{\mu,\nu}(Df(x, y), \xi(x), t + s) \geq_{L^*} \tau\{\rho_{\mu,\nu}(\phi(x), \varphi(x), t^q), \rho_{\mu,\nu}(\phi(y), \varphi(y), s^q)\} \tag{2}$$

for all $t, s > 0, q > \frac{1}{2}, x, y, z \in \mathcal{X}$. Then there exists a unique quadratic functional equation $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho'_{\mu,\nu}(Q(x) - f(x), \xi(x), t) \geq_{L^*} \rho_{\mu,\nu}\left(\phi(x), \varphi(x), \left(\frac{(a^2 - \alpha)t}{2}\right)^q\right) \tag{3}$$

for all $x \in X$ and $t > 0$.

Proof. Putting $y = 0$ and $t = s$ in (2), we have

$$\rho'_{\mu,\nu}(f(ax) - a^2 f(x), \xi(x), t) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t^q) \tag{4}$$

for all $x \in \mathcal{X}, t > 0$. Therefore

$$\rho'_{\mu,\nu}\left(\frac{1}{a^2}f(ax) - f(x), \xi(x), \frac{t}{a^2}\right) \geq_{L^*} \rho_{\mu,\nu}\left(\phi(x), \varphi(x), t^q\right) \tag{5}$$

and

$$\rho'_{\mu,\nu}\left(\frac{1}{a^2}f(ax) - f(x), \xi(x), \frac{t^p}{a^2}\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{6}$$

for all $x \in X$, $t > 0$ and $p = \frac{1}{q}$. Now, replacing x by ax in (6), then by assumption (1) we obtain that

$$\rho'_{\mu,\nu}\left(\frac{1}{a^4}f(a^2x) - \frac{1}{a^2}f(ax), \xi(ax), \frac{t^p}{a^4}\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(ax), \varphi(ax), t)$$
$$\geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{7}$$

for all $x \in X$ and $t > 0$. By comparing (6) and (7) and using property (4) of IF2NS, we obtain that

$$\rho'_{\mu,\nu}\left(\frac{1}{a^4}f(a^2x) - f(x), \xi(x), \frac{t^p}{a^2} + \frac{\alpha t^p}{a^4}\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{8}$$

for all $x \in X$ and $t > 0$. Again by replacing $x = ax$ in (8) we have

$$\rho'_{\mu,\nu}\left(\frac{1}{a^4}f(a^3x) - f(ax), \xi(ax), \frac{t^p}{a^2} + \frac{\alpha t^p}{a^4}\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(ax), \varphi(ax), t) \tag{9}$$

for all $x \in X$ and $t > 0$. Thus

$$\rho'_{\mu,\nu}\left(\frac{1}{a^6}f(a^3x) - \frac{1}{a^2}f(ax), \xi(x), \frac{\alpha t^p}{a^4} + \frac{\alpha^2 t^p}{a^6}\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{10}$$

for all $x \in X$ and $t > 0$. So by comparing (6) and (10) we obtain that

$$\rho'_{\mu,\nu}\left(\frac{1}{a^6}f(a^3x) - f(x), \xi(x), \frac{t^p}{a^2} + \frac{\alpha t^p}{a^4} + \frac{\alpha^2 t^p}{a^6}\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{11}$$

for all $x \in X$ and $t > 0$. Following this process, we obtain that

$$\rho'_{\mu,\nu}\left(\frac{1}{a^{2n}}f(a^nx) - f(x), \xi(x), \sum_{k=1}^{n}\alpha^{k-1}a^{-2k}t^p\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{12}$$

for all $x \in X$ and $t > 0$, $n \in \mathcal{N}$. If $m \in \mathcal{N}$, $n > m$, then $n - m \in \mathcal{N}$. Replacing n by $n - m$ in (12), gives

$$\rho'_{\mu,\nu}\left(a^{2m-2n}f(a^{n-m}x) - f(x), \xi(x), \sum_{k=1}^{n-m}\alpha^{k-1}a^{-2k}t^p\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{13}$$

for all $x \in X$ and $t > 0$. By replacing $x = a^m$ in (13), we obtain that

$$\rho'_{\mu,\nu}\left(\frac{f(a^nx)}{a^{2n}} - \frac{f(a^mx)}{a^{2m}}, \xi(a^mx), a^{-2m}\sum_{k=1}^{n-m}\alpha^{k-1}a^{-2k}t^p\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(a^mx), \varphi(a^mx), t)$$
$$\geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t)$$

for all $x \in X$ and $t > 0$. Thus

$$\rho'_{\mu,\nu}\left(\frac{f(a^nx)}{a^{2n}} - \frac{f(a^mx)}{a^{2m}}, \xi(x), a^{-2m}\sum_{k=m+1}^{n}\alpha^{k-1}a^{-2k}t^p\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{14}$$

for all $x \in X$ and $t > 0$. Since $\sum_{k=1}^{\infty}\alpha^{k-1}a^{-2k}$ is convergent, $\left\{\frac{f(a^nx)}{a^{2n}}\right\}$ is a Cauchy sequence in intuitionistic fuzzy 2-Banach space $(\mathcal{Y}, \rho'_{\mu,\nu}, \tau')$. Thus this sequence converges to some $Q(x) \in \mathcal{Y}$. It means

$$Q(x) = \rho'_{\mu,\nu} - \lim_{n \to \infty}\frac{f(a^nx)}{a^{2n}} \tag{15}$$

for all $x \in X$. Furthermore, by putting $m = 0$ in (14) we have

$$\rho'_{\mu,\nu}\left(\frac{f(a^n x)}{a^{2n}} - f(x), \xi(x), \sum_{k=1}^{n} \alpha^{k-1} a^{-2k} t^p\right) \geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), t) \tag{16}$$

for all $x \in X$. So

$$\rho'_{\mu,\nu}\left(\frac{f(a^n x)}{a^{2n}} - f(x), \xi(x), t\right) \geq_{L^*} \rho_{\mu,\nu}\left(\phi(x), \varphi(x), \frac{t^q}{(\sum_{k=1}^{n} \alpha^{k-1} a^{-2k})^q}\right)$$

for all $x \in X$ and $t > 0$. Next we will show that Q is a quadratic function. Let $x, y \in X$. Then we have

$$\rho'_{\mu,\nu}(DQ(x,y), \xi(x), t) \geq_{L^*} \tau\{\rho'_{\mu,\nu}\left(Q(ax+by) - \frac{f(a^n(ax+by))}{a^{2n}}, \xi(x), \frac{t}{7}\right)$$

$$, \rho'_{\mu,\nu}\left(Q(ax-by) - \frac{f(a^n(ax-by))}{a^{2n}}, \xi(x), \frac{t}{7}\right)$$

$$, \rho'_{\mu,\nu}\left(\frac{a}{2}\frac{f(a^n(x+y))}{a^{2n}} - \frac{a}{2}Q(x+y), \xi(x), \frac{t}{7}\right)$$

$$, \rho'_{\mu,\nu}\left(\frac{a}{2}\frac{f(a^n(x-y))}{a^{2n}} - \frac{a}{2}Q(x-y), \xi(x), \frac{t}{7}\right)$$

$$, \rho'_{\mu,\nu}\left((2a^2-a)\frac{f(a^n x)}{a^{2n}} - (2a^2-a)Q(x), \xi(x), \frac{t}{7}\right)$$

$$, \rho'_{\mu,\nu}\left((2b^2-a)\frac{f(a^n y)}{a^{2n}} - (2b^2-a)Q(y), \xi(x), \frac{t}{7}\right)$$

$$, \rho'_{\mu,\nu}\left(\frac{f(a^n(ax+by))}{a^{2n}} + \frac{f(a^n(ax-by))}{a^{2n}} - \frac{a}{2}\frac{f(a^n(x+y))}{a^{2n}}\right.$$

$$\left(-\frac{a}{2}\frac{f(a^n(x-y))}{a^{2n}} - (2b^2-a)\frac{f(a^n x)}{a^{2n}}a^{2n} - (2b^2-a)\frac{f(a^n x)}{a^{2n}}a^{2n}, \xi(x), \frac{t}{7}\right)\}$$

The first six terms on the right hand side of the above inequality tend to 1_{L^*} as $n \to \infty$ and the seventh term, by (2), is greater than or equal to

$$\tau\left\{\rho_{\mu,\nu}\left(\phi(x), \varphi(x), \frac{a^{2n}}{\alpha^n}\frac{t^q}{14^q}\right), \rho_{\mu,\nu}\left(\phi(y), \varphi(y), \frac{a^{2n}}{\alpha^n}\frac{t^q}{14^q}\right)\right\} \tag{17}$$

which tends to 1_{L^*} as $n \to \infty$. Therefore

$$\rho'_{\mu,\nu}(Q(ax+by) + Q(ax-by) - \frac{a}{2}Q(x+y) - \frac{a}{2}Q(x-y)$$

$$-(2a^2-a)Q(x) - (2b^2-a)Q(y), \xi(x), t) = 1_{L^*}$$

for each $x, y \in X$ and $t > 0$. So by property ρ_2, we have

$$Q(ax+by) + Q(ax-by) - \frac{a}{2}Q(x+y) - \frac{a}{2}Q(x-y) - (2a^2-a)Q(x) - (2b^2-a)Q(y) = 0$$

Therefore Q is a quadratic function, for every $x \in X$ and $t, s > 0$. By (16), for large enough n, we have

$$\rho'_{\mu,\nu}(Q(x) - f(x), \xi(x), t) \geq_{L^*} \tau\{\rho'_{\mu,\nu}\left(Q(x) - \frac{f(a^n x)}{a^{2n}}, \xi(x), \frac{t}{2}\right)$$

$$, \rho'_{\mu,\nu}\left(\frac{f(a^n x)}{a^{2n}} - f(x), \xi(x), \frac{t}{2}\right)\}$$

$$\geq_{L^*} \rho_{\mu,\nu}\left(\phi(x), \varphi(x), \frac{(\frac{t}{2})^q}{(\sum_{k=1}^{n} \alpha^{k-1}.a^{-2k})^q}\right)$$

$$= \rho_{\mu,\nu}\left(\phi(x), \varphi(x), \left(\frac{(a^2 - \alpha)t}{2}\right)^q\right)$$

Let Q' be another quadratic function from X to $\mathcal{Y}$ which satisfies in above inequality, since for each $n \in \mathbb{N}$

$$Q(a^n x) = a^{2n} Q(x) \quad , \quad Q'(a^n x) = a^{2n} Q'(x) \tag{18}$$

So we have

$$\rho'_{\mu,\nu}(Q(x) - Q'(x), \xi(x), t) = \rho'_{\mu,\nu}(Q(a^n x) - Q'(a^n x), \xi(x), a^{2n} t) \tag{19}$$

$$\geq_{L^*} \tau\{\rho'_{\mu,\nu}(Q(a^n x) - f(a^n x), \xi(x), \frac{a^{2n} t}{2})$$

$$\{\rho'_{\mu,\nu}(f(a^n x) - Q'(a^n x), \xi(x), \frac{a^{2n} t}{2})\}$$

$$\geq_{L^*} \rho_{\mu,\nu}(\phi(x), \varphi(x), (\frac{a^2 - \alpha}{2})^q . \frac{a^{2nq}}{\alpha^n} . (\frac{t}{2})^q)$$

as $n \to \infty$, since $0 < \alpha < a^2$ so we have

$$\rho'_{\mu,\nu}(Q(x) - Q'(x), \xi(x), t) \geq_{L^*} 1_{L^*} \tag{20}$$

Then $Q(x) = Q'(x)$ for all $x \in X$. This complete the proof.
$\square$

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