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On Double Sequence Spaces Defined by an Orlicz Function on a Seminormed Space

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Abstract. In this paper we introduce and study the double sequence space $m^{''}(M, \phi, q)$ by using the Orlicz function M. Also we obtain some inclusion results involving the space $m^{''}(M, \phi, q)$.

1. Introduction

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg [2] got interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or $l_p(1 \leq< p < \infty)$. Subsequently Lindenstrauss and Tzafriri [3] investigated Orlicz sequence spaces in more detail and they proved that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p(1 \leq< p\infty)$. Parashar and Choudhary [5] have introduced and discussed some properties of the four sequence spaces defined by using an Orlicz function M, which generalized the sequence space l_M and strongly summable sequence spaces $[C, 1, p], [C, 1, p]_0$ and $[C, 1, p]_\infty$. The Orlicz sequence find a number of useful applications in the theory of nonlinear integral equations. Whereas the Orlicz sequence spaces are the generalizations of l_p-spaces, the L_p spaces find themselves enveloped in Orlicz spaces.

The sequence space $m(\phi)$ was introduced by Sargent [9]. He studied some of its properties and obtained its relationship with the space ℓ_p. In this paper we introduce and study the double sequence space $m^{''}(M, \phi, q)$ by using the Orlicz function. Also we obtain some inclusion results involving the space $m^{''}(M, \phi, q)$.

2. Definitions and Background

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of x, if there exists a constant $K > 0$, such that $M(2x) \leq KM(x)$ for all $x \geq 0$. The Δ_2-condition is equivalent to $M(Lx) \leq KLM(x)$, for all values of $x > 0$ and for $L > 1$.

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An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x \eta(t)dt$, where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, for $t > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

If the convexity of the Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called as modulus function, introduced by Ruckle [8] and studied by Maddox [4] and others.

Remark 2.1. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Let ω'' denote the set of all double sequences of real numbers. In 1900, Pringsheim presented the following definition for the convergence of double sequences.

Definition 2.2. ([Pringsheim, [6]]) A double sequence $x = [x_{k,l}]$ has Pringsheim limit L (denoted by P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $\left|x_{k,l} - L\right| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as "P-convergent".

We shall denote the space of all P-convergent sequences by c''.

By a bounded double sequence we shall mean there exists a positive number M such that $|x_{k,l}| < M$ for all (k, l).

We shall also denote the set of all bounded double sequences by l''_∞. We also note in contrast to the case for single sequence, a P-convergent double sequence need not be bounded.

Let $P_{r,s}$ denotes the class of all subsets of $\mathbb{N}\mathrm{x}\mathbb{N}$, those do not contain more than (r, s) elements. Throughout $\left\{\phi_{m,n}\right\}$ represents a non-decreasing double sequence of positive real numbers such that $(m, n)\phi_{m+1,n+1} \leq (m + 1)(n + 1)\phi_{mn}$ for all $(m, n) \in \mathbb{N}\mathrm{x}\mathbb{N}$.

Throughout the article $w''(X)$ and $\ell''_\infty(X)$ denote the spaces of all double and all double bounded sequences respectively with elements in X, where (X, q) denote a seminormed space, seminormed by q. The zero sequence is denoted by $\overline{\theta} = (\theta, \theta, \theta, ...)$, where θ is the zero element of X.

In the later stage different Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [5], Et [1], Triphaty and Sen [17], Savaş [11–16] and many others.

In this article we introduce the following sequence spaces.

$$\ell''_\infty(M, q) = \left\{(x_{k,l}) \in w(X) : \sup_{k,l\geq 1} M\left(q\left(\frac{x_{k,l}}{\rho}\right)\right) < \infty, \text{ for some } \rho > 0\right\},$$

$$m''(M, \phi, q) = \left\{(x_{k,l}) \in w(X) : \sup_{r,s\geq 1,\ \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left(q\left(\frac{x_{k,l}}{\rho}\right)\right) < \infty, \text{ for some } \rho > 0\right\}.$$

3. Main Results

In this section we prove some results involving the sequence spaces $m''(M, \phi, q)$, and $l''_\infty(M, q)$.

Theorem 3.1. *$m''(M, \phi, q)$ and $\ell''_\infty(M, q)$ are linear spaces.*

Proof. Let $(x_{k,l}), (y_{k,l}) \in m''(M, \phi, q)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists positive numbers ρ_1 and ρ_2 such that

$$\sup_{r,s\geq 1,\ \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left(q\left(\frac{x_{k,l}}{\rho_1}\right)\right) < \infty$$

and

$$\sup_{r,s\geq 1,\,\sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left(q\left(\frac{x_{k,l}}{\rho_2}\right)\right) < \infty.$$

Let $\rho_3 = \max\left(2\left|\alpha\right|\rho_1, 2\left|\beta\right|\rho_2\right)$. Since q is a semi-norm and M is a non-decreasing convex function , we have

$$\begin{aligned}
\sum_{k,l\in\sigma} M\left[q\left(\frac{\alpha x_{k,l}+\beta y_{k,l}}{\rho_3}\right)\right] &\leq \sum_{k,l\in\sigma} M\left[q\left(\frac{\alpha x_{k,l}}{\rho_3}\right)+q\left(\frac{\beta y_{k,l}}{\rho_3}\right)\right] \\
&\leq \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}}{\rho_1}\right)\right]+\sum_{k,l\in\sigma} M\left[q\left(\frac{y_{k,l}}{\rho_2}\right)\right] \\
&\Rightarrow \sup_{r,s\geq 1,\,\sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{\alpha x_{k,l}+\beta y_{k,l}}{\rho_3}\right)\right] \\
&\leq \sup_{r,s\geq 1,\,\sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}}{\rho_1}\right)\right] \\
&+ \sup_{r,s\geq 1,\,\sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{y_{k,l}}{\rho_2}\right)\right] \\
&< \infty \\
&\Rightarrow (\alpha x_{k,l}+\beta y_{k,l}) \in m^{''}(M,\phi,q).
\end{aligned}$$

Hence $m^{''}(M,\phi,q)$ is a linear space.

The proof for the case $\ell_\infty^{''}(M,q)$ is a routine work in view of the above proof. □

Theorem 3.2. *The space $m^{''}(M,\phi,q)$ is a seminormed space, seminormed by*

$$f\left((x_{k,l})\right) = \inf\left\{\rho > 0 : \sup_{r,s\geq 1,\,\sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}}{\rho}\right)\right] \leq 1\right\}.$$

Proof. Obviously, $f\left((x_{k,l})\right) \geq 0$ for all $(x_{k,l}) \in m^{''}(M,\phi,q)$ and $f(\overline{\theta}) = 0$.

Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\sup_{r,s\geq 1,\,\sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}}{\rho_1}\right)\right] \leq 1$$

and

$$\sup_{r,s\geq 1,\,\sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}}{\rho_2}\right)\right] \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned}
\sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}+y_{k,l}}{\rho}\right)\right] &= \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}+y_{k,l}}{\rho_1+\rho_2}\right)\right] \\
&\leq \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} \left\{\frac{\rho_1}{\rho_1+\rho_2} M\left(q\left(\frac{x_{k,l}}{\rho_1}\right)\right)\right. \\
&\qquad \left. + \frac{\rho_2}{\rho_1+\rho_2} M\left(q\left(\frac{y_{k,l}}{\rho_2}\right)\right)\right\} \\
&\leq \frac{\rho_1}{\rho_1+\rho_2} \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left(q\left(\frac{x_{k,l}}{\rho_1}\right)\right) \\
&\quad + \frac{\rho_2}{\rho_1+\rho_2} \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left(q\left(\frac{y_{k,l}}{\rho_2}\right)\right) \\
&\leq 1.
\end{aligned}$$

Since the ρ's are nonnegative, so we have

$$\begin{aligned}
f(x+y) &= \inf\left\{\rho > 0 : \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}+y_{k,l}}{\rho}\right)\right] \leq 1\right\} \\
&\leq \inf\left\{\rho_1 > 0 : \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}}{\rho_1}\right)\right] \leq 1\right\} \\
&\quad + \inf\left\{\rho_2 > 0 : \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{y_{k,l}}{\rho_2}\right)\right] \leq 1\right\} \\
&= f(x) + f(y).
\end{aligned}$$

Next for $\lambda \in \mathbb{C}$, without loss of generality, let $\lambda \neq 0$, then

$$\begin{aligned}
f(\lambda(x_{k,l})) &= \inf\left\{\rho > 0 : \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{\lambda x_{k,l}}{\rho}\right)\right] \leq 1\right\} \\
&= \inf\left\{|\lambda| r > 0 : \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}}{r}\right)\right] \leq 1\right\} \\
&= |\lambda| \inf\left\{r > 0 : \sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M\left[q\left(\frac{x_{k,l}}{r}\right)\right] \leq 1, \text{, where } r = \frac{\rho}{|\lambda|}\right\} \\
&= |\lambda| f((x_{k,l})).
\end{aligned}$$

This completes the proof of the theorem. □

The proof of the following theorem is similar to the previous theorem, so we state the result without proof.

Proposition 3.3. *The space $\ell''_\infty(M,q)$ is a seminormed space, seminormed by*

$$g((x_{k,l})) = \inf\left\{\rho > 0 : \sup_{k,l\geq 1} M\left[q\left(\frac{x_{k,l}}{\rho}\right)\right] \leq 1\right\}.$$

Theorem 3.4. *$m''(M,\phi,q) \subseteq m''(M,\psi,q)$ if and only if $\sup_{r,s\geq 1} \frac{\phi_{r,s}}{\psi_{r,s}} < \infty$.*

Proof. Let $\sup_{r,s\geq 1}\frac{\phi_{r,s}}{\psi_{r,s}}<\infty$ and $(x_{k,l})\in m''(M,\phi,q)$. Then

$$\sup_{r,s\geq 1,\,\sigma\in P_{r,s}}\frac{1}{\phi_{r,s}}\sum_{k,l\in\sigma}M\left[q\left(\frac{x_{k,l}}{\rho}\right)\right]<\infty,\text{ for some }\rho>0$$

so we have

$$\sup_{r,s\geq 1,\,\sigma\in P_{r,s}}\frac{1}{\psi_{r,s}}\sum_{k,l\in\sigma}M\left[q\left(\frac{x_{k,l}}{\rho}\right)\right]\leq\left\{\sup_{r,s\geq 1}\frac{\phi_{r,s}}{\psi_{r,s}}\right\}\left\{\sup_{r,s\geq 1,\,\sigma\in P_{r,s}}\frac{1}{\phi_{r,s}}\sum_{k,l\in\sigma}M\left[q\left(\frac{x_{k,l}}{\rho}\right)\right]\right\}$$
$$<\quad\infty.$$

Thus $(x_{k,l})\in m''(M,\psi,q)$.

Therefore $m''(M,\phi,q)\subseteq m''(M,\psi,q)$.

Conversely, let $m''(M,\phi,q)\subseteq m''(M,\psi,q)$. Suppose that $\sup_{r,s\geq 1}\frac{\phi_{r,s}}{\psi_{r,s}}=\infty$. Then there exists a sequence of naturals $\left\{r_i s_j\right\}$ such that $\lim_{i,j\to\infty}\frac{\phi_{r_i s_j}}{\psi_{r_i s_j}}=\infty$. Let $(x_{k,l})\in m(M,\phi,q)$. Then there exists $\rho>0$ such that

$$\sup_{r,s\geq 1,\,\sigma\in P_{r,s}}\frac{1}{\phi_{r,s}}\sum_{k,l\in\sigma}M\left[q\left(\frac{x_{k,l}}{\rho}\right)\right]<\infty.$$

Now we have

$$\begin{aligned}\sup_{r,s\geq 1,\,\sigma\in P_{r,s}}\frac{1}{\psi_{r,s}}\sum_{k,l\in\sigma}M\left[q\left(\frac{x_{k,l}}{\rho}\right)\right]\quad&\geq\quad\left\{\sup_{i,j\geq 1}\frac{\phi_{r_i s_j}}{\psi_{r_i s_j}}\right\}\left\{\sup_{i,j\geq 1,\,\sigma\in P_{r_i s_j}}\frac{1}{\phi_{r_i s_j}}\sum_{k,l\in\sigma}M\left[q\left(\frac{x_{k,l}}{\rho}\right)\right]\right\}\\&=\quad\infty.\end{aligned}$$

Therefore $(x_{k,l})\notin m''(M,\psi,q)$. As such we arrive at a contradiction.

Hence $\sup_{r,s\geq 1}\frac{\phi_{r,s}}{\psi_{r,s}}<\infty$. □

The following result follows from Theorem 3.4.

Corollary 3.5. *Let M be an Orlicz function. Then $m''(M,\phi,q)=m''(M,\psi,q)$ if and only if $\sup_{r,s\geq 1}\eta_{r,s}<\infty$ and $\sup_{r,s\geq 1}\eta_{r,s}^{-1}<\infty$, where $\eta_{r,s}=\frac{\phi_{r,s}}{\psi_{r,s}}$ for all $r,s=1,2,3,\ldots$.*

Theorem 3.6. *Let M, M_1, M_2 be Orlicz functions satisfying Δ_2- condition. Then*
(i) $m''(M_1,\phi,q)\subseteq m''(M\circ M_1,\phi,q)$,
(ii) $m''(M_1,\phi,q)\cap m''(M_2,\phi,q)\subseteq m''(M_1+M_2,\phi,q)$.

Proof. (i) Let $(x_{k,l})\in m''(M_1,\phi,q)$. Then there exists $\rho>0$ such that

$$\sup_{r,s\geq 1,\,\sigma\in P_{r,s}}\frac{1}{\phi_{r,s}}\sum_{k,l\in\sigma}M_1\left[q\left(\frac{x_{k,l}}{\rho}\right)\right]<\infty.$$

Let $0<\varepsilon<1$ and δ with $0<\delta<1$ such that $M(t)<\varepsilon$ for $0\leq t<\delta$.

Let $y_{k,l}=M_1\left(q\left(\frac{x_{k,l}}{\rho}\right)\right)$ and for any $\sigma\in P_{r,s}$, let

$$\sum_{k,l\in\sigma}M\left(y_{k,l}\right)=\sum_{1}M\left(y_{k,l}\right)+\sum_{2}M\left(y_{k,l}\right),$$

where the first summation is over $y_{k,l}\leq\delta$ and the second is over $y_{k,l}>\delta$.

By the remark we have

$$\sum_1 M(y_{k,l}) \leq M(1) \sum_1 (y_{k,l}) + M(2) \sum_2 (y_{k,l}). \tag{1}$$

For $y_{k,l} > \delta$

$$y_{k,l} < \frac{y_{k,l}}{\delta} \leq 1 + \frac{y_{k,l}}{\delta},$$

since M is non-decreasing and convex, so

$$M(y_{k,l}) < M\left(1 + \frac{y_{k,l}}{\delta}\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(\frac{2y_{k,l}}{\delta}\right).$$

Since M satisfies $\Delta_2 - condition$, so

$$\begin{aligned} M(y_{k,l}) &< \frac{1}{2}K\frac{y_{k,l}}{\delta}M(2) + \frac{1}{2}K\frac{y_{k,l}}{\delta}M(2) \\ &= K\frac{y_{k,l}}{\delta}M(2). \end{aligned}$$

Therefore,

$$\sum_2 M(y_{k,l}) \leq \max\left(1, K\delta^{-1}M(2)\right) \sum_2 (y_{k,l}). \tag{2}$$

By 1 and 2 we have $(x_{k,l}) \in m''(M \circ M_1, \phi, q)$.
Thus $m''(M_1, \phi, q) \subseteq m''(M \circ M_1, \phi, q)$.
(ii) $(x_{k,l}) \in m''(M_1, \phi, q) \cap m''(M_2, \phi, q)$.
Then there exists $\rho > 0$ such that

$$\sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M_1\left[q\left(\frac{x_{k,l}}{\rho}\right)\right] < \infty$$

and

$$\sup_{r,s\geq 1,\, \sigma\in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l\in\sigma} M_2\left[q\left(\frac{x_{k,l}}{\rho}\right)\right] < \infty.$$

The rest of the proof follows from the equality

$$\sum_{k,l\in\sigma} (M_1 + M_2)\left[q\left(\frac{x_{k,l}}{\rho}\right)\right] = \sum_{k,l\in\sigma} M_1\left[q\left(\frac{x_{k,l}}{\rho}\right)\right] + \sum_{k,l\in\sigma} M_2\left[q\left(\frac{x_{k,l}}{\rho}\right)\right].$$

Taking $M_1(x) = x$ in Theorem 3.6 (i) we have the following result. □

Corollary 3.7. *Let M be an Orlicz function satisfying Δ_2- condition. Then*

$$m''(\phi, q) \subseteq m''(M, \phi, q).$$

We now have from Theorem 3.4 and Corollary 3.7:

Corollary 3.8. *Let M be an Orlicz function satisfying Δ_2- condition. Then*

$$m''(\phi, q) \subseteq m''(M, \psi, q) \quad \text{if and only if} \quad \sup_{r,s\geq 1} \frac{\phi_{r,s}}{\psi_{r,s}} < \infty.$$

Theorem 3.9. *$\ell_1^{''}(M,q) \subseteq m^{''}(M,\phi,q) \subseteq \ell_\infty^{''}(M,q)$, where*

$$\ell_1^{''}(M,q) = \left\{ (x_{k,l}) \in w(X) : \sum_{k,l=1,1}^{\infty,\infty} M\left(q\left(\frac{x_{k,l}}{\rho} \right) \right) < \infty, \textit{for some } \rho > 0 \right\}.$$

Proof. Let $(x_{k,l}) \in \ell_1^{''}(M,q)$. Then we have

$$\sum_{k,l=1,1}^{\infty,\infty} M\left[q\left(\frac{x_{k,l}}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0. \tag{3}$$

Since $\left(\phi_{m,n}\right)$ is monotonic increasing, so we have

$$\begin{aligned} \frac{1}{\phi_{r,s}} \sum_{k,l \in \sigma} M\left[q\left(\frac{x_{k,l}}{\rho} \right) \right] &\le \frac{1}{\phi_{1,1}} \sum_{k,l \in \sigma} M\left[q\left(\frac{x_{k,l}}{\rho} \right) \right] \\ &\le \frac{1}{\phi_{1,1}} \sum_{k,l=1,1}^{\infty,\infty} M\left[q\left(\frac{x_{k,l}}{\rho} \right) \right] \\ &< \infty. \end{aligned}$$

Thus,

$$\sup_{r,s \ge 1,\, \sigma \in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l \in \sigma} M\left[q\left(\frac{x_{k,l}}{\rho} \right) \right] < \infty.$$

Hence, $(x_{k,l}) \in m^{''}(M,\phi,q)$.
Therefore $\ell_1^{''}(M,q) \subseteq m^{''}(M,\phi,q)$.
Next let $(x_{k,l}) \in m^{''}(M,\phi,q)$. Then we have

$$\begin{aligned} \sup_{r,s \ge 1,\, \sigma \in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k \in \sigma} M\left[q\left(\frac{x_{k,l}}{\rho} \right) \right] &< \infty, \text{ for some } \rho > 0. \\ &\Rightarrow \sup_{k,l \in \mathbb{N}\times\mathbb{N}} \frac{1}{\phi_{1,1}} M\left[q\left(\frac{x_{k,l}}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0, \\ &\quad \text{(on taking cardinality of } \sigma \text{ to be 1)} \\ &\Rightarrow (x_{k,l}) \in \ell_\infty^{''}(M,q). \end{aligned}$$

Therefore $m^{''}(M,\phi,q) \subseteq \ell_\infty^{''}(M,q)$.
This completes the proof of the theorem. □

Finally we conclude this paper by stating the following theorem. We omit the proof since it involves known arguments.

Theorem 3.10. *Let (X,q) be complete, then $m^{''}(M,\phi,q)$ is also complete.*

If one considers a normed linear space $(X, \|\cdot\|)$ instead of a seminormed space (X,q), then one will get $m^{''}(M,\phi,\|\cdot\|)$, which will be a normed linear space, normed by

$$\left\|(x_{k,l})\right\|_M = \inf \left\{ \rho > 0 : \sup_{r,s \ge 1,\, \sigma \in P_{r,s}} \frac{1}{\phi_{r,s}} \sum_{k,l \in \sigma} M\left(\left\| \frac{x_{k,l}}{\rho} \right\| \right) \le 1 \right\}.$$

The space $m^{''}(M,\phi,\|\cdot\|)$ will be a Banach space if X is a Banach space.

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