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On the Position of Abstract Density Topologies in the Lattice of All Topologies

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Abstract. The connection between classical notions of abstract density topologies, semi-open sets and the relation of semi-correspondence, introduced by Levine, Crossley and Hildebrand, is demonstrated.

This work refers to well known concept of abstract density topologies, semi-open sets and some relation connected with these notions. Abstract density topologies were introduced by O. Haupt and Ch. Pauc in 1952 ([8]). They defined such topologies via so called lower density operators and the properties of these topologies were extensively examined by many mathematicians. Some necessary and sufficient conditions for the topology to be an abstract density topology were formulated by J. Hejduk in [10]. He also considered the problem of regularity of such topological spaces (see [11]).

The notion of semi-open sets appeared in 1963 in the article of by N. Levine ([12]). He also defined the relation of semi-correspondence between topologies. Both definitions were used in the studies of semi-topological properties (that means properties which are preserved under semi-homeomorphism, such as for instance separability, being T_2, connectedness, see [4] for details), of semi-continuity ([14]) and quasi-continuity ([3]). A generalization of semi-open sets was discussed also by E. Ekici in [5]. He defined δ-semi-open, a-open, e^*-open sets (and some other) and examined the mutual relations between them. In [6] there was introduced the notion of semi-open sets with respect to some ideal I (so called semi-I-open) and studied the properties of these sets. The research was continued in [15].

In the mentioned articles one can observe, that studies of the new concepts were conducted in the direction of improving knowledge and describing properties of these new objects. In the following paper we will present another attitude to the notion of abstract density topologies, semi-open sets and semi-correspondence. We will show that there is a very close and strict connection between all those classical concepts and abstract density topologies play a crucial role in the family of all topologies.

1. Two Equivalence Relations in the Class of All Topologies on the Given Space *X*.

For an arbitrary topological space $(X.\mathcal{T})$ let us denote:

- $\mathcal{NI}(X,\mathcal{T})$ - the family of sets with nonempty interior,
- $\mathcal{ND}(X,\mathcal{T})$ - the family of nowhere dense sets,

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- $\mathcal{NB}(X,\mathcal{T})$ - the family of sets of nowhere dense boundary

in a space $(X,\mathcal{T})$.

Let $A \subset X$. For the convenience, by $\mathrm{Int}_\alpha(A)$ and $\mathrm{Cl}_\alpha(A)$ we denote the interior and the closure of A with respect to the topology $\mathcal{T}_\alpha$ for certain α and omit the subscript if it does not lead to misunderstanding.

The subset A of the topological space $(X,\mathcal{T})$ is **semi-open** if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. Let us denote by $SO(X,\mathcal{T})$ the family of all semi-open sets with respect to the topology $\mathcal{T}$.

Consider non-empty space X with two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$.

Definition 1.1. *We say that*

- *$\mathcal{T}_1$ and $\mathcal{T}_2$ are* **similar** *($\mathcal{T}_2 \simeq_s \mathcal{T}_1$) iff $\mathcal{NI}(X,\mathcal{T}_1) = \mathcal{NI}(X,\mathcal{T}_2)$.*
- *$\mathcal{T}_1$ and $\mathcal{T}_2$ are* **semi-correspondent** *($\mathcal{T}_2 \simeq_{sc} \mathcal{T}_1$) iff $SO(X,\mathcal{T}_1) = SO(X,\mathcal{T}_2)$.*

Obviously, $\simeq_s$ and $\simeq_{sc}$ are the equivalence relations. It is evident that if $\mathcal{T}$ is a topology on X then $\mathcal{T} \subset SO(X,\mathcal{T})$. If $\mathcal{A}$ is a subfamily of $SO(X,\mathcal{T})$, then $\bigcup\mathcal{A} \in SO(X,\mathcal{T})$. It is easy to observe that

Proposition 1.2. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Then*

$$\mathcal{T}_2 \simeq_{sc} \mathcal{T}_1 \Longrightarrow \mathcal{T}_2 \simeq_s \mathcal{T}_1.$$

Indeed. Let $G \in \mathcal{T}_1$, $G \neq \emptyset$. Hence $G \in SO(\mathcal{T}_1) = SO(\mathcal{T}_2)$. As a nonempty semi open set, G has a nonempty interior in $\mathcal{T}_2$.

The relation of similarity was investigated among others in [2] and [14] (under the name of π-relation). The relation of semi-correspondence introduced by N. Levine, was investigated also by S. G. Crossley and S. K. Hildebrand in [4] and T. R. Hamlett in [7]. Another approach to this theory with different terminology was given by O. Njåstad in [13]. Below we present some facts from their notes using the notation given above. First recall that a subset M of a partially ordered set L is called a **join-semilattice** if it has a least upper bound for any nonempty finite subset and is called **convex** if the condition ($x \le z \le y$ and $x, y \in M$) implies $z \in M$ for any $x, y, z \in L$.

Proposition 1.3.

1. *Equivalence classes of the relation $\simeq_{sc}$ are convex join-semilattices of the lattice of all topologies ([13], Proposition 10).*
2. *Every equivalence class of the relation $\simeq_{sc}$ has it's greatest element (named $\mathcal{F}(\mathcal{T})$ in [4] or α-topology in [13]). It need not have the minimal element ([4], Example 2.1).*
3. *If a topology $\mathcal{T}$ is regular, then it is the minimal element of its equivalence class ([13], Proposition 11).*
4. *If $(Y,\mathcal{T}_Y)$ is a regular space, and $\mathcal{T}_1 \simeq_{sc} \mathcal{T}_2$ then*

$$C((X,\mathcal{T}_1),(Y,\mathcal{T}_Y)) = C((X,\mathcal{T}_2),(Y,\mathcal{T}_Y))$$

where $C(X,Y)$ stands for the family of continuous functions from X to Y ([13], Proposition 8).

Notice, that if we exchange the relation $\simeq_{sc}$ into $\simeq_s$ then the condition (4) does not hold (see [2]).

2. Relations Between Abstract Density Topologies

Let $\mathcal{A}$ be an algebra of sets in 2^X and $\mathcal{I}$ be a proper ideal of sets contained in $\mathcal{A}$. Our considerations in this section are focused on the space $(X,\mathcal{A},\mathcal{I})$ and abstract density topologies given on X. We remind some basic information connected with this notion (compare with [10] and [11]). If $A \triangle B \in \mathcal{I}$ then we will write $A \sim B$.

Definition 2.1. *We will say that an operator $\Phi\colon \mathcal{A} \to 2^X$ is a* lower density operator, *if for any $A, B \in \mathcal{A}$ it fulfills the following conditions:*

(a) $\Phi(\emptyset) = \emptyset, \quad \Phi(X) = X$;

(b) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;

(c) $A \sim B \Rightarrow \Phi(A) = \Phi(B)$;

(d) $A \sim \Phi(A)$ *(the analogue of the Lebesgue Density Theorem).*

We say that the pair $(\mathcal{A}, \mathcal{I})$ satisfies the *hull property* if whenever $A \subseteq X$, there is a $B \in \mathcal{A}$ such that $A \subseteq B$ and if $C \in \mathcal{A}$ and $A \subseteq C$, then $B \setminus C \in \mathcal{I}$. If B is a measurable hull of the set $X \setminus A$, then the set $X \setminus B$ is called a *measurable kernel* of A. Of course the hull and the kernel can be determined accurate to the set from $\mathcal{I}$.

Theorem 2.2 ([10], see also [11]). *If $\Phi\colon \mathcal{A} \to 2^X$ is a lower density operator on $(X, \mathcal{A}, \mathcal{I})$ and the pair $(\mathcal{A}, \mathcal{I})$ satisfies the hull property, then the family*

$$\mathcal{T}_\Phi = \{A \in \mathcal{A}\colon A \subset \Phi(A)\}$$

is a topology on X.

This topology will be called the abstract density topology (shortly - ADT) or topology generated by the operator Φ.

In the further investigation some properties of abstract density topologies will be needed (compare [10], [2]):

Proposition 2.3. *Let Φ be the lower density operator in the space $(X, \mathcal{A}, \mathcal{I})$, and let $\mathcal{T}_\Phi$ be the topology generated by Φ.*

1. *If $A \subset X$ then the interior of A in the topology $\mathcal{T}_\Phi$ is of the form $Int_{\mathcal{T}_\Phi}(A) = A \cap \Phi(B)$, where B is a measurable kernel of A. In particular, if $A \in \mathcal{A}$ then $Int_{\mathcal{T}_\Phi}(A) = A \cap \Phi(A)$.*
2. *If a measurable kernel B of $A \subset X$ does not belong to the ideal $\mathcal{I}$, then $A \in \mathcal{NI}(X, \mathcal{T}_\Phi)$.*
3. *$\mathcal{A} = \mathcal{NB}(X, \mathcal{T}_\Phi)$ and $\mathcal{I} = \mathcal{ND}(X, \mathcal{T}_\Phi)$.*

Let the triple $(X, \mathcal{A}, \mathcal{I})$ satisfy the hull property. Let Φ_1, Φ_2 - lower density operators with respect to $(X, \mathcal{A}, \mathcal{I})$. From [2], Theorem 4 it follows that:

Proposition 2.4. *$\mathcal{T}_1 \simeq_s \mathcal{T}_2$ if and only if $\mathcal{ND}(X, \mathcal{T}_1) = \mathcal{ND}(X, \mathcal{T}_2)$ and $\mathcal{NB}(X, \mathcal{T}_1) = \mathcal{NB}(X, \mathcal{T}_2)$.*

Moreover,

Proposition 2.5. *If topologies $\mathcal{T}_{\Phi_1}$ and $\mathcal{T}_{\Phi_2}$ are generated by lower density operators Φ_1 and Φ_2 on the space $(X, \mathcal{A}, \mathcal{I})$, then*

$$\mathcal{T}_{\Phi_1} \simeq_s \mathcal{T}_{\Phi_2}.$$

In general, the semi-correspondent topologies may differ a lot. In [4], Example 1.5, the authors presented two semi-correspondent topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X such that $(X, \mathcal{T}_1)$ is completely normal, paracompact, Lindelöf and metrizable, and $(X, \mathcal{T}_2)$ satisfies none of these properties. The situation changes when we consider abstract density topologies.

Theorem 2.6. *Let Φ be a lower density operator in the space $(X, \mathcal{A}, \mathcal{I})$. Then*

$$A \in SO(X, \mathcal{T}_\Phi) \iff A \in \mathcal{A} \wedge A \cap \Phi(A') = \emptyset.$$

Proof. Let $A \in SO(X, \mathcal{T}_\Phi)$. Then there exists a set $G \in \mathcal{T}_\Phi$ such that $G \subset A \subset \mathrm{Cl}_\Phi(G)$. Since Φ is the lower density operator, we have $\mathrm{Cl}_\Phi(G) \setminus G \in \mathcal{I} \subset \mathcal{A}$. Hence $A \in \mathcal{A}$. Suppose, that there exists a point $x_0 \in X$ such that $x_0 \in A \cap \Phi(A')$. Then $x_0 \in \Phi(G')$ and $x_0 \notin \Phi(G)$. From the assumption $G \subset \Phi(G)$ it follows that $x_0 \notin G$. So $x_0 \in G' \cap \Phi(G') = \mathrm{Int}_\Phi(G')$. Hence $x_0 \notin \mathrm{Cl}_\Phi(G)$ which is a contradiction with $x_0 \in A$.

Assume now, that $A \in \mathcal{A}$, $A \cap \Phi(A') = \emptyset$. Suppose, that $A \notin SO(X, \mathcal{T}_\Phi)$. Then there exists $x_0 \in A$ such that $x_0 \notin \mathrm{Cl}_\Phi(\mathrm{Int}_\Phi(A))$. Hence $x_0 \in \mathrm{Int}_\Phi\big((\mathrm{Int}_\Phi(A))'\big)$. In particular $x_0 \in \Phi((\mathrm{Int}_\Phi(A))'\big)$. But $(\mathrm{Int}_\Phi(A))' \sim A'$, so $x_0 \in \Phi(A')$. Therefore $A \cap \Phi(A') \neq \emptyset$ contrary to the assumption. □

Proposition 2.7. *If topologies $\mathcal{T}_{\Phi_1}$ and $\mathcal{T}_{\Phi_2}$ on $(X, \mathcal{A}, \mathcal{I})$ are generated by lower density operators Φ_1 and Φ_2, then*

$$\mathcal{T}_{\Phi_1} \simeq_{sc} \mathcal{T}_{\Phi_2} \iff \mathcal{T}_{\Phi_1} = \mathcal{T}_{\Phi_2} \iff \Phi_1 = \Phi_2.$$

Proof. We will prove the first equivalence. Assume $\mathcal{T}_{\Phi_1} \simeq_{sc} \mathcal{T}_{\Phi_2}$. Suppose, that $A \in \mathcal{T}_{\Phi_1} \setminus \mathcal{T}_{\Phi_2}$. Then there exists a point $x_0 \in A$ such that $x_0 \notin \Phi_2(A)$. There are two possibilities.

- If $x_0 \in \Phi_2(A')$, then $x_0 \in A \cap \Phi_2(A')$ and from Theorem 2.6 we obtain $A \notin SO(X, \mathcal{T}_{\Phi_2})$. Therefore, $\mathcal{T}_{\Phi_2} \not\simeq_{sc} \mathcal{T}_{\Phi_1}$.
- If $x_0 \notin \Phi_2(A')$, then $x_0 \notin \Phi_2(A) \cup \Phi_2(A')$. Therefore $x_0 \in \big((A \setminus \Phi_2(A)) \cup (A' \setminus \Phi_2(A'))\big) \in \mathcal{I}$. Hence $\{x_0\} \in \mathcal{I}$. Take $B = \mathrm{Int}_{\Phi_2}(A') \cup \{x_0\}$. It is evident that $B \in \mathcal{A}$. Moreover, $B \in SO(X, \mathcal{T}_{\Phi_2}) \setminus SO(X, \mathcal{T}_{\Phi_1})$. Indeed, $B \cap \Phi_2(B') = \emptyset$, hence $B \in SO(X, \mathcal{T}_{\Phi_2})$. Simultaneously, since $\{x_0\} \in \mathcal{I}$, we have $B' \sim A$. Hence $x_0 \in \Phi_1(B')$. Therefore, $B \cap \Phi_1(B') \supset \{x_0\}$ and $B \notin SO(X, \mathcal{T}_{\Phi_1})$. Since that $\mathcal{T}_{\Phi_2} \not\simeq_{sc} \mathcal{T}_{\Phi_1}$, which finishes the proof of the first equivalence.

The proof of the second equivalence one can find in [9]. □

3. Operation $\widehat{\mathcal{T}}$.

In this section we will show that ADTs play a very important role in the equivalent classes of the relation $\simeq_{sc}$. Take an arbitrary topological space $(X, \mathcal{T})$. Then $\mathcal{NB}(X, \mathcal{T})$ is an algebra, $\mathcal{ND}(X, \mathcal{T}) \subset \mathcal{NB}$ is an ideal, the pair $(\mathcal{NB}, \mathcal{ND})$ satisfies the hull property and the operator

$$\Psi(A) = \mathrm{Int}_{\mathcal{T}}(\mathrm{Cl}_{\mathcal{T}}(A))$$

is the lower density operator (compare [2], Theorem 11). Let us denote the topology $\mathcal{T}_\Psi$ by $\widehat{\mathcal{T}}$.

Proposition 3.1. *Let $\mathcal{T}$ be a topology. Then*

$$\mathcal{T} \subset \widehat{\mathcal{T}}.$$

Indeed, if $A \in \mathcal{T}$ then $A = \mathrm{Int}_{\mathcal{T}}(A) \subset \mathrm{Int}_{\mathcal{T}}(\mathrm{Cl}_{\mathcal{T}}(A)) = \Psi(A)$ hence we have $A \in \widehat{\mathcal{T}}$.

Theorem 3.2. *For any topology $\mathcal{T}$*

$$\mathcal{T} \simeq_{sc} \widehat{\mathcal{T}}.$$

Proof. Let $A \in \mathcal{NB}$. By virtue of the Theorem 2.6

$$A \in SO(X, \widehat{\mathcal{T}}) \iff A \cap \Psi(A') = \emptyset \iff A \subset \big(\Psi(A')\big)' \iff A \subset \big(\mathrm{Int}_{\mathcal{T}}(\mathrm{Cl}_{\mathcal{T}}(A'))\big)'$$
$$\iff A \subset \mathrm{Cl}_{\mathcal{T}}(\mathrm{Int}_{\mathcal{T}}(A)) \iff A \in SO(X, \mathcal{T}).$$

□

Corollary 3.3. *Let $\mathcal{T}_1, \mathcal{T}_2$ be arbitrary topologies on X. Then*

$$\mathcal{T}_1 \simeq_{sc} \mathcal{T}_2 \iff \widehat{\mathcal{T}_1} = \widehat{\mathcal{T}_2}.$$

Proof. Sufficiency. Let $\mathcal{T}_1 \simeq_{sc} \mathcal{T}_2$. Hence $\mathcal{T}_1 \simeq_s \mathcal{T}_2$. Then the families $\mathcal{NB}$ and $\mathcal{ND}$ with respect to both topologies are equal respectively (Proposition 2.4). Since that $\widehat{\mathcal{T}_1}$ and $\widehat{\mathcal{T}_2}$ are two abstract density topologies on the same space $(X, \mathcal{NB}, \mathcal{ND})$ and $\widehat{\mathcal{T}_1} \simeq_{sc} \widehat{\mathcal{T}_2}$. Hence $\widehat{\mathcal{T}_1} = \widehat{\mathcal{T}_2}$ by virtue of Proposition 2.7.

Necessity. Since $\mathcal{T}_1 \simeq_{sc} \widehat{\mathcal{T}_1} = \widehat{\mathcal{T}_2} \simeq_{sc} \mathcal{T}_2$ we have $\mathcal{T}_1 \simeq_{sc} \mathcal{T}_2$. □

Corollary 3.4. *In every equivalence class of the relation $\simeq_{sc}$ there is exactly one ADT. It is also the greatest (with respect to inclusion) element of this equivalence class.*

Using the notation from [4] we have $\mathcal{F}(\mathcal{T}) = \widehat{\mathcal{T}}$. In [13] it was stated that this greatest element is an α-topology (α-topology is the family of all sets such that $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ with respect to a given topology $\mathcal{T}$). None of the authors uses the notion of abstract density topology and saw no connection with it. Moreover, Njåstad gave the characterization of α-topology which coincides with the results obtained by Hejduk in [9] which can be formulated as follows:

Corollary 3.5. *The topology $\mathcal{T}$ is an abstract density topology iff all it's nowhere dense sets are closed (hence it is nodec, in the sense of van Douwen).*

Recall that a topological space X is called submaximal if every dense subset of X is open. Different equivalent conditions for a space to be submaximal and to be nodec are given in [1], Theorem 1.2 and [13], Corollary to Proposition 4, respectively. In particular, they imply that every submaximal space is nodec. From this it follows that abstract density topologies may be a useful tool in the studies of submaximal spaces.

Summarizing, the space of all topologies on X is divided into the equivalence classes of the relation $\simeq_s$. Each of those classes is divided more narrowly into the equivalence classes of the relation $\simeq_{sc}$. The equivalence class of $\simeq_s$ is determined by (and - it determines) the pair $(\mathcal{A}, \mathcal{I})$. The equivalence class of $\simeq_{sc}$ is determined by (and - it determines) the lower density operator. From Proposition 1.3 it follows that if the space $(X, \mathcal{T})$ (where $\mathcal{T}$ is an abstract density topology) is regular, then the equivalent class of the relation $\simeq_{sc}$ consists of one element only:

$$[\mathcal{T}]_{\simeq_{sc}} = \{\mathcal{T}\}.$$

4. Examples

Example 4.1. *The natural topology $\mathcal{T}_{nat}$ on $\mathbb{R}$ is regular. Hence $\mathcal{T}_{nat}$ is the coarsest element of it's class of equivalence of $\simeq_{sc}$. But $\mathcal{T}_{nat}$ is not abstract density topology, because there exists non-closed nowhere dense sets. The finest element of $[\mathcal{T}_{nat}]_{\simeq_{sc}}$ is the topology of the form*

$$\widehat{\mathcal{T}}_{nat} = \{G \setminus P : G \in \mathcal{T}_{nat}, P \in \mathcal{ND}(\mathcal{T}_{nat})\}.$$

Observe, that the equivalence class $[\mathcal{T}_{nat}]_{\simeq_{sc}}$ contains more than two elements. Let $\mathcal{N}$ be the $\sigma-$ideal of Lebesgue null sets. Since $\mathcal{T}_{nat}$ is Lindelöf, the family

$$\mathcal{S} = \{G \setminus P : G \in \mathcal{T}_{nat}, P \in \mathcal{ND}(\mathcal{T}_{nat}) \cap \mathcal{N}\}$$

is the topology, strictly finer than $\mathcal{T}_{nat}$. At the same time $\mathcal{S}$ is coarser than $\widehat{\mathcal{T}}$. Since $[\mathcal{T}_{nat}]_{\simeq_{sc}}$ is convex sublattice, $\mathcal{S} \in [\mathcal{T}_{nat}]_{\simeq_{sc}}$.

Example 4.2. *The ordinary density topology $\mathcal{T}_d$ on $\mathbb{R}$ is an abstract density topology, hence it is the finest element of it's equivalence class of the relation $\simeq_{sc}$. At the same time $\mathcal{T}_d$ is regular, even completely regular. Since that $\mathcal{T}_d$ is also the coarsest element of it's class. As a result*

$$[\mathcal{T}_d]_{\simeq_{sc}} = \{\mathcal{T}_d\}.$$

Example 4.3. *Take $X = [0, 1]$. Let $\mathbf{x} = (x_n)$ be a strictly decreasing sequence of points from $(0, 1)$, tending to 0. The family*

$$\mathcal{T}_{\mathbf{x}} = \{\emptyset, X\} \cup \{[0, x_n) : n \in \mathbb{N}\}$$

forms a topology. It is easy to observe that

1. *a set $A \subset X$ has a nonempty interior ($A \in \mathcal{NI}(X, \mathcal{T}_{\mathbf{x}})$) iff there exists $\varepsilon > 0$ such that $[0, \varepsilon) \subset A$;*
2. *every nonempty open set is dense;*
3. *$SO(X, \mathcal{T}_{\mathbf{x}}) = \mathcal{NI}(X, \mathcal{T}_{\mathbf{x}}) \cup \{\emptyset\}$;*
4. *if $\mathbf{y} = (y_n)$ is a strictly decreasing and tending to 0 sequence such that $y_n \in (0, 1)$ for $n \in \mathbb{N}$ then $\mathcal{T}_{\mathbf{x}} \simeq_{sc} \mathcal{T}_{\mathbf{y}}$;*
5. *if range$(\mathbf{x}) \cap$ range$(\mathbf{y}) = \emptyset$, then $\mathcal{T}_{\mathbf{x}} \cap \mathcal{T}_{\mathbf{y}} = \{\emptyset, X\} \notin [\mathcal{T}_{\mathbf{x}}]_{\simeq_{sc}}$;*

6. *the class* $[\mathcal{T}_{\mathbf{x}}]_{\simeq_{sc}}$ *hasn't got the coarsest element;*
7. *a set* A *is nowhere dense iff* $[0,\varepsilon) \subset A'$ *for some positive* ε*;*
8. *a set* A *has a nowhere dense boundary iff* $[0,\varepsilon) \subset A'$ *or* $[0,\varepsilon) \subset A$ *for some positive* ε*;*
9. $\Psi(A) = \begin{cases} X & \text{if} \quad \inf A = 0, \\ \emptyset & \text{if} \quad \inf A > 0; \end{cases}$
10. $\widehat{\mathcal{T}_{\mathbf{x}}} = SO(X, \mathcal{T}_{\mathbf{x}})$*;*
11. $\widehat{\mathcal{T}_{\mathbf{x}}}$ *and* $\mathcal{T}_{\mathbf{x}}$ *are not homeomorphic, since* $\widehat{\mathcal{T}_{\mathbf{x}}}$ *is of the cardinality* $2^{\mathfrak{c}}$ *whence the cardinality of* $\mathcal{T}_{\mathbf{x}}$ *is* $\aleph_0$*;*
12. $C((X, \mathcal{T}_{\mathbf{x}}), (\mathbb{R}, \mathcal{T}_{nat}))$ *consists of constant functions only.*

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