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Some Remarks on Topologized Groups

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Abstract. We define and study classes of topologized groups, in particular paratopological groups, related to precompactness and to several boundedness properties of topological groups and uniform spaces.

1. Introduction

In the last two decades a considerable number of results concerning various kinds of boundedness in topological groups appeared in the literature: M-boundedness, H-boundedness, R-boundedness, S-boundedness. These classes of groups have been defined by Kočinac in 1998, and the class of M-bounded groups was independently introduced by Okunev and Tkachenko in 1998 under the name o-bounded groups. For more information about these classes see, for example, [3, 4, 9–11, 16, 19, 23]. In this paper we introduce in a similar way several properties in the class of topologized groups. Our study is mainly related to paratopological groups and differences between these properties in topological and paratopological groups. For more details on paratopological groups see [2, 20–22] and also the papers [1, 5, 8, 13–15, 17, 18].

2. Preliminaries and Definitions

Let X be a nonempty set. A family $\mathbb{U}$ of subsets of $X \times X$ satisfying conditions

(U1) each $U \in \mathbb{U}$ contains the diagonal $\Delta_X = \{(x, x) : x \in X\}$ of X;

(U2) if $U, V \in \mathbb{U}$, then $U \cap V \in \mathbb{U}$;

(U3) if $U \in \mathbb{U}$ and $V \supset U$, then $V \in \mathbb{U}$

is called a *pre-uniformity* on X.

A pre-uniformity $\mathbb{U}$ on X is called a *quasi-uniformity* if $\mathbb{U}$ satisfies also

(U4) for each $U \in \mathbb{U}$ there is $V \in \mathbb{U}$ with $V \circ V \subset U$, where $V \circ V = \{(x, y) \in X \times X : \exists z \in V$ such that $(x, z) \in V, (z, y) \in V\}$.

A quasi-uniformity $\mathbb{U}$ on X is a *uniformity* if

(U5) for each $U \in \mathbb{U}$, $U^{-1} := \{(x, y) \in X \times X : (y, x) \in U\} \in \mathbb{U}$.

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Elements of the pre-uniformity $\mathbb{U}$ are called *entourages* (of the diagonal). For any entourage $U \in \mathbb{U}$, a point $x \in X$ and a subset A of X one defines the set

$$U[x] := \{y \in X : (x, y) \in U\}$$

called the *U-ball with the center x*, and the set

$$U[A] := \bigcup_{a \in A} U[a]$$

called the *U-neighborhood* of A. We refer the reader to [6, 7] for more information on pre-uniform spaces.

Definition 2.1. A pre-uniform space $(X, \mathbb{U})$ is called

(1) *Menger bounded* (or M-*bounded* for short) if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of entourages there is a sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that $X = \bigcup_{n \in \mathbb{N}} U_n[F_n]$;

(2) *Hurewicz bounded* (or H-*bounded*) if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of entourages there is a sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many $U_n[F_n]$;

(3) *Rothberger bounded* (or R-*bounded*) if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of entourages there is a sequence $\langle x_n : n \in \mathbb{N}\rangle$ of elements of X such that $X = \bigcup_{n \in \mathbb{N}} U_n[x_n]$;

(4) *Scheepers bounded* (or S-*bounded*) if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of entourages there is a sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of X such that each finite subset $A \subset X$ is contained in some set $U_n[F_n]$.

Evidently,

$$\text{H–bounded} \implies \text{S–bounded} \implies \text{M–bounded} \impliedby \text{R–bounded}.$$

A group $(G, \cdot)$ equipped with a topology τ is called a *topologized group*. By $\mathcal{N}_{e_G}$ we denote a local base at the identity element $e_G \in G$ with respect to the topology τ. By $G^- = (G, \cdot, \tau^{-1})$ we denote the topologized group (called the *conjugate of* G) in which $\{U^{-1} : U \in \mathcal{N}_{e_G}\}$ is a neighbourhood system at e_{G^-}. If $(G, \cdot, \tau)$ is a topologized group, then we often say simply "a topologized group G".

Each topologized group G carries two natural pre-uniformities: $\mathcal{L}$ generated by

$$\mathcal{B}_{\mathcal{L}} = \{L_U : U \in \mathcal{N}_{e_G}\}, \text{ where } L_U = \{(x, y) \in G \times G : x \in y \cdot U\}$$

and $\mathcal{R}$ generated by

$$\mathcal{B}_{\mathcal{R}} = \{R_U : U \in \mathcal{N}_e\}, \text{ where } R_U = \{(x, y) \in G \times G : x \in U \cdot y\}.$$

For more information on canonical pre-uniformities on topologized groups see [6, 7].

Definition 2.2. A topologized group $(G, \cdot, \tau)$ is said to be M-*bounded* (H-*bounded*, R-*bounded*, S-*bounded*, respectively) if the left pre-uniformity $\mathcal{L}$ on G is M-bounded (H-bounded, R-bounded, S-bounded, respectively).

A topologized group $(G, \cdot, \tau)$ is a *semitopological group* (resp. *paratopological group*) if the group operation $(x, y) \mapsto x \cdot y$ from $G \times G \to G$ is a separately (resp. jointly) continuous mapping. A paratopological group G in which the mapping $x \mapsto x^{-1}$ from G to G is continuous is called a *topological group*.

For a topologized group $(G, \cdot, \tau)$ by G^* (respectively G_*) we denote the group G equipped with the weakest (respectively the strongest) group topology τ^* (respectively τ_*) such that the identity mapping $(G^*, \tau^*) \to (G, \tau)$ (respectively $(G, \tau) \to (G_*, \tau_*)$) is continuous. It is known that for a paratopological group $(G, \cdot, \tau)$, G^* is the topological group $(G, \cdot, \tau \vee \tau^{-1})$ (called the *topological group associated to* G). Observe that the local base at the identity element $e_{G^*} \in G^*$ is the collection $\{U \cap U^{-1} : U \in \mathcal{N}_{e_G}\}$. For an abelian (more generally, 2-oscillating [5]) paratopological group $(G, \cdot, \tau)$ the topology of G_* (called the *group reflection* of G) is the topology $\tau \wedge \tau^{-1}$. For non-abelian paratopological groups G the topology of G_* does not coincide with $\tau \wedge \tau^{-1}$, in general (see [5, Example 1]).

Definition 2.3. A topologized group $(G, \cdot, \tau)$ is said to be

- M^*–*bounded* (H^*–*bounded*, R^*–*bounded*, S^*–*bounded*) if the topological group G^* is M–bounded (H–bounded, R–bounded, S–bounded);
- M_*–*bounded* (H_*–*bounded*, R_*–*bounded*, S_*–*bounded*) if the topological group G_* is M–bounded (H–bounded, R–bounded, S–bounded).

From the definitions above and the fact that for any topologized group G the identity mappings $G^* \to G$ and $G \to G_*$ are (uniformly) continuous, we have the following diagram:

$$\begin{array}{ccccc}
\mathsf{H}^*\text{–bounded} & \Longrightarrow & \mathsf{H}\text{–bounded} & \Longrightarrow & \mathsf{H}_*\text{–bounded} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\mathsf{S}^*\text{–bounded} & \Longrightarrow & \mathsf{S}\text{–bounded} & \Longrightarrow & \mathsf{S}_*\text{–bounded} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\mathsf{M}^*\text{–bounded} & \Longrightarrow & \mathsf{M}\text{–bounded} & \Longrightarrow & \mathsf{M}_*\text{–bounded} \\
\Uparrow & & \Uparrow & & \Uparrow \\
\mathsf{R}^*\text{–bounded} & \Longrightarrow & \mathsf{R}\text{–bounded} & \Longrightarrow & \mathsf{R}_*\text{–bounded}
\end{array}$$

In the next sections we will give some examples related to this diagram.

3. First Facts

Recall that a paratopological group G is said to be *precompact* (*pre-Lindelöf*, also called *ω-narrow* in [2, 22]) if for every neighborhood U of the neutral element of G, there is a finite (countable) subset F of G such that $F \cdot U = G = U \cdot F$. A subset A of G is precompact in G if for any $U \in \mathcal{N}_{e_G}$ there is a finite set $E \subset A$ such that $A \subset E \cdot U$. G is σ-precompact if it is a countable union of precompact sets.

Evidently, that each (σ-)precompact (in particular, every σ-compact) paratopological group is H-bounded, and each M-bounded paratopological group is pre-Lindelöf.

Example 3.1. The Sorgenfrey line $\mathbb{S}$ is an example of H-bounded (hence M-bounded) paratopological additive group which is not Menger (see [11] for the definition). This paratopological group is not R-bounded, too. (Take the sequence $\langle [0, 1/2^n) : n \in \mathbb{N}\rangle$ of neighbourhoods of the identity element $0 \in \mathbb{S}$.) The unit interval $\mathbb{I} = [0, 1]$ is precompact subset of $\mathbb{S}$, and its square is precompact in $\mathbb{S}^2$ [12]. The paratopological group $\mathbb{Q}_S$ of rational numbers with the topology inherited from $\mathbb{S}$ is H-bounded and R-bounded.

Let $\mathbb{K}$ denote the unit Sorgenfrey circle group, i.e. the unit circle $\mathbb{T}$ equipped with the Sorgenfrey topology: a local base at the neutral element $1 \in \mathbb{K}$ consists of the sets $U_n = \{e^{i\pi\varphi} : 0 \le \varphi < 1/n\}$. The (Abelian Tychonoff) paratopological multiplicative group $\mathbb{K}$ is precompact and thus H-bounded and M-bounded. The same is true for the group $\mathbb{K}^{\mathfrak{c}^+}$ ([22]), where $\mathfrak{c} = 2^\omega$ is the continuum.

As we have already mentioned every M^*–bounded (H^*–bounded) paratopological group is M-bounded (H-bounded). The converse is not true. The topological groups $\mathbb{S}^*$ associated to $\mathbb{S}$ and $\mathbb{K}^*$ associated to $\mathbb{K}$ are discrete and uncountable and so they are not M-bounded.

Recall that a topological space X is a *P-space* if the intersection of any countable family of open sets is an open set.

Proposition 3.2. *Every pre-Lindelöf paratopological group G which is a P-space is* R*-bounded.*

Proof. Let $\langle U_n : n \in \mathbb{N}\rangle$ be a sequence of neighbourhoods of $e_G \in G$. As G is a P-space, $U = \bigcap_{n\in\mathbb{N}} U_n$ is an open neighbourhood of e_G. Because of pre-Lindelöfness of G there is a countable set $A = \{a_n : n \in \mathbb{N}\} \subset G$ such that $G = A \cdot U$. Then we have $G = \bigcup_{n\in\mathbb{N}} a_n \cdot U_n$ so that the sequence $\langle a_n : n \in \mathbb{N}\rangle$ witnesses for $\langle U_n : n \in \mathbb{N}\rangle$ that G is R-bounded. □

4. Subgroups and Subsets

It is known that subgroups of M-bounded topological groups are also M-bounded [4]. But it is not the case for M-bounded paratopological groups.

Example 4.1. According to a result by Banakh and Ravsky every discrete Abelian group can be embedded as a subgroup into a precompact Hausdorff paratopological group [5, Corollary 5].

Here are two concrete examples.

The Sorgenfrey plane $\mathbb{S}^2$ is an M-bounded paratopological group. However, its subgroup $A = \{(x, -x) : x \in \mathbb{R}\}$ is discrete and uncountable, so that it cannot be M-bounded.

The paratopological group $\mathbb{K}^2$ is H-bounded (hence M-bounded), being precompact. But its uncountable subgroup $D = \{(x, x^{-1}) : x \in \mathbb{K}\}$ is closed and discrete and thus it is not M-bounded.

Observe that the group reflection of $\mathbb{S}^2$ is the Euclidean plane $\mathbb{R}^2$, and $A_* = A$, so that the group A_* with the topology inherited from $\mathbb{S}^2_* = \mathbb{R}^2$ is an M-bounded topological group since it is topologically isomorphic to $\mathbb{R}$.

Theorem 4.2. *If a paratopological group $(G, \cdot, \tau)$ is M-bounded and H is a dense subgroup of G, then H is M-bounded.*

Proof. Let $\langle O_n : n \in \mathbb{N}\rangle$ be a sequence of elements of $e_H \in H$. For each $n \in \mathbb{N}$ pick $U_n \in \mathcal{N}_{e_G}$ such that $O_n = U_n \cap H$; then choose for each n an $W_n \in \mathcal{N}_{e_G}$ such that $W_n \cdot W_n \subset U_n$. As G is M-bounded there is a sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of G such that $\bigcup_{n\in\mathbb{N}} F_n \cdot W_n = G$. Since H is dense in G, for each $n \in \mathbb{N}$ and each $x \in F_n$ there is $a_x \in H \cap W_n \cdot x^{-1}$. Let $A_n = \{a_x^{-1} : x \in F_n\}$, $n \in \mathbb{N}$. We get the sequence $\langle A_n : n \in \mathbb{N}\rangle$ of finite subsets of H which witnesses that H is M-bounded.

First we have $G = \bigcup_{n\in\mathbb{N}} F_n \cdot W_n \subset \bigcup_{n\in\mathbb{N}} A_n \cdot U_n$. This implies

$$H \subset \bigcup_{n\in\mathbb{N}} (A_n \cdot U_n) \cap H = \bigcup_{n\in\mathbb{N}} A_n \cdot (U_n \cap H) = \bigcup_{n\in\mathbb{N}} A_n \cdot O_n.$$

□

By a minor modification in the proof of this theorem we have

Theorem 4.3. *If a paratopological group $(G, \cdot, \tau)$ is H-bounded (R-bounded) and H is a dense subgroup in G, then H is H-bounded (R-bounded).*

Remark 4.4. The paratopological group $\mathbb{Q}_S$ is R-bounded and dense in $\mathbb{S}$, but $\mathbb{S}$ is not R-bounded.

5. Products

We first characterize paratopological groups which are M-bounded in all finite powers.

Theorem 5.1. *For a paratopological group $(G, \cdot, \tau)$ the following are equivalent:*

(1) *All finite powers of G are M-bounded;*

(2) *G is S-bounded.*

Proof. (1) ⇒ (2) Let $\langle U_n : n \in \mathbb{N}\rangle$ be a sequence from $\mathcal{N}_{e_G}$. Partition $\mathbb{N}$ into infinitely many pairwise disjoint infinite subsets M_k, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, $\langle U_n : n \in M_k\rangle$ is a sequence of neighbourhoods of e_G. Apply (1), to choose for each $k \in \mathbb{N}$ and each $n \in M_k$ a finite subset F_n of G such that the set $\{(F_n)^k \cdot (U_n)^k : n \in M_k\}$ covers G^k. (Note $(U_n)^k$ denotes the product $U_n \times \ldots U_n$, k times, and not the group product.) We prove that $\{F_n \cdot U_n : n \in \mathbb{N}\}$ is an ω-cover of G, i.e. that each finite subset of G is contained in $F_m \cdot U_m$ for some $m \in \mathbb{N}$.

Let $S = \{x_1, \ldots, x_p\}$ be a finite subset of G. Then $x = (x_1, \ldots, x_p) \in G^p$. Pick an $n \in M_p$ such that $x \in (F_n)^p \cdot (U_n)^p$. Then, $S \subset F_n \cdot U_n$.

(2) ⇒ (1) We prove that for a fixed $k \in \mathbb{N}$, G^k is M-bounded. Let $\langle U_n : n \in \mathbb{N}\rangle$ be a sequence in $\mathcal{N}_{e_{G^k}}$. For each n choose a $V_n \in \mathcal{N}_{e_G}$ such that $(V_n)^k \subset U_n$. By (2) there is a sequence $\langle E_n : n \in \mathbb{N}\rangle$ of finite sets in G such that $\{E_n \cdot V_n : n \in \mathbb{N}\}$ is an ω-cover of G. Then $\{(E_n)^k \cdot (V_n)^k : n \in \mathbb{N}\}$ is an open cover of G^k, hence $\{(E_n)^k \cdot U_n : n \in \mathbb{N}\}$ covers G^k, i.e. (1) is satisfied. □

Example 5.2. There is an S-bounded paratopological group G such that $G^{\mathbb{N}}$ is not M-bounded.

Let $\mathbb{S}$ be the Sorgenfrey line. Then all finite powers of $\mathbb{S}$ are M-bounded. Therefore, $\mathbb{S}$ is S-bounded.

Let us prove that $\mathbb{S}^{\mathbb{N}}$ is not M-bounded. Define the sequence $\langle U_n : n \in \mathbb{N}\rangle$ of neighbourhoud of $\mathbf{0} \in \mathbb{S}^{\mathbb{N}}$ by

$$U_n = [0, 1/2^n) \times \ldots \times [0, 1/2^n) \times \prod_{i \in \mathbb{N}\setminus\{1,2,\ldots,n\}} \mathbb{S}_i, \ (\mathbb{S}_i = \mathbb{S}, i > n).$$

We claim that for any sequence $\langle F_n : n \in \mathbb{N}\rangle$ of finite subsets of $\mathbb{S}^{\mathbb{N}}$ we have $\mathbb{S}^{\mathbb{N}} \neq \bigcup_{n\in\mathbb{N}} F_n \cdot U_n$. For $n \in \mathbb{N}$, let p_n denote the projection of $\mathbb{S}^{\mathbb{N}}$ onto the n-th coordinate space $\mathbb{S}_n = \mathbb{S}$. Evidently, each set $\mathbb{S} \setminus p_n(F_n) \cdot p_n(U_n)$ is nonempty; pick a point x_n in this set. Then the point $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ does not belong to $\bigcup_{n\in\mathbb{N}} F_n \cdot U_n$.

Remark 5.3. Note that the space $\mathbb{K}$ is S-bounded, and that $\mathbb{K}^{\mathfrak{c}^+}$ is also M-bounded (being precompact; see [22]).

The product of two M-bounded topologized groups need not be M-bounded. It is well known for M-bounded topological groups (see [4, 16]).

The following two theorems describe the behaviour of M-boundedness of paratopological groups under product. The proof of the first of them is routine, so omitted.

Theorem 5.4. *Let G be an M-bounded paratopological group and H a precompact paratopological group. Then $G \times H$ is an M-bounded paratopological group.*

Call a subset A of a topologized group G M-bounded if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of neighbourhoods of $e_G \in G$ there are finite sets $F_n \subset A$, $n \in \mathbb{N}$, such that $A \subset \bigcup_{n\in\mathbb{N}} F_n \cdot U_n$. The group G is said to be *hereditarily* M-*bounded* if each subset of G is M-bounded. Of course, similar definitions can be applied to other kinds of boundedness.

Theorem 5.5. *Let $(G, \cdot, \tau)$ and (H, σ) be paratopological groups such that G is M*–bounded, and (H, σ) a hereditarily precompact paratopological group. Then the product $(G \times H, \tau^* \times \sigma)$ is hereditarily M-bounded.*

Proof. Let $S \subset G \times H$ and let $\langle U_n : n \in \mathbb{N}\rangle$ be a sequence of neighbourhoods of e_G and $\langle V_n : n \in \mathbb{N}\rangle$ be a sequence of neighbourhoods of e_H. Since G is M*-bounded, there are finite sets F_n in G such that $\bigcup_{n\in\mathbb{N}} F_n \cdot U_n^* = G$, where $U_n^* = U_n \cap U_n^{-1}$. For each $n \in \mathbb{N}$ and each $a \in F_n$ define

$$B(n, a) = \{h \in H : \text{ there is } g \text{ in a } U_n^* \text{ such that } (g, h) \in S\}.$$

Fix $n \in \mathbb{N}$ and $a \in F_n$. Since (H, σ) is hereditarily precompact, hence $B(n, a)$ is precompact, there is a finite set $E(n, a) \subset B(n, a)$ such that $B(n, a) \subset E(n, a) \cdot V_n$.

For each $y \in E(n, a)$ there is $g(y, n, a) \in a \cdot U_n^*$ such that $(g(y, n, a), y) \in S$. The set

$$A(n, a) = \{(g(y, n, a), y) : y \in E(n, a)\},$$

is a finite subset of S.

Take any $(g, h) \in S$. There are $k \in \mathbb{N}$ and $a \in F_k$ such that $g \in a \cdot U_k^*$. Therefore $h \in B(k, a)$ and there is some $b \in B(k, a)$ such that $h \in b \cdot V_k$. Thus $(g(b, k, a), b) \in A(k, a)$, where $g(b, k, a) \in a \cdot U_k^*$. Using the symmetry of U_k^*, we get

$$(g, h) \in g(b, k, a) \cdot (U_k^*)^2 \times b \cdot V_k.$$

This means that the finite sets $A_k = \bigcup_{a\in F_k} A(k, a)$ witness that $(S, (\tau^* \times \sigma) \restriction S)$ is M-bounded. □

Problem 5.6. *If paratopological groups G and H are such that G is hereditarily* M*-bounded and H is hereditarily precompact, is then the product $G \times H$ hereditarily* M*-bounded?*[1)]

6. Relations with Games

To each property of topologized groups (and, more general, of pre-uniform spaces) defined above, one can correspond, in a natural way, a topological game. We define here a game (similar to the game introduced in [9] for topological groups) associated to M-boundedness and denoted M_G; games associated to other properties are defined in a similar way.

The game M_G is played in the following way: Players ONE and TWO play a round for each positive integer. In the n-th round, ONE chooses an open neighborhood U_n of e_G, and then TWO responds by choosing a finite set $A_n \subset G$. A play

$$U_1, A_1; \dots, U_n, A_n, \cdots$$

is won by TWO of $G = \bigcup_{n\in\mathbb{N}} A_n \cdot U_n$; otherwise, ONE wins.

Call a topologized group G *strictly* M*-bounded* if TWO has a winning strategy in the game M_G. Evidently, every strictly M-bounded topologized group is M-bounded.

Theorem 6.1. *There is an* M*-bounded paratopological group which is not strictly* M*-bounded.*

Proof. Let $\mathbb{S}$ be the Sorgenfrey line, and $\mathbb{S}^\mathbb{N}$ its countable power. The additive group $\mathbb{S}^\mathbb{N}$ is a paratopological group. If $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}^\mathbb{N}$, let $n_1 < n_2 < \dots$ be an enumeration of the set $\{n \in \mathbb{N} : x_n \neq 0\}$. Define

$$X := \{\mathbf{x} \in \mathbb{S}^\mathbb{N} : \lim_{k\to\infty} \frac{x_{n_k}}{n_{k+1}} = 0\}$$

and the subgroup $G = \langle X \rangle$ of $\mathbb{S}^\mathbb{N}$ generated by X.

Claim 1. G is an M-bounded paratopological group.

(The proof of this part is almost a repetition of the corresponding proof of Example 6.1 in [9] and it is given here only for the sake of completeness.) Let $\langle U_n : n \in \mathbb{N}\rangle$ be a sequence of neighbourhoods of the identity element $e_G \in G$. For each $n \in \mathbb{N}$ pick a neighbourhood V_n of e_G, $V_n = \prod_{i=1}^k V_{n,i} \times \prod_{i>k} \mathbb{S}_i$, such that $V_n + \dots + V_n \subset U_n$ (n times taken V_n). Define an increasing sequence $m_1 < m_2 < \dots$ in $\mathbb{N}$ such that $\{j \in \mathbb{N} : V_{n,j} \neq \mathbb{S} = \mathbb{S}_j\} \subset \{1, 2, \dots, m_n\}$ for each n. To each m_i associate the set $B_i \subset \mathbb{S}^\mathbb{N}$ defined by

$$B_i = \prod_{j\in\mathbb{N}} A_{i,j}, \quad \text{where } A_{i,j} = [-m_{i+1}, m_{i+1}), \text{ for } j \le m_i, \text{ and } A_{i,j} = \mathbb{S}, \text{ for } j > m_i.$$

Observe that $X \subset \bigcup_{n\in\mathbb{N}} B_n$. [Indeed, if $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in X$. Then $\lim_{k\to\infty} \frac{x_{n_k}}{n_{k+1}} = 0$ implies that there $s \in \mathbb{N}$ such that $|x_{n_k}| < n_{k+1}$ for all $k \ge s$. Put $t = \max\{x_{n_1}, \dots, x_{n_s}\}$ and choose i^* and ℓ in $\mathbb{N}$ such that $m_{i^*} > t$ and $m_{i^*} \le n_\ell < m_{i^*+1}$. Let $k \in \mathbb{N}$ be arbitrary. If $n_k > m_{i^*}$, then $A_{i^*,n_k} = \mathbb{S}$, hence $x_{n_k} \in A_{i^*,n_k}$; if $k < s$ and $n_k \le m_{i^*}$, then $|x_{n_k}| \le t < m_{i^*}$ so that $x_{n_k} \in A_{i^*,n_k}$; if $k \ge s$ and $n_k \le m_{i^*}$, then $k < \ell$, hence $|x_{n_k}| < n_{k+1} < m_{i^*+1}$, which implies $x_{n_k} \in A_{i^*,n_k}$, and thus $|x_j| \in A_{i^*,j}$ for each $j \in \mathbb{N}$. So, $x \in B_{i^*}$.]

For each $n \in \mathbb{N}$ choose a finite set $F_n \subset G$ such that $B_n \subset F_n + V_n$. Let $E_n = F_n + \dots + F_n$ (n times F_n). We prove that the sequence $\langle E_n : n \in \mathbb{N}\rangle$ witnesses for $\langle U_n : n \in \mathbb{N}\rangle$ that G is M-bounded. Let $g \in G$; then $g = x_1 + \dots + x_r$, $x_i \in X$ for each $i \le r$. We have that for each $i \le r$, $x_i \in B_{k_i}$ for some $k_i \in \{1, 2, \dots, r\}$. If we put $k = \max\{k_1, \dots, k_r, r\}$, then $x_i \in B_k$ for each $i \le r$. It follows

$$g = x_1 + \dots + x_r \in r(F_k + V_k) \subset k(F_k + V_k) \subset E_k + U_k.$$

[1)] T. Banakh remarked that the product $\mathbb{S} \times \mathbb{K}$ gives a negative answer to Problem 5.6.

Claim 2. G is not strictly M-bounded.

We prove that ONE has a winning strategy in the game. Let the first move of ONE be a neighbourhood $U_1 = G \cap ([-1,1) \times \prod_{i\geq 2} \mathbb{S}_i)$ of e_G, where $\mathbb{S}_i = \mathbb{S}$ for each i. If a finite set $F_1 \subset G$ is TWO's response, then ONE takes $x_1 \in \mathbb{S} \setminus p_1(F_1 + U_1)$, where $p_1 : \mathbb{S}^{\mathbb{N}} \to \mathbb{S}_1$ is the projection, pick an integer n_2 with such $2|x_1| < n_2$, and plays the second move $U_2 = G \cap ([-1,1)_1 \times \ldots \times [-1,1)_{n_2} \times \prod_{i>n_2} \mathbb{S}_i$. If TWO responses by choosing a finite set $F_2 \subset G$, then ONE pick an integer n_3, takes a point x_{n_2} such that $3|x_{n_2}| < n_3$, pick a point $x_{n_3} \in \mathbb{S} \setminus \bigcup_{i=1,2} p_i(F_i + U_i)$, and plays the third move $U_3 = [-1,1)_1 \times \ldots \times [-1,1)_{n_3}) \times \prod_{i>n_3} \mathbb{S}_i$. Two chooses a finite set $F_3 \subset G$. And so on.

Take the point $\mathbf{q} = (q_n)_{n\in\mathbb{N}} \in \mathbb{S}^{\mathbb{N}}$ defined so that $q_{n_i} = x_{n_i}$ for $i = 1, 2, \ldots$, and $q_n = 0$, otherwise. Evidently, $\mathbf{q} \in X$ and $\mathbf{q} \notin \bigcup_{n\in\mathbb{N}}(F_n + U_n)$, which gives $G \setminus \bigcup_{n\in\mathbb{N}}(F_n + U_n) \neq \emptyset$. Therefore, ONE has a winning strategy in the game, i.e. G is not strictly M-bounded. □

It is known that R-bounded σ-compact metrizable topological groups can be characterized game-theoretically:

Theorem 6.2. ([4]) *Let G be a σ-compact metrizable topological group. Then G is* R*-bounded if and only if* ONE *has no winning strategy in the corresponding game* R_G.

[ONE and TWO play a round per each natural number n. In the n–th round ONE chooses $U_n \in \mathcal{N}_{e_G}$, and TWO responds by choosing a point $x_n \in G$. A play $U_1, x_1; \ldots; U_n, x_n; \ldots$ is won by TWO if $\{x_n \cdot U_n : n \in \mathbb{N}\}$ covers G; otherwise, ONE wins.]

Problem 6.3. *Identify classes of paratopological groups in which* R*-boundedness can be characterized game-theoretically.*

Remark 6.4. There is a paratopological group which is not a topological group, but player TWO has a winning strategy in the Rothberger-type game R_G. The space $\mathbb{Q}$ of rational numbers with the Sorgenfrey topology is a metrizable paratopological non-topological group [17], and since it is countable, player TWO has a winning strategy in R_G. In fact, any countable paratopological group which is not a topological group serves as such an example.

7. Concluding Remarks

In some classes of topological groups R-boundedness has been characterized Ramsey-theoretically. For example, in [19] it was done for the class of σ-compact metrizable groups.

Recall the notion of *ordinary partition symbol*. For families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a set X and for positive integers n and k the symbol

$$\mathcal{A} \to (\mathcal{B})^n_k$$

denotes the statement:

> For each $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{1, \ldots, k\}$ there are a set $B \subseteq A$ and a $j \in \{1, \ldots, k\}$ such that for each $Y \in [B]^n$, $f(Y) = j$, and $B \in \mathcal{B}$.

$[A]^n$ denotes the set of n-element subsets of A.

A natural problem is the following.

Problem 7.1. *In which classes of paratopological groups* R*-boundedness can be characterized Ramsey-theoretically?*

One kind of boundedness in pre-uniform spaces and topologized groups has not been considered here; it is GN-boundedness which we define for paratopological groups: A paratopological group G is GN-*bounded* (abbreviation for Gerlits-Nagy bounded) if for each sequence $\langle U_n : n \in \mathbb{N}\rangle$ of neighbourhoods of e_G there are $x_n \in G$, $n \in \mathbb{N}$, such that each $g \in G$ belongs to $x_n \cdot U_n$ for all but finitely many n. It would be interesting to study this kind of boundedness.

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