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A Study on Motion of a Robot End-Effector Using the Curvature Theory of Dual Unit Hyperbolic Spherical Curves

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Abstract. In this paper we study the motion of a robot end-effector by using the curvature theory of a dual unit hyperbolic spherical curve which corresponds to a timelike ruled surface with timelike ruling generated by a line fixed in the end-effector. In this way, the linear and angular differential properties of the motion of a robot end-effector such as velocities and accelerations which are important information in robot trajectory planning are determined. Moreover, the motion of a robot end-effector which moves on the surface of a right circular hyperboloid of one sheet is examined as a practical example.

1. Introduction

Since robot end-effectors can be used in many areas such as transportation, painting, and medical science, accurate trajectory planning of a robot end-effector becomes an important research area of robotics and engineering. Describing the path of a robot end-effector and determining the linear and angular time-dependent properties are interesting problems in robot trajectory planning. The most common two methods to represent a path for robot trajectory planning are the joint interpolating method and the cartesian interpolating method [3, 13]. But these methods are based on matrix representation that is required intense computation, so they are not efficient for trajectory planning. Ryuh and Pennock proposed a method based on the curvature theory of a ruled surface generated by a line fixed in the end-effector for accurate trajectory planning [15–17]. Their method [16] was the first attempt to apply the curvature theory of a ruled surface to the study of the motion of a robot end-effector.

The research area of the motion of a robot end-effector also attracts the authors' attention in Lorentzian space, for example, Ekici et al. study the motion of a robot end-effector in Lorentzian space by using the curvature theory of a timelike ruled surface with timelike ruling [4].

As a robot end-effector moves on a specified trajectory in Lorentzian space, a line called tool line fixed in the end effector generates a ruled surface. In this paper, it is assumed that this ruled surface is a timelike ruled surface with timelike ruling. We first mention three reference frames which are the tool frame, the surface frame, and the generator trihedron used to study the motion of a robot end-effector in Lorentzian

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space. From E.Study mapping (or transference principle), this ruled surface corresponds to a dual unit hyperbolic spherical curve. Then, we give the well-known dual Darboux frame of the dual curve in dual Lorentzian space. By defining a dual frame called dual tool frame on the robot end-effector and using the relations between the dual Darboux frame and the dual tool frame, the differential properties of the motion of a robot end-effector such as linear and angular velocities, and linear and angular accelerations which are important for robot trajectory planning are determined. Moreover, we give some corollaries for special cases of the motion of a robot end-effector and an example of the study of a robot end-effector which moves on the surface of a right circular hyperboloid of one sheet.

2. Preliminaries

In this section we give a brief summary of basic concepts for the reader who is not familiar with Lorentzian space, dual space, and dual Lorentzian space.

Lorentzian space IR^3_1 is the vector space IR^3 provided with the following Lorentzian inner product

$$\langle a, b\rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3,$$

where $a, b \in IR^3$ [10]. Let $a = (a_1,\ a_2,\ a_3)$ be a vector in IR^3_1. If $\langle a, a\rangle > 0$ or $a = 0$, then a is called spacelike, if $\langle a, a\rangle < 0$, then a is called timelike, if $\langle a, a\rangle = 0$ and $a \neq 0$, then a is called null (lightlike) vector [10]. Let $a = (a_1,\ a_2,\ a_3)$ and $b = (b_1,\ b_2,\ b_3)$ be two vectors in IR^3_1, Lorentzian vector product of a and b can be defined by [20]

$$a \times b = (a_2 b_3 - a_3 b_2,\ a_1 b_3 - a_3 b_1,\ a_2 b_1 - a_1 b_2).$$

Let a and b be two spacelike vectors in IR^3_1 and they span a spacelike vector subspace, then there is a real number $\theta \geq 0$ such that $\langle a, b\rangle = \|a\|\,\|b\| \cos\theta$ and this number is called the spacelike angle between the vectors a and b [14].

A surface in Lorentzian space IR^3_1 is called a timelike surface if the induced metric on the surface is a Lorentzian metric, i.e. the normal vector on the surface is a spacelike vector [21].

The hyperbolic unit sphere can be defined by [22]

$$H^2_0 = \left\{ a = (a_1, a_2, a_3) \in IR^3_1 : \ \langle a, a\rangle = -1 \right\}.$$

A dual number can be defined as an ordered pair of real numbers $\bar{a} = (a, a^*)$, where a and a^* are called real part and dual part of the dual number, respectively. Dual numbers may be formally expressed as $\bar{a} = a + \varepsilon a^*$, where $\varepsilon = (0, 1)$ is called dual unit which satisfies the condition that $\varepsilon^2 = 0$ [23]. The set of all dual numbers is denoted by ID. Two inner operations and equality in ID are defined as follows [1, 7]:

Addition: $(a, a^*) + (b, b^*) = (a + b,\ a^* + b^*)$,

Multiplication: $(a, a^*)\ (b, b^*) = (ab,\ ab^* + a^* b)$,

Equality: $(a, a^*) = (b, b^*) \Leftrightarrow a = b,\ \ a^* = b^*$.

The set ID with the above operations is a commutative ring over the real number field. The function of a dual number $f(\bar{a})$ can be expanded in a Maclaurin series as

$$f(\bar{a}) = f(a + \varepsilon a^*) = f(a) + \varepsilon a^* f'(a),$$

where $f'(a)$ is derivative of $f(a)$ with respect to a [2]. The absolute value of a dual number $\bar{a} = a + \varepsilon a^*$ can be given as [23]

$$|\bar{a}| = |a| + \varepsilon \frac{a}{|a|} a^*, \quad (a \neq 0).$$

Similar to the dual numbers, a dual vector can be defined as an ordered pair of two real vectors (a, a^*). Dual vectors can also be expressed as $\tilde{a} = a + \varepsilon a^*$, where $a, a^* \in IR^3$ and $\varepsilon^2 = 0$ [18]. The set of all dual vectors is a module over the ring ID and is called dual space or ID-module, and can be denoted by ID^3.

Let $\tilde{a}= a+\varepsilon a^*$ and $\tilde{b}= b+\varepsilon b^*$ be two dual vectors in ID^3, the dual Lorentzian inner product and the dual Lorentzian vector product can be defined, respectively, as [22]

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \langle a, b\rangle + \varepsilon \left(\langle a, b^*\rangle + \langle a^*, b\rangle\right)$$

and

$$\tilde{a} \times \tilde{b} = a \times b + \varepsilon \left(a \times b^* + a^* \times b\right),$$

where $\langle \ , \ \rangle$ and $\times$ are the Lorentzian inner product and the Lorentzian vector product, respectively. A dual vector $\tilde{a}= a+\varepsilon a^*$ is said to be timelike (resp., spacelike, lightlike) if a is timelike (resp., spacelike, lightlike) [22]. The set of all dual Lorentzian vectors is called dual Lorentzian space and can be denoted by ID^3_1 [22]. The norm of a dual vector $\tilde{\mathbf{a}}$ is given by [7, 23]

$$\|\tilde{a}\| = \|a\| + \varepsilon \frac{\langle a, a^*\rangle}{\|a\|}, \quad (\|a\| \neq 0).$$

Let $\tilde{a} = a + \varepsilon a^* \in ID^3_1$. $\tilde{a}$ is said to be dual unit timelike (resp. spacelike) vector if the vectors a and a^* satisfy the following equations [12]:

$$< a, a >= -1 \ (resp. \ < a, a >= 1), \ \ < a , a^* > = 0.$$

The set of all dual timelike unit vectors is called dual hyperbolic unit sphere and it can be denoted by $\tilde{H}^2_0$ [22].

Let $\tilde{a}$ and $\tilde{b}$ be two spacelike vectors in ID^3_1 that span a dual spacelike vector subspace. Then the dual angle between $\tilde{a}$ and $\tilde{b}$ is defined by $< \tilde{a}, \tilde{b} >= \|\tilde{a}\| \left\|\tilde{b}\right\| \cos\bar{\theta}$. The dual number $\bar{\theta} = \theta + \varepsilon\theta^*$ is called the dual spacelike angle [5].

3. Reference Frames of a Robot End-Effector in Lorentzian Space

In this section we introduce three reference frames, namely, a tool frame, a surface frame and a generator trihedron. These frames were described in detail and used to study of the motion of a robot end-effector in real space by Ryuh and Pennock [16, 17] and in Lorentzian space by Ekici et. al. [4]. Before these papers, Karger and Novak [8] introduced the generator trihedron as "the Frenet frame of directing cone of a ruled surface". Moreover, Onder and Ugurlu [11] gave the generator trihedron as "Frenet frame" for timelike ruled surface with timelike ruling. This section sheds light on the geometrical meaning of the dual method to be used in section 5.

A robot trajectory can be defined as the path of a point fixed in a robot end-effector and the orientation of the robot end-effector along the path [17]. The point is called the tool center point and denoted by TCP. The tool frame strictly attached to the robot end-effector consists of three mutually orthonormal vectors: the orientation vector O, the approach vector A, and the normal vector N (see Figure 1). The tool center point can be chosen to be the origin of the tool frame [15]. Since it is assumed that the ruled surface is a timelike ruled surface with timelike ruling, O is a timelike vector and A and N are spacelike vectors.

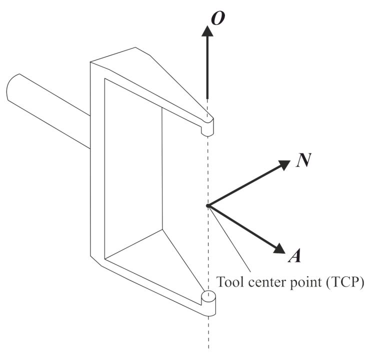

Figure 1 A robot end-effector and its tool frame.

As a robot end-effector moves on a trajectory in Lorentzian space IR_1^3, a line called tool line which passes through TCP and whose direction vector is parallel to the orientation vector O, generates a ruled surface [4]. In this study, this ruled surface is assumed to be a timelike ruled surface with timelike ruling and can be expressed by

$$X(t, v) = \alpha(t) + v\, R(t),$$

where $\alpha(t)$ which is the specified trajectory of the robot end-effector is the directrix of the ruled surface, $R(t)$ called ruling is a vector of constant magnitude parallel to the orientation vector O, and t is the parameter of time. For simplicity in the formulations, the normalized parameter s which is the arc-length parameter of the spherical image curve of R can be used instead of the time parameter t and it can be found as [7]

$$s(t) = \int_0^t \left\| \frac{dR}{dt} \right\| dt.$$

As a robot end-effector moves on a trajectory, the approach vector A may not be always perpendicular to the ruled surface. As shown in Figure 2, there may be an angle between the approach vector A and the surface normal vector which may be denoted by S_n. This angle is called spin angle and denoted by η [4].

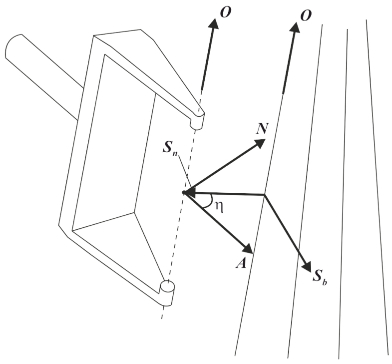

Figure 2 The spin angle.

Now, we introduce the surface frame $\{O,\ S_n,\ S_b\}$ to describe the orientation of the tool frame relative to the ruled surface. The spacelike surface normal vector S_n can be expressed as

$$S_n = \left. \frac{X_v \times X_s}{\|X_v \times X_s\|} \right|_{v=0},$$

where X_v and X_s are derivations of X with respect to v and s, respectively [4]. The spacelike surface binormal vector can also be given as

$$S_b = -O \times S_n.$$

The tool frame relative to the surface frame can be expressed as

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\eta & -\sin\eta \\ 0 & \sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix}, \tag{1}$$

where η called spacelike spin angle is the angle between A and S_n [4].

The generator trihedron can be defined on the line of striction of the ruled surface which can be expressed as $c(s) = \alpha(s) - \mu(s)R(s)$, where $\mu = \langle \alpha', R' \rangle$, and it consists of three orthonormal vectors: the timelike generator vector e, the spacelike central normal vector t, and the spacelike central tangent vector g. These vectors can be given as

$$e = \frac{R}{\|R\|},\ t = R',\ g = -e \times t,$$

respectively, where the prime indicates differentiation with respect to s which is the arc-length parameter of spherical image curve of R [11]. The distance from the line of striction to the directrix is $\mu\|R\|$. The orientation of the surface frame relative to the generator trihedron can be given in matrix form as

$$\begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\sigma & -\sin\sigma \\ 0 & \sin\sigma & \cos\sigma \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix}, \tag{2}$$

where σ is a spacelike angle between S_n and t [4]. By substituting equation (1) into equation (2), the relation between the tool frame and the generator trihedron can be found as

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix},$$

where

$$\varphi = \eta + \sigma. \tag{3}$$

The relationships between three reference frames which are the tool frame, the surface frame and the generator trihedron are shown in Figure 3, when a robot end-effector moves on a specified trajectory.

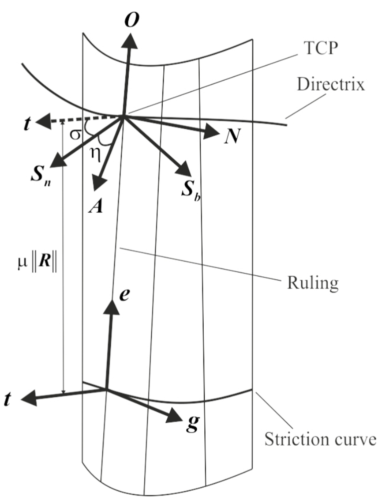

Figure 3 Three reference frames.

4. Dual Darboux Frame of Dual Unit Hyperbolic Spherical Curves

There exists one-to-one correspondence called E.Study mapping or transference principle between dual unit vectors in dual space ID^3 and the directed lines in line space IR^3 [9, 19]. By taking 3-dimensional Lorentzian space IR^3_1 instead of IR^3, this correspondence can be stated as follows: a point on the dual hyperbolic unit sphere $\tilde{H}^2_0$ corresponds to the directed timelike line in Lorentzian space IR^3_1 [22]. Then, the dual unit spherical curve lying fully on $\tilde{H}^2_0$ represents a timelike ruled surface with timelike ruling.

The timelike ruled surface with timelike ruling on which a robot end-effector moves can be given by the equation

$$X(s, v) = \alpha(s) + v\, R(s),$$

where $\alpha(s)$ is the trajectory of the robot end-effector, $R(s)$ which is a vector of constant magnitude is the ruling of the ruled surface and s is the normalized parameter discussed in Section 3. The ruled surface corresponds to a dual curve on dual unit hyperbolic sphere and it can be expressed by $\tilde{e}(s) = e(s) + \varepsilon e^*(s)$, where $e(s)$ is the timelike generator vector of the ruled surface as defined in Section 3 and e^* is called moment vector of e and it can be given by $e^* = c \times e$, where c is the line of striction of the ruled surface.

Now, we give the dual Darboux frame (or dual geodesic frame) of the dual unit hyperbolic spherical curve which was described by Onder and Ugurlu [12] in detail. The dual Darboux frame consists of three orthonormal dual unit vectors. The first dual unit vector is the dual curve itself, i.e. $\tilde{e}(s)$. The dual arc-length parameter of the dual curve $\tilde{e}$ is given by [12]

$$\bar{s} = \int_0^s \| \tilde{e}'(u) \| du = \int_0^s (1 - \varepsilon\Delta) du = s - \varepsilon \int_0^s \Delta du, \tag{4}$$

where $\Delta = \det(c', e, t)$. By differentiating (4), we have $\bar{s}' = 1 - \varepsilon\Delta$. The second dual unit vector of the dual Darboux frame can be given by [12]

$$\tilde{t} = \frac{d\tilde{e}}{d\bar{s}} = \frac{\tilde{e}'}{\bar{s}'} = \frac{\tilde{e}'}{1 - \varepsilon\Delta} = t + \varepsilon(c \times t),$$

where t is the spacelike central normal vector. The third dual unit vector $\tilde{g}$ can also be given as [12]

$$\tilde{g} = -\tilde{e} \times \tilde{t} = g + \varepsilon(c \times g),$$

where g is the spacelike central tangent vector. Note that $\tilde{e}$ is a dual timelike vector and $\tilde{t}$ and $\tilde{g}$ are dual spacelike vectors. The derivation formulae of the dual Darboux frame can be expressed in matrix form

$$\frac{d}{d\bar{s}} \begin{bmatrix} \tilde{e} \\ \tilde{t} \\ \tilde{g} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \bar{\gamma} \\ 0 & -\bar{\gamma} & 0 \end{bmatrix} \begin{bmatrix} \tilde{e} \\ \tilde{t} \\ \tilde{g} \end{bmatrix}, \tag{5}$$

where $\bar{\gamma}$ is called dual geodesic curvature [12]. The dual Darboux instantaneous rotation vector of the dual Darboux frame can also be found as [12]

$$\tilde{w}_e = -\bar{\gamma}\tilde{e} - \tilde{g}.$$

5. Differential Properties of the Motion of a Robot End-Effector

In this section we determine the linear and angular differential properties of the motion of a robot end-effector by using the curvature theory of a dual unit hyperbolic spherical curve corresponds to the ruled surface generated by a line fixed in the robot end-effector.

First, we define a dual frame called dual tool frame which can be defined by three orthonormal dual unit vectors correspond to three lines pass through the tool center point of the robot end-effector. These lines may be called the orientation line, the approach line and the normal line, and their direction vectors are the timelike orientation vector O, the spacelike approach vector A and the spacelike normal vector N, respectively (see Figure 4). The dual unit vectors correspond to the orientation line, the approach line and the normal line can be denoted by $\tilde{O}$, $\tilde{A}$ and $\tilde{N}$ which may be called dual timelike orientation vector, dual spacelike approach vector and dual spacelike normal vector, respectively.

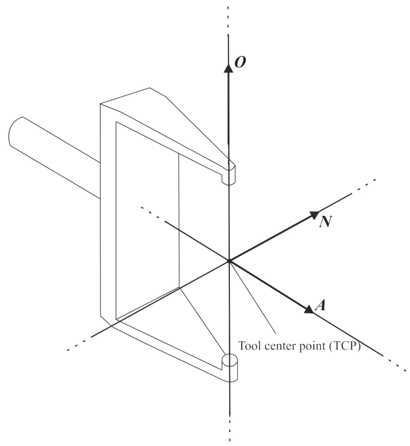

Figure 4 The dual tool frame of a robot end-effector.

Let the dual spacelike angle between the dual unit spacelike vectors $\tilde{A}$ and $\tilde{t}$ be $\bar{\varphi} = \varphi + \varepsilon\, \varphi^*$, where φ and φ^* are spacelike angle and Lorentzian shortest distance between the lines which correspond to the dual unit vectors $\tilde{A}$ and $\tilde{t}$, respectively. From Section 3, we know that φ and φ^* are the real spacelike angle in equation (3) and the distance from the line of striction to the directrix which equals to $\mu\|R\|$, respectively. The relations between the dual Darboux frame and the dual tool frame can be given in matrix form as

$$\begin{bmatrix} \tilde{O} \\ \tilde{A} \\ \tilde{N} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\bar{\varphi} & -\sin\bar{\varphi} \\ 0 & \sin\bar{\varphi} & \cos\bar{\varphi} \end{bmatrix} \begin{bmatrix} \tilde{e} \\ \tilde{t} \\ \tilde{g} \end{bmatrix}. \tag{6}$$

By differentiating equation (6) and substituting equation (5) into the result, we have

$$\frac{d}{d\bar{s}} \begin{bmatrix} \tilde{O} \\ \tilde{A} \\ \tilde{N} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cos\bar{\varphi} & -\bar{\delta}\sin\bar{\varphi} & -\bar{\delta}\cos\bar{\varphi} \\ \sin\bar{\varphi} & \bar{\delta}\cos\bar{\varphi} & -\bar{\delta}\sin\bar{\varphi} \end{bmatrix} \begin{bmatrix} \tilde{e} \\ \tilde{t} \\ \tilde{g} \end{bmatrix},$$

where $\bar{\delta} = \bar{\varphi}' - \bar{\gamma}$ and $\bar{s}$ is the dual arc-length parameter of $\tilde{e}$(or $\tilde{O}$). By using equation (6), derivation formulas of the dual tool frame can be obtained in terms of itself as

$$\frac{d}{d\bar{s}} \begin{bmatrix} \tilde{O} \\ \tilde{A} \\ \tilde{N} \end{bmatrix} = \begin{bmatrix} 0 & \cos\bar{\varphi} & \sin\bar{\varphi} \\ \cos\bar{\varphi} & 0 & -\bar{\delta} \\ \sin\bar{\varphi} & \bar{\delta} & 0 \end{bmatrix} \begin{bmatrix} \tilde{O} \\ \tilde{A} \\ \tilde{N} \end{bmatrix}.$$

Then the dual instantaneous rotation vector of the dual tool frame can be found as

$$\tilde{w}_O = \bar{\delta}\, \tilde{O} + \sin\bar{\varphi}\, \tilde{A} - \cos\bar{\varphi}\, \tilde{N}.$$

By using equation (6), the dual instantaneous rotation vector of the dual tool frame can be expressed in terms of the dual Darboux frame as

$$\tilde{w}_O = \bar{\delta}\, \tilde{e} - \tilde{g}. \tag{7}$$

We also have the relation between the dual instantaneous rotation vector of the dual tool frame and the dual Darboux vector of the dual Darboux frame as

$$\tilde{w}_O = \bar{\varphi}'\, \tilde{e} + \tilde{w}_e.$$

It is seen that the dual vector $\tilde{w}_O = w_O + \varepsilon w_O^*$ is similar to the dual Pfaff vector in dual spherical motion [6] and it can be considered as dual velocity vector of the robot end-effector. The dual tool frame rotates along the axis $\frac{\tilde{w}_O}{\|\tilde{w}_O\|}$ with the dual angle $\|\tilde{w}_O\|$. This dual Lorentzian motion includes both rotational and translational motion in real Lorentzian space. w_O and w_O^* correspond to the instantaneous angular velocity vector and the instantaneous linear velocity vector, respectively. By separating equation (7) into the real and dual parts, these vectors can be obtained as

$$w_O = \delta e - g \tag{8}$$

and

$$w_O^* = \delta e^* + \delta^* e - g^*, \tag{9}$$

respectively. By differentiating equation (7) and using equation (5), the derivative of the dual instantaneous rotation vector of the dual tool frame can be obtained as

$$\tilde{w}_O^{'} = \bar{\delta}' \, \tilde{e} + \bar{\varphi}' \, \tilde{t}. \tag{10}$$

By separating equation (10) into real and dual parts, the instantaneous angular acceleration vector and the instantaneous translational acceleration vector can be found as

$$w_O^{'} = \delta' \, e + \varphi' \, t \tag{11}$$

and

$$w_O^{*'} = \delta' \, e^* + \delta^{*'} e + \varphi' \, t^* + \varphi^{*'} t, \tag{12}$$

respectively. The linear and angular differential properties of the motion of a robot end-effector which are velocities and accelerations given in equations (8), (9), (11) and (12) are found in terms of the arc-length parameter of the spherical image curve of R, i.e. s. In order to determine the time dependent differential properties of the motion, these vectors should be associated with t which is the parameter of time. Now, we can give the following corollaries concerning the time dependent differential properties of the motion of a robot end-effector which moves on a specified trajectory in Lorentzian space.

Corollary 5.1. *Let a robot end-effector moves on a timelike ruled surface with timelike ruling with the spacelike spin angle η in Lorentzian space. Then the linear and angular velocities of the robot end-effector can be given, respectively, as*

$$v_L = w_O^* \, \dot{s} \tag{13}$$

and

$$v_A = w_O \, \dot{s}, \tag{14}$$

where w_O^* and w_O are given by equations (9) *and* (8), *respectively, and the dot indicates differentiation with respect to time, i.e.* $\dot{s} = \frac{ds}{dt}$.

Corollary 5.2. *Let a robot end-effector moves on a timelike ruled surface with timelike ruling with the spacelike spin angle η in Lorentzian space. Then the linear and angular accelerations of the robot end-effector can be given, respectively, as*

$$a_L = w_O^* \, \ddot{s} + w_O^{*'} \dot{s}^2 \tag{15}$$

and

$$a_A = w_O \, \ddot{s} + w_O^{'} \dot{s}^2 \tag{16}$$

where $w_O^{*'}$ and $w_O^{'}$ are given by equations (12) *and* (11), *respectively, and* $\ddot{s} = \frac{d^2 s}{dt^2}$.

Now we discuss some special cases that one can encounter in the study of the motion of a robot end-effector in Lorentzian space. For example, a robot end-effector can move on a specified trajectory such that it is always perpendicular to the ruled surface, namely, the spacelike spin angle η can be equal to zero. More general, the spacelike spin angle η may be constant during the motion. Furthermore, a specified trajectory which a robot end-effector follows may be the line of striction of the ruled surface, in other words, the directrix and the line of striction of the ruled surface can be the same curve. For these cases, we can give the following corollaries.

Corollary 5.3. *Let a robot end-effector moves on a timelike ruled surface with timelike ruling with the spacelike spin angle η in Lorentzian space. If the spacelike spin angle η is constant, then the linear and angular velocities of the robot end-effector can be expressed as*

$$v_L = ((\sigma' - \gamma)e^* + \delta^* e - g^*)\ \dot{s}$$

and

$$v_A = ((\sigma' - \gamma)e - g)\ \dot{s},$$

respectively.

Corollary 5.4. *Let a robot end-effector moves on a timelike ruled surface with timelike ruling with the spacelike spin angle η in Lorentzian space. If the spacelike spin angle η is constant, then the linear and angular accelerations of the robot end-effector can be expressed as*

$$a_L = ((\sigma' - \gamma)e^* + \delta^* e - g^*)\ \ddot{s} + ((\sigma'' - \gamma')\ e^* + \delta^{*'} e + \sigma'\ t^* + \varphi^{*'} t)\dot{s}^2$$

and

$$a_A = ((\sigma' - \gamma)e - g)\ \ddot{s} + ((\sigma'' - \gamma')\ e + \varphi'\ t)\ \dot{s}^2,$$

respectively.

Corollary 5.5. *Let a robot end-effector moves on a timelike ruled surface with timelike ruling with the spacelike spin angle η in Lorentzian space. If the specified trajectory is also the line of striction of the ruled surface which robot end-effector moves on, then the linear and angular velocities of the robot end-effector can be expressed as*

$$v_L = ((\eta' - \gamma)e^* - \gamma^* e - g^*)\ \dot{s}$$

and

$$v_A = ((\eta' - \gamma)e - g)\ \dot{s},$$

respectively.

Corollary 5.6. *Let a robot end-effector moves on a timelike ruled surface with timelike ruling with the spacelike spin angle η in Lorentzian space. If the specified trajectory is also the line of striction of the ruled surface which robot end-effector moves on, then the linear and angular accelerations of the robot end-effector can be expressed as*

$$a_L = ((\eta' - \gamma)e^* - \gamma^* e - g^*)\ \ddot{s} + ((\eta'' - \gamma')\ e^* - \gamma^{*'} e + \eta'\ t^*)\dot{s}^2$$

and

$$a_A = ((\eta' - \gamma)e - g)\ \ddot{s} + ((\eta'' - \gamma')\ e + \eta'\ t)\ \dot{s}^2,$$

respectively.

6. Example

Let a robot end-effector moves on the surface of a right circular hyperboloid of one sheet (see Figure 5) given by the following equation

$$X(t,v) = (cv,\ \sin t + v\cos t,\ \cos t - v\sin t),$$

with a spin angle η, where $|c| > 1$ and t is the parameter of time. The directrix and the ruling of the hyperboloid of one sheet are $\alpha(t) = (0,\ \sin t,\ \cos t)$ and $R(t) = (c,\ \cos t,\ -\sin t)$, respectively. Since $\mu = \langle \alpha', R' \rangle = 0$, the directrix is also the line of striction, i.e. $c(t) = \alpha(t)$. A dual curve corresponds to the hyperboloid of one sheet and it can be expressed as

$$\tilde{e}(s) = \frac{1}{m}\left[(c,\ \cos s,\ -\sin s)\ + \varepsilon\,(-1,\ -c\cos s,\ c\sin s)\right],$$

where $m = \sqrt{c^2-1}$ and s is the arc-length of the spherical image curve of R. The second and third elements of the dual Darboux frame can be found as

$$\tilde{t}(s) = (0,\ -\sin s,\ -\cos s),$$

and

$$\tilde{g}(s) = \frac{1}{m}\left[(1, c\cos s,\ -c\sin s)\ + \varepsilon\,(-c, -\cos s,\ \sin s)\right],$$

respectively. By using equation (5), it is seen that the dual geodesic curvature can be found as $\bar{\gamma} = \gamma + \varepsilon\gamma^* = -\frac{c}{\sqrt{c^2-1}} + \varepsilon\frac{\sqrt{c^2-1}+c^2}{c^2-1}$. Since the directrix is also the line of striction, the distance between the directrix and the line of striction equals to zero, i.e. $\varphi^* = 0$, and the spacelike normal vector S_n and the spacelike central normal vector t are the same vectors, i.e. $\sigma = 0$. Thus, we have $\bar{\varphi} = \eta$ and $\bar{\delta} = \left(\eta' + \frac{c}{\sqrt{c^2-1}}\right) - \varepsilon\left(\frac{\sqrt{c^2-1}+c^2}{c^2-1}\right)$. The dual instantaneous rotation vector of the dual tool frame can be found as

$$\tilde{w}_O = w_O + \varepsilon w_O^* = \frac{1}{m}\left(\left(\eta' + \frac{c}{\sqrt{c^2-1}}\right)(c,\ \cos s,\ -\sin s) + (1,\ c\cos s,\ -c\sin s)\right)$$

$$+\varepsilon\frac{1}{m}\left(\left(\eta' + \frac{c}{\sqrt{c^2-1}}\right)(-1,\ -c\cos s,\ c\sin s) - \left(\frac{\sqrt{c^2-1}+c^2}{c^2-1}\right)(c,\ \cos s,\ -\sin s) - (-c, -\cos s, \sin s)\right),$$

where $m = \sqrt{c^2-1}$. The linear and angular velocities of the robot end-effector can be obtained by substituting w_O and w_O^* into equations (13) and (14), respectively. By differentiating the dual instantaneous rotation vector, we have

$$\tilde{w}_O' = w_O' + \varepsilon w_O^{*'} = \left(\frac{\eta'' c}{m},\ \frac{\eta''}{m}\cos s - \eta'\sin s,\ -\frac{\eta''}{m}\sin s - \eta'\cos s\right) + \varepsilon\frac{\eta''}{m}(-1,\ -c\cos s,\ c\sin s).$$

Finally, the linear and angular accelerations of the robot end-effector can be obtained by substituting w_O' and $w_O^{*'}$ into equations (15) and (16), respectively.

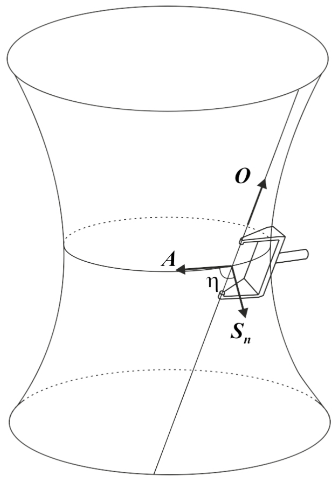

Figure 5 A robot end-effector which moves on the surface of a right circular hyperboloid of one sheet.

7. Conclusions

This paper presents a dual method to study the motion of a robot end-effector which moves on a specified trajectory in Lorentzian space IR^3_1. By using the dual curvature theory of a dual unit hyperbolic spherical curve, corresponding to the ruled surface generated by a line fixed in the robot end-effector in Lorentzian space, the linear and angular differential properties which are velocities and accelerations of the motion of a robot end-effector can be determined. This dual method provides simplicity in expression and it is believed that it reduces computation time in computer programming. It is hoped that this paper can contribute to the research area of robot trajectory planning.

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