Filomat 30:3 (2016), 569–582
DOI 10.2298/FIL1603569S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Convergence and Data Dependence Results of an Iteration Process in a Hyperbolic Space

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Abstract. In this paper we prove the strong and $\triangle$-convergence theorems of an iteration process of Khan et al. (J. Appl. Math. Comput. 35 (2011) 607-616) for three finite families of total asymptotically nonexpansive nonself mappings in a hyperbolic space. Moreover we obtain the data dependence result of this iteration for contractive-like mappings under some suitable conditions. Also we present some examples to support the results proved herein. Our results extend and improve some recent results announced in the current literature.

1. Introduction

Most of the problems in various disciplines of science are nonlinear in nature, whereas fixed point theory proposed in the setting of normed linear spaces or Banach spaces majorly depends on the linear structure of the underlying spaces. A nonlinear framework for fixed point theory is a metric space embedded with a "convex structure". The class of hyperbolic spaces, nonlinear in nature, is prominent among non-positively curved spaces and provides rich geometrical structures for different results with applications in topology, graph theory, multivalued analysis and metric fixed point theory. The study of hyperbolic spaces has been largely motivated and dominated by questions about hyperbolic groups, one of the main objects of study in geometric group theory.

Khan et al. [18] considered the following iteration process in a Banach space:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n S y_n, \\ y_n = (1-\beta_n)x_n + \beta_n Q x_n, \quad n \in \mathbb{N}, \end{cases} \tag{1}$$

where K is a nonempty subset of a real Banach space X and T, S, Q are three self-mappings on K and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0,1]$. It is worth mentioning that the iteration process (1) coincides with the iteration process of Khan et al. [18] when $Q = T$. Moreover, this iteration is reduced to the S-iteration process of Agarwal et al. [1] when $T = S = Q$. It is also reduced to Ishikawa iteration in [17] when $T = I, S = Q$, Mann iteration in [23] when $T = Q = I$ and Picard iteration when $T = S, Q = I$ (where I is the identity mapping).

2010 *Mathematics Subject Classification*. Primary 47H09; Secondary 47H10, 49M05

Keywords. Hyperbolic space, total asymptotically nonexpansive nonself mapping, $\triangle$-convergence, strong convergence, data dependence, contractive-like mapping

Received: 24 June 2015; Revised: 20 December 2015; Accepted: 22 December 2015

Communicated by Ljubiša D.R. Kočinac and Ekrem Savaş

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In this paper, we study the convergence and the data dependence of the iteration process (1) in a hyperbolic space. This paper contains four sections. In Section 2, we recollect basic definitions and a detailed overview of the fundamental results. In Section 3, we prove some results related to the strong and $\triangle$-convergence of the iteration process (1) for three finite families of total asymptotically nonexpansive nonself mappings and also give some illustrative examples in support of our theorems. In Section 4, we prove the data dependence result of the iteration process (1) for contractive-like mappings. Our results can be viewed as refinement and generalization of several well-known results in CAT(0) and uniformly convex Banach spaces.

2. Preliminaries and Lemmas

We start this section with the concept of hyperbolic space introduced by Kohlenbach [20] which is more restrictive than the hyperbolic type introduced in Goebel and Kirk [11] and more general than the concept of hyperbolic space in Reich and Shafrir [26].

A hyperbolic space [20] is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is a function satisfying

(W1) $d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y)$,
(W2) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2|\, d(x, y)$,
(W3) $W(x, y, \lambda) = W(y, x, (1 - \lambda))$,
(W4) $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w)$

for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$.

If a space satisfies only (W1), it coincides with the convex metric space introduced by Takahashi [32]. A subset K of a hyperbolic space X is *convex* if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. The class of hyperbolic space in [20] contains all normed linear spaces and convex subsets thereof, $\mathbb{R}$-trees in the sense of Tits, the Hilbert ball with the hyperbolic metric (see [12]), Cartesian products of Hilbert balls, Hadamard manifolds (see [26, 27]) and CAT(0) spaces in the sense of Gromov (see [5]).

A hyperbolic space (X, d, W) is said to be *uniformly convex* [28] if for all $u, x, y \in X, r > 0$ and $\varepsilon \in (0, 2]$, there exists a constant $\delta \in (0, 1]$ such that

$$\left.\begin{array}{r} d(x,u) \leq r \\ d(y,u) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Longrightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)r.$$

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ is called *modulus of uniform convexity* if $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$. The function η is *monotone* if it decreases with r (for a fixed ε).

Imoru and Olantinwo [16] gave the following contractive definition.

Definition 2.1. Let T be a self mapping on a metric space X. The mapping T is called a contractive-like mapping if there exist a constant $a \in [0, 1)$ and a strictly increasing and continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq ad(x, y) + \varphi(d(x, Tx)). \tag{2}$$

This mapping is more general than those considered by Berinde [2, 3], Harder and Hicks [15], Zamfirescu [35], Osilike and Udomene [24].

A contractive-like mapping need not have a fixed point, even if X is a complete. For example, let $X = [0, \infty), d(x, y) = \left|x - y\right|$ and define T by

$$Tx = \begin{cases} 1, & \text{if} \quad 0 \leq x \leq 0.8, \\ 0.6, & \text{if} \quad 0.8 < x < +\infty. \end{cases}$$

It is proved in [14] that T is a contractive-like mapping. But the mapping T has no fixed point.

By using (2), it is obvious that if a contractive-like mapping has a fixed point then it is unique.

Let K be a nonempty subset of a metric space (X, d) and $T : K \to X$ be a nonself mapping. Denote by $F(T) = \{x \in K : Tx = x\}$, the set of fixed points of T. A nonself mapping T is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in K.$$

Recall that K is said to be a *retract* of X, if there exists a continuous mapping $P : X \to K$ such that $Px = x, \forall x \in K$. A mapping $P : X \to K$ is said to be a *retraction* if $P^2 = P$. If P is a retraction, then $Py = y$ for all y in the range of P.

Definition 2.2. ([34]) Let K be a nonempty subset of a metric space (X, d) and P be a nonexpansive retraction of X onto K. A nonself mapping $T : K \to X$ is said to be

(i) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq k_n d(x, y), \quad \forall n \geq 1, x, y \in K;$$

(ii) $(\{v_n\}, \{\mu_n\}, \zeta)$-*total asymptotically nonexpansive* if there exist nonnegative real sequences $\{v_n\}, \{\mu_n\}$ with $v_n \to 0, \mu_n \to 0$ and a strictly increasing continuous function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq d(x, y) + v_n \zeta(d(x, y)) + \mu_n, \quad \forall n \geq 1, x, y \in K; \tag{3}$$

(iii) *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq L d(x, y), \quad \forall n \geq 1, x, y \in K.$$

Remark 2.3. From the above definitions, it is known that each nonexpansive nonself mapping is an asymptotically nonexpansive nonself mapping with $k_n = 1$, $\forall n \geq 1$ and each asymptotically nonexpansive nonself mapping is a total asymptotically nonexpansive nonself mapping with $v_n = k_n - 1$, $\mu_n = 0$, $\forall n \geq 1$, $\zeta(t) = t$, $\forall t \geq 0$. Moreover, each asymptotically nonexpansive nonself mapping is a uniformly L-Lipschitzian nonself mapping with $L = \sup_{n \in \mathbb{N}}\{k_n\}$. However, the converse of these statements is not true, in general.

The concept of $\triangle$-convergence in a metric space was introduced by Lim [22] and its analogue in a CAT(0) space has been investigated by Dhompongsa and Panyanak [9]. In order to define the concept of $\triangle$-convergence in the general setup of hyperbolic space, we first collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, we define a continuous functional $r(., \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\}.$$

The *asymptotic radius* $r_K(\{x_n\})$ of $\{x_n\}$ with respect to a subset K of X is given by

$$r_K(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in K\}.$$

The *asymptotic center* of $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

The *asymptotic center* of $A_K(\{x_n\})$ of $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K(\{x_n\}) = \{x \in K : r(x, \{x_n\}) = r_K(\{x_n\})\}.$$

Recall that a sequence $\{x_n\}$ in X is said to be $\triangle$-*convergent* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and call x as $\triangle$-limit of $\{x_n\}$.

Remark 2.4. (i) Let K be a nonempty closed convex subset of a hyperbolic space X and $\{x_n\}$ be a bounded sequence in K. In what follows, we define

$$\{x_n\} \rightharpoonup w \quad \Leftrightarrow \quad \Phi(w) = \inf_{x\in K} \Phi(x),$$

where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.

(ii) It is easy to see that $\{x_n\} \rightharpoonup w$ if and only if $A_K(\{x_n\}) = \{w\}$.

It is well known that one of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder's demiclosedness principle [6] which states that if K is a nonempty closed convex subset of a uniformly convex Banach space X and $T : K \to X$ is a nonexpansive mapping, then $I - T$ is *demiclosed* at 0, that is, for any sequence $\{x_n\}$ in K if $x_n \to x$ weakly and $(I - T)x_n \to 0$ strongly, then $(I - T)x = 0$. Chang *et al.* [7] proved the demiclosedness principle for total asymptotically nonexpansive nonself mappings in a CAT(0) space. Very recently, Wan [33] proved the demiclosedness principle for these mappings in a hyperbolic space as follows.

Lemma 2.5. ([33, Theorem 1]) *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and K be a nonempty closed convex subset of X. Let $T : K \to X$ be a uniformly L-Lipschitzian and total asymptotically nonexpansive nonself mapping and P be a nonexpansive retraction of X onto K. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $x_n \rightharpoonup p$. Then $Tp = p$.*

In the sequel, we shall need the following results.

Lemma 2.6. ([21, Proposition 3.3]) *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and K be a nonempty closed convex subset of X. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to K.*

Lemma 2.7. ([19, Lemma 2.5]) *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \le r, \quad \limsup_{n\to\infty} d(y_n, x) \le r, \quad \lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$$

for some $r \ge 0$, then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 2.8. ([25, Lemma 2]) *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \ \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.9. ([31]) *Let $\{a_n\}$ be a nonnegative sequence for which one assumes that there exists an $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,*

$$a_{n+1} \le (1 - r_n)a_n + r_n t_n$$

is satisfied, where $r_n \in (0, 1)$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} r_n = \infty$ and $t_n \ge 0, \forall n \in \mathbb{N}$. Then the following holds:

$$0 \le \limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} t_n.$$

3. Some Strong and $\triangle$-Convergence Theorems for Total Asymptotically Nonexpansive Nonself Mappings

First, we define the iteration process (1) for three finite families of nonself mappings in a hyperbolic space:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = PW(T_n(PT_n)^{n-1}x_n, S_n(PS_n)^{n-1}y_n, \alpha_n), \\ y_n = PW(x_n, Q_n(PQ_n)^{n-1}x_n, \beta_n), \quad n \in \mathbb{N}, \end{cases} \tag{4}$$

where $T_n = T_{n(\bmod N)}, S_n = S_{n(\bmod N)}$ and $Q_n = Q_{n(\bmod N)}$ (here the function mod N takes values in $\{1, 2, ..., N\}$.) and for each $i = 1, 2, ..., N, T_i : K \to X$ is a uniformly L_i-Lipschitzian and $(\{v_n^{(i)}\}, \{\mu_n^{(i)}\}, \zeta^{(i)})$-total asymptotically nonexpansive mapping, $S_i : K \to X$ is a uniformly $L_i^{'}$-Lipschitzian and $(\{v_n^{'(i)}\}, \{\mu_n^{'(i)}\}, \zeta'^{(i)})$-total asymptotically nonexpansive mapping and $Q_i : K \to X$ is a uniformly $L_i^{''}$-Lipschitzian and $(\{v_n^{''(i)}\}, \{\mu_n^{''(i)}\}, \zeta''^{(i)})$-total asymptotically nonexpansive mapping.

Remark 3.1. In fact, letting

$$\begin{aligned} L &= \max\{L_i, L_i^{'}, L_i^{''}; i = 1, 2, ..., N\}, v_n = \max\{v_n^{(i)}, v_n^{'(i)}, v_n^{''(i)}; i = 1, 2, ..., N\}, \\ \mu_n &= \max\{\mu_n^{(i)}, \mu_n^{'(i)}, \mu_n^{''(i)}; i = 1, 2, ..., N\}, \zeta = \max\{\zeta^{(i)}, \zeta'^{(i)}, \zeta''^{(i)}; i = 1, 2, ..., N\}, \end{aligned}$$

then $\{T_i\}_{i=1}^N$, $\{S_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$ are three finite families of uniformly L-Lipschitzian and $(\{v_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive nonself mappings.

From now on for three finite families $\{T_i\}_{i=1}^N$, $\{S_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$, we set $F = \cap_{i=1}^N (F(T_i) \cap F(S_i) \cap F(Q_i)) \neq \emptyset$.

We prove the $\triangle$-convergence theorem of the iterative sequence $\{x_n\}$ defined by (4) for three finite families of total asymptotically nonexpansive nonself mappings in a hyperbolic space.

Theorem 3.2. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i\}_{i=1}^N$, $\{S_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$ be three finite families of uniformly L-Lipschitzian and $(\{v_n\}, \{\mu_n\}, \zeta)$-total asymptotically nonexpansive nonself mappings and P be a nonexpansive retraction of X onto K. If the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} v_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$;

(ii) there exist constants $a, b \in (0, 1)$ such that $\{\alpha_n\}, \{\beta_n\} \subset [a, b]$;

(iii) there exists a constant $M > 0$ such that $\zeta(r) \leq Mr, \forall r \geq 0$;

(iv) $d(x_n, S_n(PS_n)^{n-1}x_n) \leq d(T_n(PT_n)^{n-1}x_n, S_n(PS_n)^{n-1}x_n)$,

then the sequence $\{x_n\}$ defined by (4), $\triangle$-converges to a point in F.

Proof. We divide our proof into three steps.

Step 1. First we prove that for each $p \in F$,

$$\lim_{n\to\infty} d(x_n, p) \text{ exists.} \tag{5}$$

In fact, by (W1), (3), (4) and the condition (iii), we get

$$\begin{aligned} d(y_n, p) &= d(PW(x_n, Q_n(PQ_n)^{n-1}x_n, \beta_n), p) \\ &\leq d(W(x_n, Q_n(PQ_n)^{n-1}x_n, \beta_n), p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(Q_n(PQ_n)^{n-1}x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n\{d(x_n, p) + v_n\zeta(d(x_n, p)) + \mu_n\} \\ &\leq (1 + \beta_n v_n M)d(x_n, p) + \beta_n\mu_n \end{aligned} \tag{6}$$

and

$$\begin{aligned} d(x_{n+1},p) &= d(PW(T_n(PT_n)^{n-1}x_n, S_n(PS_n)^{n-1}y_n,\alpha_n),p) \\ &\le d(W(T_n(PT_n)^{n-1}x_n, S_n(PS_n)^{n-1}y_n,\alpha_n),p) \\ &\le (1-\alpha_n)d(T_n(PT_n)^{n-1}x_n,p)+\alpha_n d(S_n(PS_n)^{n-1}y_n,p) \\ &\le (1-\alpha_n)\{d(x_n,p)+v_n\zeta(d(x_n,p))+\mu_n\}+\alpha_n\{d(y_n,p)+v_n\zeta(d(y_n,p))+\mu_n\} \\ &\le (1-\alpha_n)(1+v_nM)d(x_n,p)+\alpha_n(1+v_nM)d(y_n,p)+\mu_n. \end{aligned} \tag{7}$$

By substituting (6) into (7) and simplifying it, we have

$$d(x_{n+1},p)\le(1+\sigma_n)d(x_n,p)+\xi_n,\quad \forall n\ge 1, \tag{8}$$

where $\sigma_n = v_nM(1+\alpha_n\beta_n(1+v_nM))$ and $\xi_n = \mu_n\,(1+\alpha_n\beta_n(1+v_nM))$. Furthermore, using the condition (i), we obtain

$$\sum_{n=1}^{\infty}\sigma_n<\infty \text{ and } \sum_{n=1}^{\infty}\xi_n<\infty.$$

By Lemma 2.8, we get $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F$.

Step 2. Next we prove that

$$\lim_{n\to\infty} d(x_n,T_ix_n)=\lim_{n\to\infty} d(x_n,S_ix_n)=\lim_{n\to\infty} d(x_n,Q_ix_n)=0 \text{ for each } i=1,2,...,N. \tag{9}$$

In fact, it follows from (5) that $\lim_{n\to\infty} d(x_n,p)$ exists for each given $p\in F$. We may assume that

$$\lim_{n\to\infty} d(x_n,p)=r\ge 0. \tag{10}$$

By (6) and (10), we have

$$\limsup_{n\to\infty} d(y_n,p)\le r. \tag{11}$$

Noting

$$\begin{aligned} d(S_n(PS_n)^{n-1}y_n,p) &\le d(y_n,p)+v_n\zeta(d(y_n,p))+\mu_n \\ &\le (1+v_nM)d(y_n,p)+\mu_n,\quad \forall n\ge 1, \end{aligned}$$

by (11) we have

$$\limsup_{n\to\infty} d(S_n(PS_n)^{n-1}y_n,p)\le r. \tag{12}$$

Similarly, by (10) we obtain

$$\limsup_{n\to\infty} d(T_n(PT_n)^{n-1}x_n,p)\le r. \tag{13}$$

In addition, by (7) and (8) we get

$$d(x_{n+1},p)\le d(W(T_n(PT_n)^{n-1}x_n,S_n(PS_n)^{n-1}y_n,\alpha_n),p)\le(1+\sigma_n)d(x_n,p)+\xi_n$$

which yields that

$$\lim_{n\to\infty} d(W(T_n(PT_n)^{n-1}x_n,S_n(PS_n)^{n-1}y_n,\alpha_n),p)=r. \tag{14}$$

With the help of (12)-(14) and Lemma 2.7, we have

$$\lim_{n\to\infty} d(T_n(PT_n)^{n-1}x_n,S_n(PS_n)^{n-1}y_n)=0. \tag{15}$$

On the other hand, since

$$\begin{aligned} d(x_{n+1},p) &\le (1-\alpha_n)d(T_n(PT_n)^{n-1}x_n,p)+\alpha_n d(S_n(PS_n)^{n-1}y_n,p) \\ &\le (1-\alpha_n)\{d(T_n(PT_n)^{n-1}x_n,S_n(PS_n)^{n-1}y_n)+d(S_n(PS_n)^{n-1}y_n,p)\}+\alpha_n d(S_n(PS_n)^{n-1}y_n,p) \\ &\le (1-\alpha_n)d(T_n(PT_n)^{n-1}x_n,S_n(PS_n)^{n-1}y_n)+(1+v_nM)d(y_n,p)+\mu_n, \end{aligned}$$

for all $n \ge 1$, we have $\liminf_{n\to\infty} d(y_n,p) \ge r$. Combined with (11), it yields that $\lim_{n\to\infty} d(y_n,p) = r$. Then it follows from (6) and (10) that

$$\lim_{n\to\infty} d(W(x_n,Q_n(PQ_n)^{n-1}x_n,\beta_n),p) = r. \tag{16}$$

Noting

$$\begin{aligned} d(Q_n(PQ_n)^{n-1}x_n,p) &\le d(x_n,p)+v_n\zeta(d(x_n,p))+\mu_n \\ &\le (1+v_nM)d(x_n,p)+\mu_n, \quad \forall n\ge 1, \end{aligned}$$

by (10) we have

$$\limsup_{n\to\infty} d(Q_n(PQ_n)^{n-1}x_n,p) \le r. \tag{17}$$

With the help of (10), (16), (17) and Lemma 2.7, we have

$$\lim_{n\to\infty} d(x_n,Q_n(PQ_n)^{n-1}x_n) = 0. \tag{18}$$

By virtue of (18), we get

$$\begin{aligned} d(x_n,y_n) &= d(x_n,PW(x_n,Q_n(PQ_n)^{n-1}x_n,\beta_n)) \\ &\le d(x_n,W(x_n,Q_n(PQ_n)^{n-1}x_n,\beta_n)) \\ &\le \beta_n d(x_n,Q_n(PQ_n)^{n-1}x_n) \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Hence

$$\begin{aligned} d(S_n(PS_n)^{n-1}x_n,S_n(PS_n)^{n-1}y_n) &\le d(x_n,y_n)+v_n\zeta(d(x_n,y_n))+\mu_n \\ &\le (1+v_nM)d(x_n,y_n)+\mu_n \\ &\to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{19}$$

From the condition (iv), we have

$$\begin{aligned} d(x_n,S_n(PS_n)^{n-1}x_n) &\le d(T_n(PT_n)^{n-1}x_n,S_n(PS_n)^{n-1}x_n) \\ &\le d(T_n(PT_n)^{n-1}x_n,S_n(PS_n)^{n-1}y_n)+d(S_n(PS_n)^{n-1}y_n,S_n(PS_n)^{n-1}x_n). \end{aligned}$$

It follows from (15) and (19) that

$$\lim_{n\to\infty} d(x_n,S_n(PS_n)^{n-1}x_n) = 0. \tag{20}$$

Now

$$d(x_n,T_n(PT_n)^{n-1}x_n) \le d(x_n,S_n(PS_n)^{n-1}x_n)+d(S_n(PS_n)^{n-1}x_n,S_n(PS_n)^{n-1}y_n)+d(S_n(PS_n)^{n-1}y_n,T_n(PT_n)^{n-1}x_n)$$

implies by (15), (19) and (20), we have

$$\lim_{n\to\infty} d(x_n,T_n(PT_n)^{n-1}x_n) = 0. \tag{21}$$

Moreover, it follows from (19)-(21) that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(PW(T_n(PT_n)^{n-1}x_n, S_n(PS_n)^{n-1}y_n, \alpha_n), x_n) \\ &\leq d(W(T_n(PT_n)^{n-1}x_n, S_n(PS_n)^{n-1}y_n, \alpha_n), x_n) \\ &\leq (1-\alpha_n)d(T_n(PT_n)^{n-1}x_n, x_n) + \alpha_n d(S_n(PS_n)^{n-1}y_n, x_n) \\ &\leq (1-\alpha_n)d(T_n(PT_n)^{n-1}x_n, x_n) + \alpha_n\{d(S_n(PS_n)^{n-1}y_n, S_n(PS_n)^{n-1}x_n) + d(S_n(PS_n)^{n-1}x_n, x_n)\} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{22}$$

Now by (21) and (22), for any $i = 1, 2, ...N$, we get

$$\begin{aligned} d(x_n, T_i x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_i(PT_i)^n x_{n+1}) + d(T_i(PT_i)^n x_{n+1}, T_i(PT_i)^n x_n) + d(T_i(PT_i)^n x_n, T_i x_n) \\ &\leq (1+L)d(x_n, x_{n+1}) + d(x_{n+1}, T_i(PT_i)^n x_{n+1}) + Ld(T_i(PT_i)^{n-1}x_n, x_n) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, S_i x_n) = 0 \text{ and } \lim_{n\to\infty} d(x_n, Q_i x_n) = 0 \text{ for each } i = 1, 2, ..., N.$$

Step 3. Now we are in a position to prove the $\triangle$-convergence of $\{x_n\}$. Since $\{x_n\}$ is bounded, by Lemma 2.6, it has a unique asymptotic center $A_K(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with $A_K(\{u_n\}) = \{u\}$. By (9), we have

$$\lim_{n\to\infty} d(u_n, T_i u_n) = \lim_{n\to\infty} d(u_n, S_i u_n) = \lim_{n\to\infty} d(u_n, Q_i u_n) = 0 \quad \text{for each } i = 1, 2, ..., N.$$

Then it follows from Remark 2.4(ii) and Lemma 2.5 that $u \in F$. By the uniqueness of asymptotic centers, we get $x = u \in F$. It implies that the sequence $\{x_n\}$ $\triangle$-converges to $x \in F$. The proof is completed. □

Example 3.3. Let $\mathbb{R}$ be the real line with the usual metric $|.|$ and $T, S, Q : \mathbb{R} \to \mathbb{R}$ be three mappings defined by $Tx = 1 - x$, $Sx = \frac{2x+1}{4}$ and $Qx = \frac{1}{2}$. The mappings T and S satisfy the condition $d(x, S^n x) \leq d(T^n x, S^n x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Additionally T, S and Q are uniformly L-Lipschitzian and total asymptotically nonexpansive mappings. Clearly, $F = \left\{\frac{1}{2}\right\}$. Set $\alpha_n = \frac{n}{2n+1}$ and $\beta_n = \frac{2n}{3n+1}$ for all $n \in \mathbb{N}$. Thus, the conditions of Theorem 3.2 are satisfied.

If we take $Q_i = T_i$ for each $i = 1, 2, ..., N$ in Theorem 3.2, we get the following corollary, yet it is new in the literature.

Corollary 3.4. *Let $X, K, \{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ be the same as in Theorem 3.2. Suppose that the conditions (i)-(iii) in Theorem 3.2 are satisfied. Then the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = PW(T_n(PT_n)^{n-1}x_n, S_n(PS_n)^{n-1}y_n, \alpha_n), \\ y_n = PW(x_n, T_n(PT_n)^{n-1}x_n, \beta_n), \quad n \in \mathbb{N}, \end{cases}$$

$\triangle$-converges to a common fixed point of $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$.

Example 3.5. Let $\mathbb{R}$ be the real line with the usual metric $|\cdot|$ and let $K = [-1, 1]$. Define two mappings $T, S : K \to K$ by

$$Tx = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1] \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0) \end{cases} \quad \text{and} \quad Sx = \begin{cases} x, & \text{if } x \in [0,1] \\ -x, & \text{if } x \in [-1,0) \end{cases}.$$

It is mentioned in [29] that both T and S are uniformly L-Lipschitzian and total asymptotically nonexpansive mappings. Clearly, $F(T) = \{0\}$ and $F(S) = \{x \in K; 0 \leq x \leq 1\}$. Set

$$\alpha_n = \frac{n}{2n+1} \text{ and } \beta_n = \frac{n}{3n+1} \text{ for all } n \geq 1. \tag{23}$$

Thus, the conditions of Corollary 3.4 are fulfilled.

Example 3.6. Let $\mathbb{R}$ be the real line with the usual metric $|\cdot|$ and let $K = [0, \infty)$. Define two mappings $T, S : K \to K$ by $Tx = \sin x$ and $Sx = x$. It is mentioned in [29] that both T and S are uniformly L-Lipschitzian and total asymptotically nonexpansive mappings. Clearly, $F(T) = \{0\}$ and $F(S) = \{x \in K; 0 \le x < \infty\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the same as in (23). So, the conditions of Corollary 3.4 are satisfied.

If we take $Q_i = S_i = T_i$ for each $i = 1, 2, ..., N$ in Theorem 3.2, we get the following corollary which is still new in the literature.

Corollary 3.7. *Let $X, K, \{T_i\}_{i=1}^N$ be the same as in Theorem 3.2. Suppose that the conditions (i)-(iii) in Theorem 3.2 are satisfied. Then the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_1 \in K, \\ x_{n+1} = PW(T_n(PT_n)^{n-1}x_n, T_n(PT_n)^{n-1}y_n, \alpha_n), \\ y_n = PW(x_n, T_n(PT_n)^{n-1}x_n, \beta_n), \quad n \in \mathbb{N}, \end{cases}$$

$\triangle$-converges to a fixed point of $\{T_i\}_{i=1}^N$.

Recall that a mapping T from a subset K of a metric space (X, d) into X is semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

By using this definition, we obtain the strong convergence theorem.

Theorem 3.8. *Under the assumptions of Theorem 3.2, if one of the mappings in $\{T_i\}_{i=1}^N$, $\{S_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$ is semi-compact, then the sequence $\{x_n\}$ defined by (4) converges strongly to a common fixed point in F.*

Proof. We can assume that the mapping T_k in $\{T_i\}_{i=1}^N$ is semi-compact. By (9) and semi-compactness of T_k, there exists a subsequence $\left\{x_{n_j}\right\} \subset \{x_n\}$ such that $\left\{x_{n_j}\right\}$ converges strongly to some point $p \in K$. Moreover, by the uniform continuity of $\{T_i\}_{i=1}^N$, we have

$$d(p, T_i p) = \lim_{j\to\infty} d(x_{n_j}, T_i x_{n_j}) = 0 \text{ for each } i = 1, 2, ..., N.$$

This implies that p is a fixed point of $\{T_i\}_{i=1}^N$. Similarly, p is a common fixed point of $\{S_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$. Then $p \in F$. It follows from (5) that $\lim_{n\to\infty} d(x_n, p)$ exists and hence $\lim_{n\to\infty} d(x_n, p) = 0$. As a result, $\{x_n\}$ converges strongly to a point p in F. The proof is completed. $\square$

Fukhar-ud-din and Khan [10] defined the condition (A) for two finite families of mappings as follows.

Let f be a nondecreasing self-mappings on $[0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$. Then two finite families $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ are said to satisfy *condition (A)* on K if

$$d(x, Tx) \ge f(d(x, F)) \text{ or } d(x, Sx) \ge f(d(x, F)) \text{ for all } x \in K$$

holds for at least one $T \in \{T_i\}_{i=1}^N$ or one $S \in \{S_i\}_{i=1}^N$, where $d(x, F) = \inf\{d(x, p): p \in F.\}$

We can modify this definition for three finite families of mappings as follows.

Three finite families $\{T_i\}_{i=1}^N$, $\{S_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$ are said to satisfy *condition (B)* on K if, for all $x \in K$,

$$d(x, Tx) \ge f(d(x, F)) \text{ or } d(x, Sx) \ge f(d(x, F)) \text{ or } d(x, Qx) \ge f(d(x, F)) \tag{24}$$

holds for at least one $T \in \{T_i\}_{i=1}^N$ or one $S \in \{S_i\}_{i=1}^N$ or one $Q \in \{Q_i\}_{i=1}^N$.

The condition (B) is reduced to the condition (A) when $Q_i = T_i$ for each $i = 1, 2, ..., N$.

We use the condition (B) to prove the strong convergence of $\{x_n\}$ defined by (4).

Theorem 3.9. *Under the assumptions of Theorem 3.2, if a triple of mappings T, S and Q in $\{T_i\}_{i=1}^N$, $\{S_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$, respectively, satisfies condition (B), then the sequence $\{x_n\}$ defined by (4) converges strongly to a common fixed point in F.*

Proof. By (9) and (24), we obtain $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function with $f(0) = 0$ and $f(r) > 0, \forall r > 0$, we have $\lim_{n\to\infty} d(x_n, F) = 0$. The rest of the proof is similar to Theorem 4 in [33] and therefore it is omitted. □

Remark 3.10. (i) Our results generalize the corresponding results of Şahin and Başarır [30] from three nonexpansive self mappings to three finite families of uniformly L-Lipschitzian and total asymptotically nonexpansive nonself mappings.

(ii) Since the iteration process (4) is reduced to the iterative scheme in [33] when $T_i = I, S_i = S, Q_i = Q$ for each $i = 1, 2, ..., N$, our results generalize the corresponding results of Wan [33].

4. Data Dependence of an Iteration Process

Data dependence of fixed points has become an important subject for research. The data dependence of various iteration processes has been studied by many authors; see [8, 14, 31].

We begin with modification of the iterative scheme (1) from a Banach space to a hyperbolic space:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = W(Tx_n, Sy_n, \alpha_n), \\ y_n = W(x_n, Qx_n, \beta_n), \quad n \in \mathbb{N}, \end{cases} \tag{25}$$

where K is a nonempty subset of a hyperbolic space X and $T, S, Q : K \to K$ are three contractive-like mappings and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$.

Remark 4.1. Since T, S and Q are contractive-like mappings, then there exist constants a_1, a_2, a_3 and functions $\varphi_1, \varphi_2, \varphi_3$ such that $d(Tx, Ty) \le a_1 d(x, y) + \varphi_1(d(x, Tx))$, $d(Sx, Sy) \le a_2 d(x, y) + \varphi_2(d(x, Sx))$ and $d(Qx, Qy) \le a_3 d(x, y) + \varphi_3(d(x, Tx))$ for all $x, y \in K$. Throughout this paper, we take $a = \max\{a_1, a_2, a_3\}$ and $\varphi = \max\{\varphi_1, \varphi_2, \varphi_3\}$ so that $d(Tx, Ty) \le ad(x, y) + \varphi(d(x, Tx))$, $d(Sx, Sy) \le ad(x, y) + \varphi(d(x, Sx))$ and $d(Qx, Qy) \le ad(x, y) + \varphi(d(x, Qx))$ for all $x, y \in K$.

We prove the strong convergence of the iterative sequence $\{x_n\}$ defined by (25) for contractive-like mappings in a hyperbolic space.

Theorem 4.2. *Let K be a nonempty closed convex subset of a hyperbolic space X, let $T, S, Q : K \to K$ be three contractive-like mappings with $F \neq \emptyset$ and $\{x_n\}$ be a sequence defined by (25) such that $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$. Then the sequence $\{x_n\}$ converges strongly to the unique common fixed point of T, S and Q.*

Proof. Let p be the unique common fixed point of T, S and Q. From (W1), (2) and (25), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(Tx_n, Sy_n, \alpha_n), p) \\ &\le (1 - \alpha_n)d(Tx_n, p) + \alpha_n d(Sy_n, p) \\ &\le (1 - \alpha_n)\{ad(x_n, p) + \varphi(d(p, Tp))\} + \alpha_n \{ad(y_n, p) + \varphi(d(p, Sp))\} \\ &= (1 - \alpha_n)ad(x_n, p) + \alpha_n ad(y_n, p). \end{aligned} \tag{26}$$

Similarly, we obtain

$$\begin{aligned} d(y_n, p) &= d(W(x_n, Qx_n, \beta_n), p) \\ &\le (1 - \beta_n)d(x_n, p) + \beta_n d(Qx_n, p) \\ &\le (1 - \beta_n)d(x_n, p) + \beta_n \{ad(x_n, p) + \varphi(d(p, Qp))\} \\ &= (1 - \beta_n)d(x_n, p) + \beta_n ad(x_n, p) \\ &= (1 - \beta_n(1 - a))d(x_n, p) \\ &\le d(x_n, p). \end{aligned} \tag{27}$$

Then from (26) and (27), we get that

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\alpha_n) a d(x_n, p) + \alpha_n a d(y_n, p) \\ &\leq (1-\alpha_n) a d(x_n, p) + \alpha_n a d(x_n, p) \\ &\leq a d(x_n, p) \\ &\;\;\vdots \\ &\leq a^{n+1} d(x_0, p). \end{aligned}$$

If $a \in (0,1)$, we obtain

$$\lim_{n\to\infty} d(x_{n+1}, p) = 0.$$

Thus we have $x_n \to p \in F$. If $a = 0$, the result is clear. This completes the proof. □

Example 4.3. Let $\mathbb{R}$ be the real line with the usual metric $|\cdot|$ and let $K = [0,1]$. Define three mappings $T, S, Q : K \to K$ by $Tx = \frac{x}{2}$, $Sx = \frac{x}{4}$ and $Qx = \frac{x}{6}$. It is clear that T, S, Q are contractive-like mappings and $F = \{0\}$. Let $\alpha_n = \beta_n = 0$ for $n = 1, 2, 3$ and $\alpha_n = \beta_n = \frac{2}{\sqrt{n}}$ for all $n \geq 4$. It is easy to see that the conditions of Theorem 4.2 are satisfied.

Example 4.4. Let $\mathbb{R}$ be the real line with the usual metric $|\cdot|$ and let $K = [0,1]$. Define three mappings $T, S, Q : K \to K$ by

$$Tx = \begin{cases} \frac{1}{6}, & x \in (0.5, 1] \\ 0, & x \in [0, 0.5] \end{cases}, \quad Sx = \begin{cases} \frac{1}{7}, & x \in (0.5, 1] \\ 0, & x \in [0, 0.5] \end{cases} \text{ and } Qx = \begin{cases} \frac{1}{8}, & x \in (0.5, 1] \\ 0, & x \in [0, 0.5] \end{cases}.$$

It is mentioned in [13, Example 2.4] that T, S and Q are contractive-like mappings. Clearly, $F = \{0\}$. Let $\alpha_n = \beta_n = 0$ for $n = 1, 2, \ldots, 15$ and $\alpha_n = \beta_n = \frac{4}{\sqrt{n}}$ for all $n \geq 16$. So, the conditions of Theorem 4.2 are satisfied.

Definition 4.5. ([4]) Let $T, \widetilde{T} : X \to X$ be two operators. We say that $\widetilde{T}$ is an approximate operator for T if, for all $x \in X$ and for a fixed $\varepsilon > 0$, we have $d(Tx, \widetilde{T}x) \leq \varepsilon$.

By using this definition, we now prove the data dependence result for the iteration process defined by (25).

Theorem 4.6. *Let X, K, T, S and Q be the same as in Theorem 4.2. Suppose that $\widetilde{T}, \widetilde{S}, \widetilde{Q}$ are approximate operators of T, S, Q as in Definition 4.5, respectively, that is, $d(Tx, \widetilde{T}x) \leq \varepsilon_1, d(Sx, \widetilde{S}x) \leq \varepsilon_2, d(Qx, \widetilde{Q}x) \leq \varepsilon_3$. Let $\{x_n\}$ and $\{u_n\}$ be two iterative sequences defined by (25) and*

$$\begin{cases} u_1 \in K, \\ u_{n+1} = W(\widetilde{T}u_n, \widetilde{S}v_n, \alpha_n), \\ v_n = W(u_n, \widetilde{Q}u_n, \beta_n), \quad n \in \mathbb{N}, \end{cases} \tag{28}$$

respectively, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n \geq \frac{1}{2}, \forall n \in \mathbb{N}$. If $p = Tp = Sp = Qp$ and $q = \widetilde{T}q = \widetilde{S}q = \widetilde{Q}q$, then we have

$$d(p, q) \leq \frac{3\varepsilon}{1-a},$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

Proof. From (W4), (2), (25) and (28), we have the following estimates:

$$\begin{aligned} d(x_{n+1},u_{n+1}) &= d(W(Tx_n,Sy_n,\alpha_n),W(\widetilde{T}u_n,\widetilde{S}v_n,\alpha_n)) \\ &\leq (1-\alpha_n)d(Tx_n,\widetilde{T}u_n)+\alpha_n d(Sy_n,\widetilde{S}v_n) \\ &\leq (1-\alpha_n)\{d(Tx_n,Tu_n)+d(Tu_n,\widetilde{T}u_n)\}+\alpha_n\{d(Sy_n,Sv_n)+d(Sv_n,\widetilde{S}v_n)\} \\ &\leq (1-\alpha_n)\{ad(x_n,u_n)+\varphi(d(x_n,Tx_n))+\varepsilon_1\}+\alpha_n\{ad(y_n,v_n)+\varphi(d(y_n,Sy_n))+\varepsilon_2\} \end{aligned} \tag{29}$$

and

$$\begin{aligned} d(y_n,v_n) &= d(W(x_n,Qx_n,\beta_n),W(u_n,\widetilde{Q}u_n,\beta_n)) \\ &\leq (1-\beta_n)d(x_n,u_n)+\beta_n d(Qx_n,\widetilde{Q}u_n) \\ &\leq (1-\beta_n)d(x_n,u_n)+\beta_n\{d(Qx_n,Qu_n)+d(Qu_n,\widetilde{Q}u_n)\} \\ &\leq (1-\beta_n)d(x_n,u_n)+\beta_n\{ad(x_n,u_n)+\varphi(d(x_n,Qx_n))+\varepsilon_3\} \\ &= (1-\beta_n(1-a))d(x_n,u_n)+\beta_n\varphi(d(x_n,Qx_n))+\beta_n\varepsilon_3. \end{aligned} \tag{30}$$

Combining (29) and (30), we get

$$\begin{aligned} d(x_{n+1},u_{n+1}) \leq\ & \{(1-\alpha_n)a+\alpha_n a(1-\beta_n(1-a))\}d(x_n,u_n)+(1-\alpha_n)\varphi(d(x_n,Tx_n))+\alpha_n\varphi(d(y_n,Sy_n)) \\ &+\alpha_n\beta_n a\varphi(d(x_n,Qx_n))+(1-\alpha_n)\varepsilon_1+\alpha_n\varepsilon_2+\alpha_n\beta_n a\varepsilon_3. \end{aligned} \tag{31}$$

Since $a\in[0,1)$ and $\{\alpha_n\},\{\beta_n\}\subset[0,1]$, we have

$$(1-\alpha_n)a\leq 1-\alpha_n,\quad 1-\beta_n(1-a)\leq 1,\quad \alpha_n\beta_n a\leq\alpha_n. \tag{32}$$

It follows from the assumption $\alpha_n\geq\frac{1}{2},\forall n\in\mathbb{N}$ that

$$1-\alpha_n\leq\alpha_n,\quad \forall n\in\mathbb{N}. \tag{33}$$

By substituting (32) and (33) into (31), we obtain

$$\begin{aligned} d(x_{n+1},u_{n+1}) \leq\ & (1-\alpha_n(1-a))d(x_n,u_n)+\alpha_n\varphi(d(x_n,Tx_n))+\alpha_n\varphi(d(y_n,Sy_n))+\alpha_n\varphi(d(x_n,Qx_n)) \\ &+\alpha_n\varepsilon_1+\alpha_n\varepsilon_2+\alpha_n\varepsilon_3, \end{aligned}$$

or, equivalently,

$$\begin{aligned} d(x_{n+1},u_{n+1}) \leq\ & (1-\alpha_n(1-a))d(x_n,u_n) \\ &+\alpha_n(1-a)\frac{\varphi(d(x_n,Tx_n))+\varphi(d(y_n,Sy_n))+\varphi(d(x_n,Qx_n))+\varepsilon_1+\varepsilon_2+\varepsilon_3}{1-a}. \end{aligned} \tag{34}$$

Now define

$$\begin{aligned} a_n &= d(x_n,u_n), \\ r_n &= \alpha_n(1-a), \\ t_n &= \frac{\varphi(d(x_n,Tx_n))+\varphi(d(y_n,Sy_n))+\varphi(d(x_n,Qx_n))+\varepsilon_1+\varepsilon_2+\varepsilon_3}{1-a}. \end{aligned}$$

Thus, (34) becomes

$$a_{n+1}\leq(1-r_n)a_n+r_nt_n. \tag{35}$$

From Theorem 4.2, it follows that $\lim_{n\to\infty}d(x_n,p)=0$ and $\lim_{n\to\infty}d(u_n,q)=0$. Since T is a contractive-like mapping and $p=Tp$,

$$\begin{aligned} 0 &\leq d(x_n,Tx_n) \\ &\leq d(x_n,p)+d(Tp,Tx_n) \\ &\leq d(x_n,p)+ad(p,x_n)+\varphi(d(p,Tp)) \\ &= (1+a)d(x_n,p)\to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{36}$$

It is easy to see from (36) that this result is also valid for $d(y_n, Sy_n)$ and $d(x_n, Qx_n)$. Since φ is continuous, we have

$$\lim_{n\to\infty} \varphi(d(x_n, Tx_n)) = \lim_{n\to\infty} \varphi(d(y_n, Sy_n)) = \lim_{n\to\infty} \varphi(d(x_n, Qx_n)) = 0.$$

Therefore, using Lemma 2.9, (35) yields

$$d(p, q) \leq \frac{3\varepsilon}{1-a},$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. □

Since the Picard, Mann, Ishikawa and S-iterative processes are special cases of the iterative scheme (1), the data dependence results of these iterative processes can be obtained similarly.

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