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η-Ricci Solitons on Lorentzian Para-Sasakian Manifolds

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Abstract. We consider η-Ricci solitons on Lorentzian para-Sasakian manifolds satisfying certain curvature conditions: $R(\xi, X) \cdot S = 0$ and $S \cdot R(\xi, X) = 0$. We prove that on a Lorentzian para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, if the Ricci curvature satisfies one of the previous conditions, the existence of η-Ricci solitons implies that (M, g) is Einstein manifold. We also conclude that in these cases there is no Ricci soliton on M with the potential vector field ξ. On the other way, if M is of constant curvature, then (M, g) is elliptic manifold. Cases when the Ricci tensor satisfies different other conditions are also discussed.

1. Introduction

In the last years, the interest in studying Ricci solitons has considerably increased. Ricci solitons were introduced by R. S. Hamilton as natural generalizations of Einstein metrics [12] and have been studied in many contexts: on Kähler manifolds [8], on contact and Lorentzian manifolds [1], [6], [14], [20], [21], on Sasakian [10], [13], α-Sasakian [14], trans-Sasakian [22] and K-contact manifolds [20], on Kenmotsu [2], [18] and f-Kenmotsu manifolds [6] etc. In paracontact geometry, Ricci solitons firstly appeared in the paper of G. Calvaruso and D. Perrone [4]. Recently, C. L. Bejan and M. Crasmareanu dealed with Ricci solitons on 3-dimensional normal paracontact manifolds [3].

A more general notion is that of *η-Ricci soliton* introduced by J. T. Cho and M. Kimura [7], which was treated by C. Călin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [5].

In the present paper we consider η-Ricci solitons on Lorentzian para-Sasakian manifolds which satisfy certain curvature properties, in particular, $R(\xi, X) \cdot S = 0$ and $S \cdot R(\xi, X) = 0$, respectively. Remark that in [18], H. G. Nagaraja and C. R. Premalatha have obtained some results on Ricci solitons satisfying conditions of the following type: $R(\xi, X) \cdot \tilde{C} = 0$, $P(\xi, X) \cdot \tilde{C} = 0$, $H(\xi, X) \cdot S = 0$, $\tilde{C}(\xi, X) \cdot S = 0$ and in [2], C. S. Bagewadi, G. Ingalahalli and S. R. Ashoka treated the cases: $R(\xi, X) \cdot B = 0$, $B(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot \bar{P} = 0$ and $\bar{P}(\xi, X) \cdot S = 0$. We also prove that a Lorentzian para-Sasakian manifold of constant curvature supporting an η-Ricci soliton is locally isometric to a sphere.

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2. Lorentzian Para-Sasakian Manifolds

Let M be an n-dimensional smooth manifold, φ a tensor field of $(1,1)$-type, ξ a vector field, η a 1-form and g a Lorentzian metric on M.

Definition 2.1. *[15] We say that (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on M if:*

1. $\varphi\xi = 0,\ \eta \circ \varphi = 0,$
2. $\eta(\xi) = -1,\ \varphi^2 = I + \eta \otimes \xi,$
3. $g(\varphi\cdot, \varphi\cdot) = g + \eta \otimes \eta,$
4. $(\nabla_X \varphi)Y = g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X$, *for any* $X, Y \in \mathfrak{X}(M)$, *where* ∇ *is the Levi-Civita connection associated to* g.

From the definition it follows that η is the g-dual of ξ, i.e.:

$$\eta(X) = g(X, \xi), \tag{1}$$

for any $X \in \mathfrak{X}(M)$, ξ satisfies:

$$g(\xi, \xi) = -1 \tag{2}$$

and φ is a g-symmetric operator, i.e.:

$$g(\varphi X, Y) = g(X, \varphi Y), \tag{3}$$

for any $X, Y \in \mathfrak{X}(M)$.

Properties of this structure are given in the next proposition.

Proposition 2.2. *On a Lorentzian para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, for any $X, Y, Z \in \mathfrak{X}(M)$, the following relations hold:*

$$\nabla_X \xi = \varphi X \tag{4}$$

$$\eta(\nabla_X \xi) = 0,\ \ \nabla_\xi \xi = 0, \tag{5}$$

$$R(X, Y)\xi = -\eta(X)Y + \eta(Y)X, \tag{6}$$

$$\eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z),\ \ \eta(R(X, Y)\xi) = 0, \tag{7}$$

$$(\nabla_X \eta)Y = (\nabla_Y \eta)X = g(\varphi X, Y),\ \ \nabla_\xi \eta = 0, \tag{8}$$

$$\mathcal{L}_\xi \varphi = 0,\ \ \mathcal{L}_\xi \eta = 0,\ \ \mathcal{L}_\xi g = 2g(\varphi\cdot, \cdot), \tag{9}$$

where R is the Riemann curvature tensor field and ∇ is the Levi-Civita connection associated to g.

Proof. The proof of the relations (4) – (8) can be found in [19].

Express now the Lie derivatives along ξ as follows:

$$(\mathcal{L}_\xi \varphi)(X) := [\xi, \varphi X] - \varphi([\xi, X]) = \nabla_\xi \varphi X - \nabla_{\varphi X} \xi - \varphi(\nabla_\xi X) + \varphi(\nabla_X \xi) = \nabla_\xi \varphi X - \varphi(\nabla_\xi X) := (\nabla_\xi \varphi)X = 0,$$

$$(\mathcal{L}_\xi \eta)(X) := \xi(\eta(X)) - \eta([\xi, X]) = \xi(g(X, \xi)) - g(\nabla_\xi X, \xi) + g(\nabla_X \xi, \xi) = g(X, \nabla_\xi \xi) + \eta(\nabla_X \xi) = 0$$

and

$$(\mathcal{L}_\xi g)(X, Y) := \xi(g(X, Y)) - g([\xi, X], Y) - g(X, [\xi, Y]) = \xi(g(X, Y)) - g(\nabla_\xi X, Y) + g(\nabla_X \xi, Y) - g(X, \nabla_\xi Y) + g(X, \nabla_Y \xi) =$$

$$= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2g(\varphi X, Y).$$

□

From this proposition we get that the $(0,2)$-tensor field:

$$\omega(X,Y) := g(X,\varphi Y), \tag{10}$$

is symmetric and satisfies:

$$\omega(\varphi X,Y) = \omega(X,\varphi Y), \ \ \omega(\varphi X,\varphi Y) = \omega(X,Y)$$

$$(\nabla_X\omega)(Y,Z) = \eta(Y)g(X,Z) + \eta(Z)g(X,Y) + 2\eta(X)\eta(Y)\eta(Z),$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Remark 2.3. *On a Lorentzian para-Sasakian manifold (M,φ,ξ,η,g) we deduce that:*
(i) The 1-form η is closed. Indeed, from $\nabla_X\xi = \varphi X$ we obtain:

$$(d\eta)(X,Y) := X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]) = X(g(Y,\xi)) - Y(g(X,\xi)) - g([X,Y],\xi) =$$

$$= X(g(Y,\xi)) - g(\nabla_XY,\xi) - Y(g(X,\xi)) + g(\nabla_YX,\xi) = g(Y,\nabla_X\xi) - g(X,\nabla_Y\xi) =$$

$$= g(Y,\varphi X) - g(X,\varphi Y) = 0.$$

(ii) The Nijenhuis tensor field associated to φ,

$$N_\varphi(X,Y) := \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y],$$

equals to:

$$N_\varphi(X,Y) = -\varphi[(\nabla_X\varphi)Y - (\nabla_Y\varphi)X] + (\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X =$$

$$= -\varphi[\eta(Y)X - \eta(X)Y] + \eta(Y)\varphi X - \eta(X)\varphi Y = 0.$$

Therefore, the structure is normal.

Example 2.4. *[19] Let $M = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Consider the linearly independent system of vector fields*

$$E_1 := e^z\frac{\partial}{\partial y}, \ \ E_2 := e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), \ \ E_3 := \frac{\partial}{\partial z}.$$

Define the Lorentzian metric g by:

$$g(E_1,E_1) = g(E_2,E_2) = -g(E_3,E_3) = 1,$$

$$g(E_1,E_2) = g(E_2,E_3) = g(E_3,E_1) = 0$$

the vector field ξ and the 1-form η by:

$$\xi := E_3, \ \ \eta(X) := g(X,E_3),$$

for any $X \in \mathfrak{X}(M)$, and the $(1,1)$-tensor field φ by:

$$\varphi E_1 = -E_1, \ \ \varphi E_2 = -E_2, \ \ \varphi E_3 = 0.$$

Using Koszul's formula for the Lorentzian metric g we obtain:

$$\nabla_{E_1}E_1 = -E_3, \ \ \nabla_{E_1}E_2 = 0, \ \ \nabla_{E_1}E_3 = -E_1, \ \ \nabla_{E_2}E_1 = 0, \ \ \nabla_{E_2}E_2 = -E_3,$$

$$\nabla_{E_2}E_3 = -E_2, \ \ \nabla_{E_3}E_1 = 0, \ \ \nabla_{E_3}E_2 = 0, \ \ \nabla_{E_3}E_3 = 0.$$

In this case, (φ,ξ,η,g) is a Lorentzian para-Sasakian structure on M.

3. Ricci and η-Ricci Solitons on $(M, \varphi, \xi, \eta, g)$

Let $(M, \varphi, \xi, \eta, g)$ be a Lorentzian para-Sasakian manifold. Consider the equation:

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{11}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor field of the metric g, and λ and μ are real constants. Writing $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$2S(X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X, Y) - 2\mu\eta(X)\eta(Y), \tag{12}$$

for any $X, Y \in \mathfrak{X}(M)$, or equivalent:

$$S(X, Y) = -g(\varphi X, Y) - \lambda g(X, Y) - \mu\eta(X)\eta(Y), \tag{13}$$

for any $X, Y \in \mathfrak{X}(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (11) is said to be an *η-Ricci soliton* on M [7]; in particular, if $\mu = 0$, (g, ξ, λ) is a *Ricci soliton* [11] and it is called *shrinking*, *steady* or *expanding* according as λ is negative, zero or positive, respectively [9].

In [16] and [17] the authors proved that on a Lorentzian para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$, the Ricci tensor field satisfies:

$$S(X, \xi) = (\dim(M) - 1)\eta(X), \tag{14}$$

$$S(\varphi X, \varphi Y) = S(X, Y) + (\dim(M) - 1)\eta(X)\eta(Y), \tag{15}$$

for any $X, Y \in \mathfrak{X}(M)$. From (13) and (14) we obtain:

$$\mu - \lambda = n - 1. \tag{16}$$

Example 3.1. *On the Lorentzian para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ considered in Example 2.4, the data (g, ξ, λ, μ) for $\lambda = -1$ and $\mu = 1$ defines an η-Ricci soliton. Indeed, the Riemann and the Ricci curvature tensor fields are given by:*

$$R(E_1, E_2)E_2 = E_1, \quad R(E_1, E_3)E_3 = -E_1, \quad R(E_2, E_1)E_1 = E_2,$$
$$R(E_2, E_3)E_3 = -E_2, \quad R(E_3, E_1)E_1 = E_3, \quad R(E_3, E_2)E_2 = E_3,$$
$$S(E_1, E_1) = S(E_2, E_2) = 2, \quad S(E_3, E_3) = -2.$$

From (13) we obtain $S(E_1, E_1) = 1 - \lambda$ and $S(E_3, E_3) = \lambda - \mu$, therefore $\lambda = -1$ and $\mu = 1$.

The next theorems formulate results in the cases when the Lorentzian para-Sasakian manifold is of constant curvature, has cyclic Ricci tensor (in particular, if the manifold is Ricci symmetric) or cyclic η-recurrent Ricci tensor.

Remark that if (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the manifold M and (M, g) is of constant curvature, then M is elliptic manifold. Indeed, suppose that $R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]$, for any $X, Y, Z \in \mathfrak{X}(M)$. Applying η to this relation and using the Proposition 2.2 we obtain $k = 1$.

Theorem 3.2. *Let (φ, ξ, η, g) be a Lorentzian para-Sasakian structure on the manifold M and let (g, ξ, λ, μ) be an η-Ricci soliton on M.*

1. *If the manifold (M, g) has cyclic Ricci tensor (i.e. $(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$, for any $X, Y, Z \in \mathfrak{X}(M)$), then $\mu = -1$ and $\lambda = -n$.*
2. *If the manifold (M, g) has cyclic η-recurrent Ricci tensor (i.e. $(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = \eta(X)S(Y, Z) + \eta(Y)S(Z, X) + \eta(Z)S(X, Y)$, for any $X, Y, Z \in \mathfrak{X}(M)$), then $\mu = -\frac{n+1}{2}$ and $\lambda = -\frac{3n-1}{2}$.*

Proof. 1. Replacing the expression of S from (13) in $(\nabla_X S)(Y,Z) := X(S(Y,Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z)$ we obtain:

$$(\nabla_X S)(Y,Z) = -g((\nabla_X \varphi)Y, Z) - \mu[\eta(Y)(\nabla_X \eta)Z + \eta(Z)(\nabla_X \eta)Y]$$

and replacing $\nabla\varphi$ and $\nabla\eta$ we get:

$$(\nabla_X S)(Y,Z) = -\eta(Y)[g(X,Z) + \mu g(\varphi X, Z)] - \eta(Z)[g(X,Y) + \mu g(\varphi X, Y)] - 2\eta(X)\eta(Y)\eta(Z). \tag{17}$$

Then:

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) =$$

$$= -2\{\eta(X)[g(Y,Z) + \mu g(\varphi Y, Z)] + \eta(Y)[g(Z,X) + \mu g(\varphi Z, X)] + \eta(Z)[g(X,Y) + \mu g(\varphi X, Y)] + 3\eta(X)\eta(Y)\eta(Z)\}.$$

For $Z := \xi$ we obtain:

$$\mu g(\varphi X, Y) + g(\varphi X, \varphi Y) = 0,$$

for any $X, Y \in \mathfrak{X}(M)$ and for $Y \mapsto \varphi Y$ we get:

$$\mu g(\varphi X, \varphi Y) + g(\varphi X, Y) = 0,$$

for any $X, Y \in \mathfrak{X}(M)$. Adding the previous two relations we have:

$$(1 + \mu)[g(\varphi X, Y) + g(\varphi X, \varphi Y)] = 0,$$

for any $X, Y \in \mathfrak{X}(M)$ and follows $\mu = -1$. From the relation (16) we get $\lambda = -n$.

2. After a computation we get:

$$(1 + \lambda)[\eta(X)g(Y,Z) + \eta(Y)g(Z,X) + \eta(Z)g(X,Y)] + (1 + \mu)[\eta(X)g(\varphi Y, Z) + \eta(Y)g(\varphi Z, X) + \eta(Z)g(\varphi X, Y)] +$$

$$+3(1 + \mu)\eta(X)\eta(Y)\eta(Z) = 0,$$

for any $X, Y, Z \in \mathfrak{X}(M)$. For $Y := \xi$ and $Z := \xi$ we obtain:

$$(2 - \lambda + 3\mu)\eta(X) = 0,$$

for any $X \in \mathfrak{X}(M)$ and follows $2 - \lambda + 3\mu = 0$. From the relation (16) we get $\mu = -\frac{n+1}{2}$ and $\lambda = -\frac{3n-1}{2}$.
□

From Theorem 3.2 we deduce:

Corollary 3.3. *On a Lorentzian para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ having cyclic Ricci tensor or cyclic η-recurrent Ricci tensor, there is no Ricci soliton with the potential vector field ξ.*

More particular cases as those from Theorem 3.2 are further considered.

Proposition 3.4. *Let (φ, ξ, η, g) be a Lorentzian para-Sasakian structure on the manifold M and let (g, ξ, λ, μ) be an η-Ricci soliton on M.*

1. *If the manifold (M, g) is Ricci symmetric (i.e. $\nabla S = 0$), then $\mu = -1$ and $\lambda = -n$.*
2. *If the Ricci tensor is η-recurrent (i.e. $\nabla S = \eta \otimes S$), then $\mu = \frac{n+3}{2}$ and $\lambda = -\frac{n-5}{2}$.*

Proof. 1. If $\nabla S = 0$, taking $Z := \xi$ in the expression of ∇S from (17) we obtain:

$$\mu g(\varphi X, Y) + g(\varphi X, \varphi Y) = 0,$$

for any $X, Y \in \mathfrak{X}(M)$ and as in the proof of Theorem 3.2 we get $\mu = -1$ and $\lambda = -n$.

2. If $\nabla S = \eta \otimes S$, from (17) we get:

$$-\eta(X)[\lambda g(Y,Z)+g(\varphi Y,Z)]+\eta(Y)[g(X,Z)+\mu g(\varphi X,Z)]+\eta(Z)[g(X,Y)+\mu g(\varphi X,Y)]+(2-\mu)\eta(X)\eta(Y)\eta(Z) = 0,$$

for any $X, Y, Z \in \mathfrak{X}(M)$. For $X := \xi$, $Y := \xi$ and $Z := \xi$ we obtain $\lambda + \mu - 4 = 0$. From the relation (16) we get $\mu = \frac{n+3}{2}$ and $\lambda = -\frac{n-5}{2}$.
□

From Proposition 3.4 we deduce:

Corollary 3.5. *If a Lorentzian para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ is Ricci symmetric or has η-recurrent Ricci tensor, then on M there is no Ricci soliton with the potential vector field ξ.*

In what follows we shall consider η-Ricci solitons requiring for the curvature to satisfy $R(\xi, X) \cdot S = 0$ and $S \cdot R(\xi, X) = 0$, respectively.

Theorem 3.6. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $R(\xi, X) \cdot S = 0$, then $\mu = -1$ and $\lambda = -n$.*

Proof. The condition that must be satisfied by S is:

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0, \tag{18}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Replacing the expression of S from (13) and from the symmetries of R we get:

$$g(\eta(Y)Z + \eta(Z)Y, \varphi X) + \mu[\eta(Y)g(X,Z) + \eta(Z)g(X,Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0, \tag{19}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

For $Z := \xi$ we have:

$$g(\varphi X, Y) + \mu[g(X,Y) + \eta(X)\eta(Y)] = 0, \tag{20}$$

for any $X, Y \in \mathfrak{X}(M)$, which is equivalent to:

$$g(\varphi X, Y) + \mu g(\varphi X, \varphi Y) = 0, \tag{21}$$

and for $Y \mapsto \varphi Y$ we get:

$$g(\varphi X, \varphi Y) + \mu g(\varphi X, Y) = 0,$$

for any $X, Y \in \mathfrak{X}(M)$. Adding the previous two relations we have:

$$(1 + \mu)[g(\varphi X, Y) + g(\varphi X, \varphi Y)] = 0,$$

for any $X, Y \in \mathfrak{X}(M)$ and follows $\mu = -1$. From the relation (16) we get $\lambda = -n$. □

From Theorem 3.6 we deduce:

Corollary 3.7. *On a Lorentzian para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ satisfying $R(\xi, X) \cdot S = 0$, there is no Ricci soliton with the potential vector field ξ.*

From the relations (13), (16) and (20) we obtain:

$$S = (\mu - \lambda)g = (n - 1)g. \tag{22}$$

Therefore:

Proposition 3.8. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $R(\xi, X) \cdot S = 0$, then (M, g) is Einstein manifold.*

Theorem 3.9. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $S(\xi, X) \cdot R = 0$, then $\mu = -1$ and $\lambda = -n$.*

Proof. The condition that must be satisfied by S is:

$$S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W - S(\xi, Y)R(X, Z)W + S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W+$$
$$+S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0, \tag{23}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$.

Taking the inner product with ξ, the relation (23) becomes:

$$S(X, R(Y, Z)W) - S(\xi, R(Y, Z)W)\eta(X) + S(X, Y)\eta(R(\xi, Z)W) - S(\xi, Y)\eta(R(X, Z)W) + S(X, Z)\eta(R(Y, \xi)W)-$$
$$-S(\xi, Z)\eta(R(Y, X)W) + S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\eta(R(Y, Z)X) = 0, \tag{24}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$.

Replacing the expression of S from (13), we get:

$$2(\lambda-\mu)\eta(X)[\eta(Y)g(Z, W)-\eta(Z)g(Y, W)]+\mu\eta(W)[\eta(Z)g(X, Y)-\eta(Y)g(X, Z)]+\lambda[g(X, Y)g(Z, W)-g(X, Z)g(Y, W)]+$$
$$+g(\varphi X, Z)[\eta(Y)\eta(W) + g(Y, W)] + g(\varphi X, Y)[\eta(Z)\eta(W) + g(Z, W)] - g(\lambda X + \varphi X, R(Y, Z)W) = 0, \tag{25}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$.

For $Z := \xi$ and $W := \xi$ we have:

$$g(\varphi X, Y) + \mu[g(X, Y) + \eta(X)\eta(Y)] = 0, \tag{26}$$

for any $X, Y \in \mathfrak{X}(M)$ and as in the proof of Theorem 3.6 we obtain $\mu = -1$ and $\lambda = -n$. □

From Theorem 3.9 we deduce:

Corollary 3.10. *On a Lorentzian para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ satisfying $S(\xi, X) \cdot R = 0$, there is no Ricci soliton with the potential vector field ξ.*

From the relations (13), (16) and (26) we obtain:

$$S = (\mu - \lambda)g = (n - 1)g. \tag{27}$$

Therefore:

Proposition 3.11. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the manifold M, (g, ξ, λ, μ) is an η-Ricci soliton on M and $S(\xi, X) \cdot R = 0$, then (M, g) is Einstein manifold.*

As a final remark concerning the existence of Ricci solitons on a Lorentzian para-Sasakian manifold $(M^n, \varphi, \xi, \eta, g)$, we conclude that there is but one Ricci soliton given for $\lambda = -n + 1$.

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