Filomat 30:3 (2016), 873–884
DOI 10.2298/FIL1603873T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Statistical Summability $(\overline{N}, P)$ of Sequences of Fuzzy Numbers

Özer Talo[a], Canan Bal[b]

[a]*Department of Mathematics, Faculty of Art and Sciences, Celal Bayar University, 45040 Manisa, Turkey*
[b]*Private Central Vocational and Technical Anatolian High School, Manisa, Turkey*

Abstract. In this paper we introduce the concept of statistical summability $(\overline{N}, p)$ of sequences of fuzzy numbers. We also present Tauberian conditions under which statistical convergence of a sequence of fuzzy numbers follows from its statistical summability $(\overline{N}, p)$. Furthermore, we prove a Korovkin-type approximation theorem for fuzzy positive linear operators by using the notion of statistical summability $(\overline{N}, p)$.

1. Introduction

Following the introduction of sequences of fuzzy numbers by Matloka [15], summability of sequences of fuzzy numbers is studied by many researcher and bacame a recent research area in fuzzy set theory. Several summablity methods have been defined for sequence of fuzzy numbers and Tauberian theorems are given for these methods. Cesàro summability method is defined by Subrahmanyam[23] and various Tauberian conditions for Cesàro summability method are given by Talo and Çakan [24], Talo and F. Başar [26], Çanak [14]. Nörlund and Riesz mean of sequences of fuzzy numbers are studied by Tripathy and Baruah [27], Çanak [13], Önder et al. [20]. Furthermore, various power series methods of summability for sequences and series of fuzzy numbers are studied by Yavuz and Talo [28], Sezer and Çanak [22], Yavuz and Çoşkun [29].

The concept of statistical convergence of sequences of real numbers was originally introduced by Fast [11], and extended to sequences of fuzzy numbers by Nuray and Savaş [19]. Altn et. al. [1] have studied the concept of statistical summability $(C; 1)$ for sequences of fuzzy numbers. Talo and Çakan [25] have recently proved necessary and sufficient Tauberian conditions under which statistical convergence follows from statistically $(C, 1)$-convergence of sequences of fuzzy numbers.

In the present paper our primary interest is to generalize the results in [1, 25] to a large class of summability methods $(\overline{N}, p)$ by weighted means. We also obtain a Korovkin-type approximation theorem for fuzzy positive linear operator by means of the concept of statistical summability $(\overline{N}, p)$.

2010 *Mathematics Subject Classification*. Primary 03E72; Secondary 40A35, 40G15
Keywords. Sequences of fuzzy numbers, summability by weigted mean, Tauberian conditions, statistical convergence, fuzzy Korovkin theory
Received: 30 August 2015; Revised 01 December 2015; Accepted: 02 December 2015
Communicated by Ljubiša D.R. Kočinac and Ekrem Savaş
Email addresses: ozertalo@hotmail.com, ozer.talo@cbu.edu.tr (Özer Talo), cananbarut45@gmail.com (Canan Bal)

2. Definitions and Notation

Let K be a subset of natural numbers $\mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$, The natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n+1}|K_n|$$

if this limit exists, where $|A|$ denotes the number of elements in A. The concept of statistical convergence was introduced by Fast [11]. A sequence $(x_k)_{k\in\mathbb{N}}$ of real (or complex) numbers is said to be *statistically convergent* to some number l if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0.$$

In this case, we write st$-\lim_{k\to\infty} x_k = l$. Basic results on statistical convergence may be found in [9, 10, 16]

We recall the basic definitions dealing with fuzzy numbers.

A *fuzzy number* is a fuzzy set on the real axis, i.e. a mapping $u : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions:

(i) u is normal, i.e. there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
(ii) u is fuzzy convex, i.e. $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$;
(iii) u is upper semi-continuous;
(iv) The set $[u]_0 := \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact,

where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.
We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and called it as the space of fuzzy numbers. *α-level set* $[u]_\alpha$ of $u \in E^1$ is defined by

$$[u]_\alpha := \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\} & , \quad \text{if } 0 < \alpha \leq 1, \\ \overline{\{t \in \mathbb{R} : u(x) > \alpha\}} & , \quad \text{if } \alpha = 0. \end{cases}$$

The set $[u]_\alpha$ is closed, bounded and non-empty interval for each $\alpha \in [0, 1]$ which is defined by $[u]_\alpha := [u^-(\alpha), u^+(\alpha)]$. $\mathbb{R}$ can be embedded in E^1, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) := \begin{cases} 1 & , \quad \text{if } x = r, \\ 0 & , \quad \text{if } x \neq r. \end{cases}$$

Let $u, v, w \in E^1$ and $k \in \mathbb{R}$. Then the operations addition and scalar multiplication are defined on E^1 by

$$\begin{aligned} u + v = w \quad &\Longleftrightarrow \quad [w]_\alpha = [u]_\alpha + [v]_\alpha \ \text{ for all } \alpha \in [0, 1] \\ &\Longleftrightarrow \quad w^-(\alpha) = u^-(\alpha) + v^-(\alpha) \ \text{ and } w^+(\alpha) = u^+(\alpha) + v^+(\alpha) \ \text{ for all } \alpha \in [0, 1], \end{aligned}$$

$$[ku]_\alpha = k[u]_\alpha \ \text{ for all } \alpha \in [0, 1].$$

The operations addition and scalar multiplication on fuzzy numbers have the following properties.

Lemma 2.1. ([8])

(i) If $\overline{0} \in E^1$ is neutral element with respect to $+$,i.e $u + \overline{0} = \overline{0} + u = u$, for all $u \in E^1$;

(ii) With respect to $\overline{0}$, none of $u \neq \overline{r}$, $r \in \mathbb{R}$ has opposite in E^1;

(iii) For any $a, b \in \mathbb{R}$ with $a, b \geq 0$ or $a, b \leq 0$, and any $u \in E^1$, we have $(a + b)u = au + bu$. For general $a, b \in \mathbb{R}$, the above property does not hold;

(iv) For any $a \in \mathbb{R}$ and any $u, v \in E^1$, we have $a(u + v) = au + av$;

(v) For any $a, b \in \mathbb{R}$ and any $u \in E^1$, we have $a(bu) = (ab)u$.

Properties (*ii*) and (*iii*) show us that $(E^1, +, \cdot)$ is not a linear space over $\mathbb{R}$.

Let W be the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e. $A := [\underline{A}, \overline{A}]$. Define the relation d on W by

$$d(A, B) := \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can be easily observed that d is a metric on W and (W, d) is a complete metric space, (see Nanda [18]). Now, we may define the metric D on E^1 by means of the Hausdorff metric d as follows

$$D(u, v) := \sup_{\alpha\in[0,1]} d([u]_\alpha, [v]_\alpha) := \sup_{\alpha\in[0,1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\}.$$

One can see that

$$D(u, \overline{0}) = \sup_{\alpha\in[0,1]} \max\{|u^-(\alpha)|, |u^+(\alpha)|\} = \max\{|u^-(0)|, |u^+(0)|\}. \tag{1}$$

Now, we may give:

Proposition 2.2. ([8]) *Let $u, v, w, z \in E^1$ and $k \in \mathbb{R}$. Then,*

(i) (E^1, D) *is a complete metric space;*

(ii) $D(ku, kv) = |k|D(u, v)$*;*

(iii) $D(u + v, w + v) = D(u, w)$*;*

(iv) $D(u + v, w + z) \le D(u, w) + D(v, z)$*;*

(v) $|D(u, \overline{0}) - D(v, \overline{0})| \le D(u, v) \le D(u, \overline{0}) + D(v, \overline{0})$*.*

One can extend the natural order relation on the real line to intervals as follows:

$$A \le B \text{ if and only if } \underline{A} \le \underline{B} \text{ and } \overline{A} \le \overline{B}.$$

Also, the partial ordering relation on E^1 is defined as follows:

$$u \le v \iff [u]_\alpha \le [v]_\alpha \text{ for all } \alpha \in [0, 1] \iff u^-(\alpha) \le v^-(\alpha) \text{ and } u^+(\alpha) \le v^+(\alpha) \text{ for all } \alpha \in [0, 1].$$

Following Matloka [15], we give some definitions concerning the sequences of fuzzy numbers. A sequence $u = (u_k)$ of fuzzy numbers is said to be *convergent* to $\mu \in E^1$, if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$D(u_n, \mu) < \varepsilon \text{ for all } n \ge n_0.$$

A sequence (u_n) of fuzzy numbers is said to be *bounded* if there exists $M > 0$ such that $D(u_n, \overline{0}) \le M$ for all $n \in \mathbb{N}$.

Statistical convergence of sequences of fuzzy numbers was introduced by Nuray and Savaş [19]. A sequence $(u_k : k = 0, 1, 2, \ldots)$ of fuzzy numbers is said to be *statistically convergent* to a fuzzy number μ_0 if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n+1} \left|\{k \le n : D(u_k, \mu_0) \ge \varepsilon\}\right| = 0.$$

In this case we write

$$\text{st-}\lim_{k\to\infty} u_k = \mu_0. \tag{2}$$

For more results on statistical convergence of sequences of fuzzy numbers we refer to [4–7, 12, 21].

3. Main Results

Let $p = (p_k : k = 0, 1, 2, \dots)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad (n \to \infty)$$

and set

$$t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k, \quad n = 0, 1, 2, \dots$$

We say that the sequence (u_k) of fuzzy numbers is *statistically summable* to fuzzy number μ_0 by the weighted mean method determined by the sequence $p = (p_k)$, briefly: statistically summable $(\overline{N}, p)$ if

$$\text{st-}\lim_{n\to\infty} t_n = \mu_0. \tag{3}$$

If $p_n = 1$ for all n in (3), we have

$$\text{st-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} u_k = \mu_0. \tag{4}$$

Then we say that the sequence (u_n) of fuzzy numbers is statistically summable $(C, 1)$ to μ_0.

Example 3.1. Let $(u_k) = (u_0, v_0, u_0, v_0, ...)$ where

$$u_0(t) = \begin{cases} t-1 & , \text{ if } 1 \le t \le 2, \\ -t+3 & , \text{ if } 2 \le t \le 3, \\ 0 & , \text{ otherwise} \end{cases}$$

and

$$v_0(t) = \begin{cases} t+1 & , \text{ if } -1 \le t \le 0, \\ -t+1 & , \text{ if } 0 \le t \le 1, \\ 0 & , \text{ otherwise.} \end{cases}$$

The sequence (u_n) is statistically summable $(C, 1)$ to $w_0 = (u_0 + v_0)/2$. But (u_n) is not statistically summable $(\overline{N}, 2^n)$.

We claim that if a sequence (u_k) of fuzzy numbers is bounded, then

$$\text{st-}\lim_{k\to\infty} u_k = \mu_0. \quad \text{implies} \quad \text{st-}\lim_{n\to\infty} t_n = \mu_0. \tag{5}$$

In fact,

$$\begin{aligned} D(t_n, \mu_0) &= D\left(\frac{1}{P_n}\sum_{k=0}^{n} p_k u_k, \mu_0\right) \\ &= D\left(\frac{1}{P_n}\sum_{k=0}^{n} p_k u_k, \frac{1}{P_n}\sum_{k=0}^{n} p_k \mu_0\right) \\ &\le \frac{1}{P_n}\sum_{k=0}^{n} p_k D(u_k, \mu_0) \end{aligned}$$

Since the real sequence of $(D(u_k, \mu_0))$ is bounded and $\text{st}-\lim_{n\to\infty} D(u_k, \mu_0) = 0$, we have

$$\text{st}-\lim_{n\to\infty} \frac{1}{P_n} \sum_{k=0}^{n} p_k D(u_k, \mu_0) = 0.$$

Thus, we have $\text{st}-\lim_{n\to\infty} D(t_n, \mu_0) = 0$. This means that $\text{st}-\lim_{n\to\infty} t_n = \mu_0$.

The converse implication in (5) is not true in general, even in real case (see [17]). Our main goal is to find conditions under which

$$\text{st}-\lim_{n\to\infty} t_n = \mu_0 \quad \text{implies} \quad \text{st}-\lim_{k\to\infty} u_k = \mu_0. \tag{6}$$

We give two-sided Tauberian conditions, each of which is necessary and sufficient in order that statistical convergence follow from statistical summability $(\overline{N}, p)$.

Throughout this paper, λ_n denotes the integral part of the product λn; i.e., $\lambda_n := [\lambda n]$.

The concepts of statistical limit inferior and superior of a sequence of real numbers were introduced by Fridy and Orhan [10] and the following lemma was proved by Móricz and Orhan [17]. We use it for the proof of our results.

Lemma 3.2. ([17, Lemma1]) *If (P_n) is a nondecreasing sequence of positive numbers, then conditions*

$$\text{st}-\liminf_{n\to\infty} \frac{P_{\lambda_n}}{P_n} > 1 \quad \textit{for every} \quad \lambda > 1 \tag{7}$$

and

$$\text{st}-\liminf_{n\to\infty} \frac{P_n}{P_{\lambda_n}} > 1 \quad \textit{for every} \quad 0 < \lambda < 1. \tag{8}$$

are equivalent.

We need the following lemmas.

Lemma 3.3. *Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and condition (7) is satisfied, and let (u_k) be a sequence of fuzzy numbers which is statistically summable $(\overline{N}, p)$ to a fuzzy number μ_0. Then for every $\lambda > 0$,*

$$\text{st}-\lim_{n\to\infty} t_{\lambda_n} = \mu_0. \tag{9}$$

The proof can be carried out in the same way as in the proof of Lemma 2 in [17].

Lemma 3.4. *Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and condition (7) is satisfied, and let (u_k) be a sequence of fuzzy numbers which is statistically summable $(\overline{N}, p)$ to a fuzzy number μ_0. Then, for every $\lambda > 1$,*

$$\text{st}-\lim_{n\to\infty} \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k u_k = \mu_0 \tag{10}$$

and for every $0 < \lambda < 1$,

$$\text{st}-\lim_{n\to\infty} \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k u_k = \mu_0. \tag{11}$$

Proof. Case $\lambda > 1$. If $P_{\lambda_n} > P_n$, then

$$\begin{aligned} D\left(\frac{1}{P_{\lambda_n} - P_n}\sum_{k=n+1}^{\lambda_n} p_k u_k, \mu_0\right) &= D\left(\frac{1}{P_{\lambda_n} - P_n}\sum_{k=n+1}^{\lambda_n} p_k u_k + t_n, \mu_0 + t_n\right) \\ &\leq D\left(\frac{1}{P_{\lambda_n} - P_n}\sum_{k=n+1}^{\lambda_n} p_k u_k, t_n\right) + D(t_n, \mu_0) \end{aligned}$$

and

$$\begin{aligned} D\left(\frac{1}{P_{\lambda_n} - P_n}\sum_{k=n+1}^{\lambda_n} p_k u_k, t_n\right) &= D\left(\frac{1}{P_{\lambda_n} - P_n}\sum_{k=n+1}^{\lambda_n} p_k u_k, \frac{1}{P_n}\sum_{k=0}^{n} p_k u_k\right) \\ &= D\left(\frac{1}{P_{\lambda_n} - P_n}\sum_{k=n+1}^{\lambda_n} p_k u_k + \frac{1}{P_{\lambda_n} - P_n}\sum_{k=0}^{n} p_k u_k, \frac{1}{P_n}\sum_{k=0}^{n} p_k u_k + \frac{1}{P_{\lambda_n} - P_n}\sum_{k=0}^{n} p_k u_k\right) \\ &= D\left(\frac{1}{P_{\lambda_n} - P_n}\sum_{k=0}^{\lambda_n} p_k u_k, \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n}\frac{1}{P_n}\sum_{k=0}^{n} p_k u_k\right) \\ &= D\left(\frac{P_{\lambda_n}}{P_{\lambda_n} - P_n}\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k u_k, \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n}\frac{1}{P_n}\sum_{k=0}^{n} p_k u_k\right) \\ &= \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n} D\left(\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k u_k, \frac{1}{P_n}\sum_{k=0}^{n} p_k u_k\right) \\ &= \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n} D\left(t_{\lambda_n}, t_n\right). \end{aligned}$$

So, we have the following inequality

$$D\left(\frac{1}{P_{\lambda_n} - P_n}\sum_{k=n+1}^{\lambda_n} p_k u_k, \mu_0\right) \leq \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n} D\left(t_{\lambda_n}, t_n\right) + D(t_n, \mu_0). \tag{12}$$

By (7) we have

$$\text{st-}\limsup_{n\to\infty} \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n} = 1 + \left\{-1 + \text{st-}\liminf_{n\to\infty} \frac{P_{\lambda_n}}{P_n}\right\}^{-1} < \infty. \tag{13}$$

Therefore (10) follows from (12), (13), Lemma 3.3 and the statistical convergence of (t_n).

Case $0 < \lambda < 1$. This time, we make use of the following inequality:

$$D\left(\frac{1}{P_n - P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n} p_k u_k, \mu_0\right) \leq \frac{P_{\lambda_n}}{P_n - P_{\lambda_n}} D\left(t_{\lambda_n}, t_n\right) + D(t_n, \mu_0) \tag{14}$$

provided $P_n > P_{\lambda_n}$. By Lemma 3.2 we obtain

$$\text{st-}\limsup_{n\to\infty} \frac{P_{\lambda_n}}{P_n - P_{\lambda_n}} = \left\{-1 + \text{st-}\liminf_{n\to\infty} \frac{P_n}{P_{\lambda_n}}\right\}^{-1} < \infty. \tag{15}$$

Therefore (11) follows from (14), (15), Lemma 3.3 and the statistical convergence of (t_n). $\square$

Now we are ready to give our main results.

Theorem 3.5. *Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and condition (7) is satisfied, and let (u_k) be a sequence of fuzzy numbers which is statistically summable $(\overline{N}, p)$ to a fuzzy number μ_0.Then (u_k) is statistically convergent to μ_0 if and only if one of the following two conditions holds: for every $\varepsilon > 0$,*

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leq N : D\left(\frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k u_k, u_n \right) \geq \varepsilon \right\} \right| = 0 \tag{16}$$

or

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leq N : D\left(\frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k u_k, u_n \right) \geq \varepsilon \right\} \right| = 0. \tag{17}$$

Proof. The necessity follows from Lemma 3.4.

Sufficiency. Assume that conditions (3) and one of (16) and (17) are satisfied. In order to prove (2), it is enough to prove that

$$\text{st}-\lim_{n\to\infty} D(u_n, t_n) = 0.$$

First, we consider the case $\lambda > 1$. Since

$$D(t_n, u_n) \leq D\left(\frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k u_k, u_n \right) + \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n} D\left(t_{\lambda_n}, t_n\right),$$

for any $\varepsilon > 0$ we have

$$\{n \leq N : D(t_n, u_n) \geq \varepsilon\} \subseteq \left\{ n \leq N : \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n} D\left(t_{\lambda_n}, t_n\right) \geq \frac{\varepsilon}{2} \right\} \cup \left\{ n \leq N : D\left(\frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k u_k, u_n \right) \geq \frac{\varepsilon}{2} \right\}.$$

Given any $\delta > 0$, by (16) there exists some $\lambda > 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leq N : D\left(\frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k u_k, u_n \right) \geq \frac{\varepsilon}{2} \right\} \right| \leq \delta. \tag{18}$$

On the other hand, by Lemma 3.3, we have

$$\limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leq N : \frac{P_{\lambda_n}}{P_{\lambda_n} - P_n} D\left(t_{\lambda_n}, t_n\right) \geq \frac{\varepsilon}{2} \right\} \right| = 0. \tag{19}$$

Combining (18) with (19) we get that

$$\limsup_{N\to\infty} \frac{1}{N+1} \left| \{ n \leq N : D\left(u_n, t_n\right) \geq \varepsilon \} \right| \leq \delta.$$

Since $\delta > 0$ is arbitrary, we conclude that for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N+1} \left| \{ n \leq N : D\left(u_n, t_n\right) \geq \varepsilon \} \right| = 0.$$

Secondly, we consider the case $0 < \lambda < 1$. Since

$$D(t_n, u_n) \leq D\left(\frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k u_k, u_n \right) + \frac{P_{\lambda_n}}{P_n - P_{\lambda_n}} D\left(t_{\lambda_n}, t_n\right)$$

for any $\varepsilon > 0$, we have

$$\{n \leq N : D(t_n, u_n) \geq \varepsilon\} \subseteq \left\{ n \leq N : \frac{P_{\lambda_n}}{P_n - P_{\lambda_n}} D\left(t_{\lambda_n}, t_n\right) \geq \frac{\varepsilon}{2} \right\} \cup \left\{ n \leq N : D\left(\frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k u_k, u_n \right) \geq \frac{\varepsilon}{2} \right\}.$$

Given any $\delta > 0$, by (17) there exist some $0 < \lambda < 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leq N : D\left(\frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k u_k, u_n \right) \geq \frac{\varepsilon}{2} \right\} \right| \leq \delta.$$

Using a similar argument as in the case $\lambda > 1$, by Lemma 3.3 and condition (11), we conclude that

$$\lim_{N\to\infty} \frac{1}{N+1} \left|\{n \leq N : D\,(u_n, t_n) \geq \varepsilon\}\right| = 0.$$

This completes the proof. □

Following Talo and Çakan [25], a sequence (u_k) of fuzzy numbers is said to be *statistically slowly oscillating* if for every $\varepsilon > 0$,

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leq N : \max_{n<k\leq\lambda_n} D(u_k, u_n) \geq \varepsilon \right\} \right| = 0 \tag{20}$$

or equivalently,

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leq N : \max_{\lambda_n<k\leq n} D(u_k, u_n) \geq \varepsilon \right\} \right| = 0. \tag{21}$$

The conditions (20) and (21) are clearly imply the conditions (16) and (17), respectively. This gives rise to the following corollary of Theorem 3.5.

Corollary 3.6. *Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and condition (7) is satisfied, and let (u_k) be a statistically slowly oscillating sequence of fuzzy numbers. Then the implication (6) hold.*

4. Application to Fuzzy Korovkin Theory

In this section we prove a fuzzy Korovkin-type theorem via the concept of statistical summability $(\overline{N}, p)$.

We denote by $C[a, b]$ the space of all continuous real functions on [a, b]. This space is equipped with the supremum norm

$$\|h\| = \sup_{x\in[a,b]} |h(x)|.$$

A fuzzy-number-valued function $f : [a, b] \to E^1$ has the parametric representation

$$[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$$

for each $x \in [a, b]$ and $\alpha \in [0, 1]$. Let $f, g : [a, b] \to E^1$ be fuzzy number valued functions. Then, the distance between f and g is given by

$$\begin{aligned} D^*(f, g) &= \sup_{x\in[a,b]} D(f(x), g(x)) \\ &= \sup_{x\in[a,b]} \sup_{\alpha\in[0,1]} \max\{|f_\alpha^-(x) - g_\alpha^-(x)|, |f_\alpha^+(x) - f_\alpha^+(x)|\}. \end{aligned}$$

A fuzzy-number-valued function $f : [a, b] \to E^1$ is said to be continuous at $x_0 \in [a, b]$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$ whenever $x \in [a, b]$ with $|x - x_0| < \delta$. If $f(x)$ is continuous at each $x \in [a, b]$, then we say $f(x)$ is continuous on $[a, b]$. The set of all fuzzy continuous functions on the interval $[a, b]$ is denoted by $C_{\mathcal{F}}[a, b]$.

Now let $L : C_{\mathcal{F}}[a, b] \to C_{\mathcal{F}}[a, b]$ be an operator. Then L is said to be fuzzy linear if, for every $\lambda_1, \lambda_2 \in \mathbb{R}$, $f_1, f_2 \in C_{\mathcal{F}}[a, b]$ and $x \in [a, b]$,

$$L(\lambda_1 f_1 + \lambda_2 f_2; x) = \lambda_1 L(f_1; x) + \lambda_2 L(f_2; x)$$

holds. Also L is called fuzzy positive linear operator if it is fuzzy linear and the condition $L(f;x) \preceq L(g;x)$ is satisfied for any $f, g \in C_{\mathcal{F}}[a,b]$ and all $x \in [a,b]$ with $f(x) \preceq g(x)$.

The fuzzy Korovkin approximation theorem has been obtained by Anastassiou[2]. Its statistical version was given by Anastassiou and Duman [3]. The fuzzy Korovkin approximation theorem states as follows:

Theorem 4.1. *Let $\{L_n\}_{n\in N}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a,b]$ into itself. Assume that there exists a corresponding sequence $\{\widetilde{L}_n\}_{n\in N}$ of positive linear operators from $C[a,b]$ into itself with the property*

$$\{L_n(f;x)\}_{\alpha}^{\pm} = \widetilde{L}_n(f_{\alpha}^{\pm};x) \tag{22}$$

for all $x \in [a,b], \alpha \in [0,1], n \in \mathbb{N}$ and $f \in C_{\mathcal{F}}[a,b]$. Assume further that

$$\lim_{n\to\infty} \| \widetilde{L}_n(e_i) - e_i \| = 0,$$

with $e_i(x) = x^i, i = 0,1,2$. Then, for all $f \in C_{\mathcal{F}}[a,b]$, we have

$$\lim_{n\to\infty} D^*(L_n(f), f) = 0.$$

Now we prove the following result by using the notion of statistical summability $(\overline{N}, p)$.

Theorem 4.2. *Let $\{L_n\}_{n\in N}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}[a,b]$ into itself. Assume that there exists a corresponding sequence $\{\widetilde{L}_n\}_{n\in N}$ of positive linear operators from $C[a,b]$ into itself with the property (22). Assume further that*

$$st - \lim_{n\to\infty} \left\| \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k(e_i) - e_i \right\| = 0 \tag{23}$$

with $e_i(x) = x^i, i = 0,1,2$. Then ,for all $f \in C_F[a,b]$, we have

$$st - \lim_{n\to\infty} D^* \left(\frac{1}{P_n} \sum_{k=0}^{n} p_k L_k(f), f \right) = 0. \tag{24}$$

Proof. Let $f \in C_{\mathcal{F}}[a,b], x \in [a,b]$ and $\alpha \in [0,1]$. By the hypothesis, since $f_{\alpha}^{\pm} \in C_{\mathcal{F}}[a,b]$, we may write, for every $\varepsilon > 0$, that there exists a number $\delta > 0$ such that $| f_{\alpha}^{\pm}(y) - f_{\alpha}^{\pm}(x) | < \varepsilon$ holds for every $y \in [a,b]$ satisfying $|y - x| < \delta$. Then we immediately get, for all $y \in [a,b]$, that

$$|f_{\alpha}^{\pm}(y) - f_{\alpha}^{\pm}(x)| \leq \varepsilon + 2M\frac{(y-x)^2}{\delta^2}, \tag{25}$$

where $M := \sup_{x\in[a,b]} D(f(x), \overline{0})$. Now using the linearity and the positivity of the operators $\widetilde{L}_n$,we have , for each $n \in \mathbb{N}$, that

$$\begin{aligned}
\left| \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k (f_{\alpha}^{\pm};x) - f_{\alpha}^{\pm}(x) \right| &\leq \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k \left(\left| f_{\alpha}^{\pm}(y) - f_{\alpha}^{\pm}(x) \right| ;x \right) + M \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k(e_0;x) - e_0(x) \right| \\
&\leq \varepsilon + (\varepsilon + M) \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k(e_0;x) - e_0(x) \right| + \frac{2M}{\delta^2} \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k((y-x)^2;x) \right|
\end{aligned}$$

which yields

$$\begin{aligned}
\left| \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k(f_{\alpha}^{\pm};x) - f_{\alpha}^{\pm}(x) \right| &\leq \varepsilon + \left(\varepsilon + M + \frac{2c^2M}{\delta^2} \right) \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k(e_0;x) - e_0(x) \right| \\
&\quad + \frac{4cM}{\delta^2} \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k(e_1;x) - e_1(x) \right| + \frac{2M}{\delta^2} \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k \widetilde{L}_k(e_2;x) - e_2(x) \right| ,
\end{aligned}$$

where $c := \max\{|a|, |b|\}$. Also letting $K(\varepsilon) := \max\left\{\varepsilon + M + \frac{2c^2M}{\delta^2}, \frac{4cM}{\delta^2}, \frac{2M}{\delta^2}\right\}$ and taking supremum over $x \in [a,b]$, the above inequallity implies that

$$\left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(f_\alpha^\pm;x) - f_\alpha^\pm(x)\right\| \leq \varepsilon + K(\varepsilon)\left\{\left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(e_0) - e_0\right\| + \left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(e_1) - e_1\right\| + \left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(e_2) - e_2\right\|\right\}. \tag{26}$$

It follows from (22) that

$$\begin{aligned} D^*\left(\frac{1}{P_n}\sum_{k=0}^{n} p_kL_k(f), f\right) &= \sup_{x\in[a,b]} D\left(\frac{1}{P_n}\sum_{k=0}^{n} p_kL_k(f;x), f(x)\right) \\ &= \sup_{x\in[a,b]}\sup_{\alpha\in[0,1]}\max\left\{\left|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(f_\alpha^-;x) - f_\alpha^-(x)\right|, \left|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(f_\alpha^+;x) - f_\alpha^+(x)\right|\right\} \\ &= \sup_{\alpha\in[0,1]}\max\left\{\left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(f_\alpha^-) - f_\alpha^-\right\|, \left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(f_\alpha^+) - f_\alpha^+\right\|\right\}. \end{aligned}$$

Combining the above equality with (26), we have

$$D^*\left(\frac{1}{P_n}\sum_{k=0}^{n} p_kL_k(f), f\right) \leq \varepsilon + K(\varepsilon)\left\{\left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(e_0) - e_0\right\| + \left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(e_1) - e_1\right\| + \left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(e_2) - e_2\right\|\right\}. \tag{27}$$

Now, for a given $\varepsilon' > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < \varepsilon'$,and also define the following sets:

$$\begin{aligned} U: &= \left\{n \in \mathbb{N} : D^*\left(\frac{1}{P_n}\sum_{k=0}^{n} p_kL_k(f), f\right) \geq \varepsilon'\right\}, \\ U_0: &= \left\{n \in \mathbb{N} : \left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(e_0) - e_0\right\| \geq \frac{\varepsilon'-\varepsilon}{3K(\varepsilon)}\right\}, \\ U_1: &= \left\{n \in \mathbb{N} : \left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(e_1) - e_1\right\| \geq \frac{\varepsilon'-\varepsilon}{3K(\varepsilon)}\right\}, \\ U_2: &= \left\{n \in \mathbb{N} : \left\|\frac{1}{P_n}\sum_{k=0}^{n} p_k\widetilde{L}_k(e_2) - e_2\right\| \geq \frac{\varepsilon'-\varepsilon}{3K(\varepsilon)}\right\}, \end{aligned}$$

Then inequality (27) gives $U \subseteq U_0 \cup U_1 \cup U_2$ and so $\delta(U) \leq \delta(U_0) + \delta(U_1) + \delta(U_2)$. Then using the hypothesis (23), we get (24). The proof is completed. □

Example 4.3. Consider the sequence of fuzzy Bernstein-type polynomials [2]

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, \tag{28}$$

where $f \in C_{\mathcal{F}}[0,1], x \in [0,1]$. Let $p_k = 1$ for all k. Then $(\overline{N}, p)$-mean is reduced to $(C,1)$-mean. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1 \quad , & \text{if k is odd,} \\ -1 \quad , & \text{if k is even.} \end{cases}$$

Observe now that, $st - \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} x_k = 0$. We define the following fuzzy positive linear operators

$$L_n(f;x) = (1 + x_n)B_n(f;x). \tag{29}$$

Then, the sequence $\{L_n\}$ satisfies conditions of Theorem 4.2. Hence, we have

$$st - \lim_{n\to\infty} D^{*}\left(\frac{1}{n}\sum_{k=0}^{n} L_k(f), f\right) = 0.$$

However, (x_n) is neither convergent nor statistical convergent to 0. So the classical fuzzy Korovkin theorem 4.1 and the statistical fuzzy Korovkin theorem ([3, Theorem 2.1.]) do not work for our operators defined by (29).

Acknowledgement

The authors would like to thank the anonymous referees for their valuable comments and suggestions to improve the quality and readability of the paper.

References

[1] Y. Altin, M. Mursaleen, H. Altinok, Statistical summability $(C, 1)$-for sequences of fuzzy real numbers and a Tauberian theorem, Journal of Intelligent & Fuzzy Systems 21 (2010) 379–384.
[2] G.A. Anastassiou, On basic fuzzy Korovkin theory, Studia Universitatis BabeÅ-Bolyai Mathematica 50 (2005) 3–10.
[3] G.A. Anastassiou, O. Duman, Statistical fuzzy approximation by fuzzy positive linear operators, Computers and Mathematics with Applications 55 (2008) 573–580.
[4] S. Aytar, M. Mammadov, S. Pehlivan, Statistical limit inferior and limit superior for sequences of fuzzy numbers, Fuzzy Sets and Systems 157 (2006) 976–985.
[5] S. Aytar, S. Pehlivan, Statistical cluster and extreme limit points of sequences of fuzzy numbers, Information Sciences 177 (2007) 3290–3296.
[6] S. Aytar, S. Pehlivan, Statistically monotonic and statistically bounded sequences of fuzzy numbers, Information Sciences 176 (2006) 734–744.
[7] S. Aytar, Statistical limit points of sequences of fuzzy numbers, Information Sciences 165 (2004) 129–138.
[8] B. Bede, S.G. Gal, Almost periodic fuzzy number valued functions, Fuzzy Sets and Systems 147 (2004) 385–403.
[9] J.A. Fridy, On statistical convergence, Analysis 5 (1985) 301–313.
[10] J. A. Fridy, C. Orhan, Statistical limit superior and limit inferior, Proceedings of the American Mathematical Society 125 (1997) 3625–3631.
[11] H. Fast, Sur la convergence statistique, Colloquium Mathematicum 2 (1951) 241–244.
[12] J.S. Kwon, On statistical and p-Cesàro convergence of fuzzy numbers, The Korean Journal of Computational & Applied Mathematics 7 (2000) 195–203.
[13] İ. Çanak, On the Riesz mean of sequences of fuzzy real numbers, Journal of Intelligent & Fuzzy Systems 26(6) (2014) 2685-2688.
[14] İ. Çanak, Tauberian theorems for Cesàro summability of sequences of fuzzy number, Journal of Intelligent & Fuzzy Systems 27 (2014) 937–942.
[15] M. Matloka, Sequence of fuzzy numbers, BUSEFAL 28 (1986) 28–37.
[16] F. Móricz, Tauberian conditions, under which statistical convergence follows from statistical summability $(C, 1)$, Journal of Mathematical Analysis and Applications 275 (2002) 277–287.
[17] F. Móricz, C. Orhan, Tauberian conditions under which statistical convergence follows from statistical summability by weighted means, Studia Scientiarum Mathematicarum Hungarica 41 (2004) 391–403.
[18] S. Nanda, On sequence of fuzzy numbers, Fuzzy Sets and Systems 33 (1989) 123–126.
[19] F. Nuray, E. Savaş, Statistical convergence of fuzzy numbers, Mathematica Slovaca 45 (1995) 269–273.
[20] Z. Önder, S.A. Sezer, İ. Çanak, A Tauberian theorem for the weighted mean method of summability of sequences of fuzzy numbers, Journal of Intelligent & Fuzzy Systems 28 (2015) 1403-1409.
[21] E. Savaş, On statistical convergent sequences of fuzzy numbers, Information Sciences 137 (2001) 277–282.
[22] S.A. Sezer, İ. Çanak, Power series methods of summability for series of fuzzy numbers and related Tauberian theorems, Soft Computing (2015) doi: 10.1007/s00500-015-1840-0.
[23] P.V. Subrahmanyam, Cesàro summability of fuzzy real numbers, Journal of Analysis 7 (1999) 159-168.
[24] Ö. Talo, C. Çakan, On the Cesàro convergence of sequences of fuzzy numbers, Applied Mathematics Letters 25 (2012) 676–681.
[25] Ö. Talo, C. Çakan, Tauberian theorems for statistically $(C, 1)$-convergent sequences of fuzzy numbers, Filomat 28 (2014) 849–858
[26] Ö. Talo, F. Başar, On the slowly decreasing sequences of fuzzy numbers, Abstract and Applied Analysis 2013 (2013), Article ID 891986, 7 pages.

[27] B.C. Tripathy, A. Baruah, Nörlund and Riesz mean of sequences of fuzzy real numbers, Applied Mathematics Letters 23 (2010) 651–655.
[28] E. Yavuz, Ö. Talo, Abel summability of sequences of fuzzy numbers, Soft Computing (2014) doi: 10.1007/s00500-014-1563-7.
[29] E. Yavuz, H. Çoşkun, On the Borel summability method of sequences of fuzzy numbers, Journal of Intelligent & Fuzzy Systems (2015) doi: 10.3233/IFS-151924