Filomat 30:4 (2016), 921–927
DOI 10.2298/FIL1604921K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Notes on Unified q-Apostol-Type Polynomials

Burak Kurt

Akdeniz University, Faculty of Educations Department of Mathematics, Antalya, TR-07058, Turkey

Abstract. Recently, many mathematicians (Karande and Thakare [6], Ozarslan [14], Ozden et. al. [15], El-Deouky et. al. [5]) have studied the unification of Bernoulli, Euler and Genocchi polynomials. They gave some recurrence relations and proved some theorems. Mahmudov [13] defined the new q-Apostol-Bernoulli and q-Apostol-Euler polynomials. Also he gave the analogous of the Srivastava-Pintér addition theorems. Kurt [8] gave the new identities and some relations for these polynomials. In this work, we give some recurrence relations for the unified q-Apostol-type polynomials related to multiple sums. By using generating functions we derive many new identities and recurrence relations associated with the q-Apostol-type Bernoulli, the q-Apostol-type Euler and the q-Apostol-type Genocchi polynomials and numbers and also the generalized Stirling type numbers of the second kind.

1. Introduction, Definitions and Notations

Throughout this paper, we always make use of the following notation; $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

We use notations and definition related to q-calculus which are given by Kac and Cheung [7].

The q-numbers and q-factorial are defined by

$$[a]_q = \begin{cases} \frac{1-q^a}{1-q}, & q \neq 1 \\ a, & q = 1 \end{cases} ,$$

if $q \in \mathbb{C}$ then $\left|q\right| < 1$, if $q \in \mathbb{R}$ then $0 < q < 1$. In limit case $\lim\limits_{q \to 1}[a]_q = a$ and $[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$, where $[0]_q! = 1$ and $n \in \mathbb{N}$. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q:q)_n}{(q:q)_{n-k}(q:q)_k}.$$

2010 *Mathematics Subject Classification*. Primary 05A10; Secondary 11B73, 11B68.

Keywords. Bernoulli numbers and polynomials; Euler polynomials and numbers; generating function; generating function; Apostol-Bernoulli number and polynomials; generalized q-Bernoulli polynomials; generalized q-Genocchi polynomials.

Received: 24 June 2015; Accepted: 28 September 2015

Communicated by Gradimir Milovanović and Yilmaz Simsek

Research supported by the Scientific Research Project Administration of Akdeniz University. The authors would like to thank the referees and editors for their many valuable comments and suggestions.

Email address: burakkurt@akdeniz.edu.tr (Burak Kurt)

The q-analogue of the function $(x+y)_q^n$ is defined by

$$(x+y)_q^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}} x^{n-k} y^k.$$

The q-binomial formula is known as

$$(1-a)_q^n = \prod_{j=0}^{n-1} \left(1 - q^j a\right) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}} (-1)^k a^k.$$

In the standard approach to the q-calculus two exponential functions are used

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)\,q^k z)}, \; 0 < |q| < 1, \; |z| < \frac{1}{|1-q|}$$

and

$$E_q(z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \left(1 + (1-q)\,q^k z\right), \; 0 < |q| < 1, \; z \in \mathbb{C}.$$

From this form, we easily see that $e_q(z)E_q(-z) = 1$. The q-derivative and the derivative of the product of two functions and the derivative of the division of two functions are given by the following equation in [7] respectively

$$D_q f(z) = \lim_{q \to 1} \frac{f(qx) - f(x)}{q(x-1)}, \; D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz) D_q (f(z)) - f(qz) D_q g(z)}{g(z) g(qz)}, \tag{1}$$

$$D_q (f(z)g(z)) = f(qz) D_q g(z) + g(z) D_q f(z).$$

Recently, many mathematicians studied the unification of the Bernoulli and Euler polynomials. Firstly, Karande *et. al.* in [6] introduced and generalized the multiplication formula. Ozden *et. al.* in [15] defined the unified Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. Lastly, El-Desoukly B. S. in [5] defined and investigated a unified family $M_n^{(r)}(x, k, \bar{\alpha}_r)$ of generalized Apostol-Bernoulli, Euler and Genocchi polynomials. He proved some recurrence relations and the addition formula for these unified family $M_n^{(r)}(x, k, \bar{\alpha}_r)$. Finally, Ozarslan in [14] introduced and proved some relations for the uniform form of the Apostol-Bernoulli, Euler and Genocchi polynomials $\mathcal{P}_{n,\beta}^{(\alpha)}(x, k, a, b)$.

We introduce the unified q-Apostol-Bernoulli, Euler and Genocchi polynomials. We give some relations and theorems.

The generalized q-Apostol-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x, y, \lambda)$, the generalized q-Apostol-Euler polynomials $\mathcal{E}_{n,q}^{(\alpha)}(x, y, \lambda)$ and the generalized q-Apostol-Genocchi polynomials $\mathcal{G}_{n,q}^{(\alpha)}(x, y, \lambda)$ are defined by Mahmudov in [13]. He defined the generalized q-Apostol-Bernoulli polynomials $\mathcal{B}_{n,q}^{(\alpha)}(x, y, \lambda)$ as:

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,q}^{(\alpha)}(x, y, \lambda) \frac{t^n}{[n]_q!} = \left(\frac{t}{\lambda e_q(t) - 1}\right)^{\alpha} e_q(tx) E_q(ty), \; \left(|t + \log \lambda| < 2\pi, \; 1^{\alpha} := 1\right)$$

where α and λ are arbitrary real or complex parameters and $x \in \mathbb{R}$.

The following unified Apostol-Bernoulli, Euler and Genocchi polynomials of order α are defined by Ozarslan in [14] as

$$\begin{aligned} f_{a,b}^{(\alpha)}(x;t,a,b) &= \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta}^{(\alpha)}(x,k,a,b)\frac{t^n}{n!}, \\ & \left(\left|t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi, x \in \mathbb{R}, k \in \mathbb{N}_0, a, b \in \mathbb{R}^+, \beta \in \mathbb{C}\right). \end{aligned} \tag{2}$$

We define the unified q-Apostol-Bernoulli, Euler and Genocchi polynomials of order α as:

Definition 1.1. *We define the following unified q-Apostol-Bernoulli, Euler and Genocchi polynomials of order α as:*

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\frac{t^n}{[n]_q!} &= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t) - a^b}\right)^\alpha e_q(tx)E_q(ty), \\ k &\in \mathbb{N}_0, a, b \in \mathbb{R}\setminus\{0\}, \alpha, \beta \in \mathbb{C}. \end{aligned} \tag{3}$$

We take the limit for $q \to 1$ and $y = 0$ in (3). This definition reduces to (2) as follow

$$\lim_{q\to 1}\sum_{n=0}^{\infty} \mathcal{P}_{n,\beta,q}^{(\alpha)}(x,0,k,a,b)\frac{t^n}{[n]_q!} = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha e^{xt}.$$

We obtain Ozarslan's definition.

We have

$$\mathcal{P}_{n,\lambda,q}^{(\alpha)}(x,y;1,1,1) = \mathcal{B}_{n,q}^{(\alpha)}(x,y;\lambda), \mathcal{P}_{n,\lambda,q}^{(\alpha)}(x,y;0,-1,1) = \mathcal{E}_{n,q}^{(\alpha)}(x,y;\lambda), \mathcal{P}_{n,\frac{\lambda}{2},q}^{(\alpha)}\left(x,y;1,-\frac{1}{2},1\right) = \mathcal{G}_{n,q}^{(\alpha)}(x,y;\lambda)$$

and

$$\lim_{q\to 1^-}\sum_{n=0}^{\infty} P_{n,\beta,q}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} P_{n,\beta,1}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{n!} = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha e^{t(x+y)}.$$

$$\begin{aligned} \lim_{q\to 1^-} P_{n,\lambda,q}^{(\alpha)}(x,y;1,1,1) &= B_n^{(\alpha)}(x+y), \lim_{q\to 1^-} P_{n,\lambda,q}^{(\alpha)}(x,y;0,-1,1) = E_n^{(\alpha)}(x+y), \\ \lim_{q\to 1^-} P_{n,\lambda,q}^{(\alpha)}\left(x,y;1,-\frac{1}{2},1\right) &= G_n^{(\alpha)}(x+y). \end{aligned}$$

Proposition 1.2. *Unified q-Apostol-Bernoulli, Euler and Genocchi polynomials satisfy the following relations:*

$$P_{n,\beta,q}^{(\alpha)}(x,y;k,a,b) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q P_{l,\beta,q}^{(\alpha)}(0,0;k,a,b)(x+y)_q^l, \tag{4}$$

$$P_{n,\beta,q}^{(\alpha)}(x,y;k,a,b) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{\frac{(n-l)(n-l-1)}{2}} P_{l,\beta,q}^{(\alpha)}(x,0;k,a,b)\, y^{n-l}, \tag{5}$$

$$= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q P_{l,\beta,q}^{(\alpha)}(0,y;k,a,b)\, x^{n-l}. \tag{6}$$

Proof. **Proof of (4):** From (3); we write as:

$$\begin{aligned}\sum_{n=0}^{\infty} P_{n,\beta,q}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{[n]_q!} &= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q(tx)E_q(ty)\\ &= \sum_{n=0}^{\infty} P_{n,\beta,q}^{(\alpha)}(0,0;k,a,b)\frac{t^n}{[n]_q!}\sum_{r=0}^{\infty}(x+y)_q^r\frac{t^r}{[r]_q!}.\end{aligned}$$

Using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$, we have (4).

Proof of (5): From (3); we write as:

$$\begin{aligned}\sum_{n=0}^{\infty} P_{n,\beta,q}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{[n]_q!} &= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} e_q(tx)E_q(ty)\\ &= \sum_{n=0}^{\infty} P_{n,\beta,q}^{(\alpha)}(x,0;k,a,b)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty} q^{\frac{m(m-1)}{2}}y^m\frac{t^m}{[m]_q!}.\end{aligned}$$

Comparing of the coefficients of $\frac{t^n}{n!}$, on both sides of above equation, we obtain the desired result. □

Remark 1.3. *From (5);*

$$P_{n,\beta,q}^{(\alpha)}(x,1;k,a,b) = \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q P_{l,\beta,q}^{(\alpha)}(x,0;k,a,b). \tag{7}$$

From (6);

$$P_{n,\beta,q}^{(\alpha)}(1,y;k,a,b) = \sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_q q^{\frac{(n-l)(n-l-1)}{2}} P_{l,\beta,q}^{(\alpha)}(0,y;k,a,b). \tag{8}$$

Corollary 1.4. *Taking $q \to 1^-$ and $k = a = b = 1$ in (7), we have*

$$B_n^{(\alpha)}(x+1) = \sum_{l=0}^{n}\binom{n}{l}B_l^{(\alpha)}(x). \tag{9}$$

Corollary 1.5. *Taking $q \to 1^-$ and $k = 0$, $a = -1$, $b = 1$ in (8), we have*

$$E_n^{(\alpha)}(x+1) = \sum_{l=0}^{n}\binom{n}{l}E_l^{(\alpha)}(x). \tag{10}$$

Show that (7) and (8) are q-analogue of (9) and (10).

Lemma 1.6. *The following relation is true:*

$$D_{q,x}P_{n,\beta,q}^{(\alpha)}(x,y;k,a,b) = [n]_q P_{n-1,\beta,q}^{(\alpha)}(x,y;k,a,b).$$

Proof. From (3), taking derivative with respect to x, we obtain;

$$\sum_{n=0}^{\infty} D_{q,x}P_{n,\beta,q}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{[n]_q!} = \left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha} D_{q,x}\left(e_q(tx)\right)E_q(ty).$$

$$= t\sum_{n=0}^{\infty} P_{n,\beta,q}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty}[n]_q P_{n-1,\beta,q}^{(\alpha)}(x,y;k,a,b)\frac{t^n}{[n]_q!}.$$

We have the result, since $P_{0,\beta,q}^{(\alpha)}(x,y;k,a,b) = 0$. □

2. Explicit Relation for the Unified Family of Generalized q-Apostol-Bernoulli, Euler and Genocchi Polynomials

In this section, we aim to obtain the explicit relations of the polynomials $\mathcal{P}^{(\alpha)}_{n,q,\beta}(x,k,a,b)$ and give the relation between the unified family of generalized Apostol-Bernolli, Euler and Genocchi polynomials and the generalized q-Stirling numbers of second kind $S(n,\nu,a,b,\beta)$ of order ν.

Theorem 2.1. *The following relation holds true:*

$$P^{(\alpha-m)}_{n,\beta,q}(x,y;k,a,b)=\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q P^{-m}_{n-l,\beta,q}(0,0;k,a,b)\,P^{(\alpha)}_{l,\beta,q}(x,y;k,a,b)\,. \tag{11}$$

Proof.

$$\sum_{n=0}^{\infty}P^{(\alpha-m)}_{n,\beta,q}(x,y;k,a,b)\frac{t^n}{[n]_q!}=\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{-m}\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}e_q(tx)E_q(ty)$$

$$=\left(\sum_{n=0}^{\infty}P^{-m}_{n,\beta,q}(0,0;k,a,b)\frac{t^n}{[n]_q!}\right)\left(\sum_{r=0}^{\infty}P^{(\alpha)}_{r,\beta,q}(x,y;k,a,b)\frac{t^r}{[r]_q!}\right).$$

Using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at (11). □

Theorem 2.2. *The following relations hold true:*

$$\beta^b P^{(\alpha)}_{n,\beta,q}(1,y;k,a,b)-a^b P^{(\alpha)}_{n,\beta,q}(0,y;k,a,b)=2^{1-k}\frac{[n]_q!}{[n-k]_q!}P^{(\alpha)}_{n-k,\beta,q}(0,y;k,a,b) \tag{12}$$

and

$$\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q\left(\beta^b P^{(\alpha)}_{l,\beta,q}(x,0;k,a,b)-a^b P^{(\alpha)}_{l,\beta,q}(x,-1;k,a,b)\right)=2^{1-k}\frac{[n]_q!}{[n-k]_q!}P^{(\alpha)}_{n-k,\beta,q}(x,0;k,a,b)\,. \tag{13}$$

Proof. From (3):

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta^b P^{(\alpha)}_{n,\beta,q}(1,y;k,a,b)\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}a^b P^{(\alpha)}_{n,\beta,q}(0,y;k,a,b)\frac{t^n}{[n]_q!}\\
=\;&\beta^b\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}e_q(t)E_q(ty)-a^b\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}E_q(ty)\\
=\;&\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}E_q(ty)\left\{\beta^b e_q(t)-a^b\right\}=2^{1-k}t^k\sum_{n=0}^{\infty}P^{(\alpha-1)}_{n,\beta,q}(0,y;k,a,b)\frac{t^n}{[n]_q!}
\end{aligned}$$

$$=2^{1-k}\sum_{n=0}^{\infty}P^{(\alpha-1)}_{n,\beta,q}(0,y;k,a,b)\frac{t^{n+k}}{[n]_q!}\frac{[n+k]_q!}{[n+k]_q!}=2^{1-k}\sum_{n=0}^{\infty}\frac{[n]_q!}{[n-k]_q!}P^{(\alpha-1)}_{n-k,\beta,q}(0,y;k,a,b)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain (12).

Proof of (13):

From (3):

$$\begin{aligned}
&\sum_{n=0}^{\infty}\beta^b P^{(\alpha)}_{n,\beta,q}(x,0;k,a,b)\frac{t^n}{[n]_q!}-\sum_{n=0}^{\infty}a^b P^{(\alpha)}_{n,\beta,q}(x,-1;k,a,b)\frac{t^n}{[n]_q!}\\
=\;&\beta^b\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}e_q(xt)-a^b\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)^{\alpha}e_q(xt)E_q(-t)
\end{aligned}$$

$$= \left(\frac{2^{1-k}t^k}{\beta^b e_q(t) - a^b}\right)^{\alpha} e_q(tx)\left\{\beta^b - a^b \frac{1}{e_q(t)}\right\} = \frac{1}{e_q(t)} 2^{1-k}t^k \left(\frac{2^{1-k}t^k}{\beta^b e_q(t) - a^b}\right)^{\alpha-1} e_q(tx)$$

$$\sum_{n=0}^{\infty}\left\{\beta^b P_{n,\beta,q}^{(\alpha)}(x,0;k,a,b) - a^b P_{n,\beta,q}^{(\alpha)}(x,-1;k,a,b)\right\}\frac{t^n}{[n]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!} = 2^{1-k}t^k\sum_{n=0}^{\infty}P_{n,\beta,q}^{(\alpha-1)}(x,0;k,a,b)\frac{t^n}{[n]_q!}$$

$$\sum_{n=0}^{\infty}\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q \left(\beta^b P_{l,\beta,q}^{(\alpha)}(x,0;k,a,b) - a^b P_{l,\beta,q}^{(\alpha)}(x,-1;k,a,b)\right)\frac{t^n}{[n]_q!}$$

$$= \quad 2^{1-k}\sum_{n=0}^{\infty}P_{n,\beta,q}^{(\alpha-1)}(x,0;k,a,b)\frac{t^{n+k}}{[n+k]_q!} = 2^{1-k}\sum_{n=k}^{\infty}\frac{[n]_q!}{[n-k]_q!}P_{n-k,\beta,q}^{(\alpha-1)}(x,0;k,a,b)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain. □

Definition 2.3. *We define the generalized q-Stirling numbers $S(n,\nu,a,b,\beta)$ of the second kind of order ν as follows:*

$$\sum_{n=0}^{\infty}S(n,\nu,a,b,\beta)\frac{t^n}{[n]_q!} = \frac{\left(\beta^b e_q(t) - a^b\right)^{\nu}}{[\nu]_q!}.$$

Theorem 2.4. *There is the following relation between the generalized q-Stirling numbers $S(n,\nu,a,b,\beta)$ of the second kind and the unified q-Apostol-Bernoulli, Euler and Genocchi polynomials $\mathcal{P}_{n,q,\beta}^{(\alpha)}(x,y,k,a,b)$;*

$$\mathcal{P}_{n-\nu k,q,\beta}^{(\alpha)}(x,y,k,a,b) = 2^{(k-1)}\frac{[\nu]_q!\,[n-\nu k]_q!}{[n]_q!}\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q \mathcal{P}_{l,q,\beta}^{(\nu-\alpha)}(x,y,k,a,b)\,S(n-l,\nu,a,b,\beta). \tag{14}$$

Proof.

$$\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\frac{t^n}{[n]_q!} = \left(\frac{2^{1-k}t^k}{\beta^b e_q(t) - a^b}\right)^{\alpha} e_q(tx)E_q(ty)\frac{\left(\beta^b e_q(t) - a^b\right)^{\nu}}{[\nu]_q!}\frac{[\nu]_q!}{\left(\beta^b e_q(t) - a^b\right)^{\nu}}$$

$$= \frac{[\nu]_q!}{\left(2^{1-k}t^k\right)^{\nu}}\sum_{l=0}^{\infty}\mathcal{P}_{l,\beta,q}^{(\alpha-\nu)}(x,y,k,a,b)\frac{t^l}{[l]_q!}\sum_{m=0}^{\infty}\mathcal{S}(m,\nu,a,b,\beta)\frac{t^m}{m!}$$

$$\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)\frac{t^{n+\nu k}}{[n]_q!} = [\nu]_q!2^{(k-1)\nu}\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q \mathcal{P}_{l,\beta,q}^{(\alpha-\nu)}(x,y,k,a,b)\,\mathcal{S}(n-l,\nu,a,b,\beta)\right\}\frac{t^n}{[n]_q!}$$

$$\sum_{n=0}^{\infty}\frac{[n]_q!}{[n-\nu k]_q!}\mathcal{P}_{n-\nu k,\beta,q}^{(\alpha)}(x,y,k,a,b)\frac{t^n}{[n]_q!} = [\nu]_q!2^{(k-1)\nu}\sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\left[\begin{array}{c} n \\ l \end{array}\right]_q \mathcal{P}_{l,\beta,q}^{(\alpha-\nu)}(x,y,k,a,b)\,\mathcal{S}(n-l,\nu,a,b,\beta)\right\}\frac{t^n}{[n]_q!}.$$

From $\mathcal{P}_{0,\beta,q}^{(\alpha)}(x,y,k,a,b) = 0, \cdots, \mathcal{P}_{n-1-\nu k,\beta,q}^{(\alpha)}(x,y,k,a,b) = 0$. Using Cauchy product, we obtain (14). □

Theorem 2.5. *There is the following relation between the unified q-Apostol-Bernoulli, Euler and Genocchi polynomials $\mathcal{P}_{n,q,\beta}^{(\alpha)}(x,y,k,a,b)$:*

$$\mathcal{P}_{n+1,q,\beta}(x,y,k,a,b) = yq^k\mathcal{P}_{n,q,\beta}(qx,qy,k,a,b) + xq^k\mathcal{P}_{n,q,\beta}(x,y,k,a,b) + [k]_q\frac{[n]_q!}{[n+k]_q!}\mathcal{P}_{n+k,q,\beta}(x,y,k,a,b)$$

$$-q^k\beta^b\sum_{l=0}^{n+k}\left[\begin{array}{c} n+k \\ l \end{array}\right]_q \mathcal{P}_{l,q,\beta}(x,y,k,a,b)\,q^l\mathcal{P}_{n+k-l,q,\beta}(1,0,k,a,b). \tag{15}$$

Proof. For $\alpha = 1$ and using (3);

$$\sum_{n=0}^{\infty} D_{q,t}\mathcal{P}_{n,q,\beta}(x,y,k,a,b)\frac{t^n}{[n]_q!} = D_{q,t}\left(\frac{2^{1-k}t^k}{\beta^b e_q(t)-a^b}\right)e_q(tx)E_q(ty) = 2^{1-k}D_{q,t}\left(\frac{t^k}{\beta^b e_q(t)-a^b}\right)e_q(tx)E_q(ty)$$

$$= 2^{1-k}\left\{\frac{\left(\beta^b e_q(qt)-a^b\right)D_{q,t}\left(t^k e_q(tx)E_q(ty)\right)-\left((qt)^k e_q(qtx)E_q(qty)D_{q,t}\left(\beta^b e_q(t)-a^b\right)\right)}{\left(\beta^b e_q(t)-a^b\right)\left(\beta^b e_q(qt)-a^b\right)}\right\}$$

$$= 2^{1-k}\left(\frac{q^k t^k\left[y e_q(tx)E_q(ty)+x e_q(tx)E_q(ty)\right]+[k]\,e_q(tx)E_q(ty)}{\left(\beta^b e_q(t)-a^b\right)}-\frac{q^k\beta^b t^k e_q(qtx)E_q(qty)}{\left(\beta^b e_q(qt)-a^b\right)}\frac{e_q(t)}{\left(\beta^b e_q(t)-a^b\right)}\right).$$

After some mathematical calculations, we have

$$= \sum_{n=0}^{\infty}\left(q^k y\mathcal{P}_{n,q,\beta}(qx,qy,k,a,b)+xq^k\mathcal{P}_{n,q,\beta}(x,y,k,a,b)\right)\frac{t^n}{[n]_q!}+\sum_{n=k}^{\infty}\left\{[k]_q\frac{[n]_q!}{[n+k]_q!}\mathcal{P}_{n+k,q,\beta}(x,y,k,a,b)\right.$$
$$\left.-q^k\beta^b\sum_{l=0}^{n+k}\begin{bmatrix} n+k \\ l \end{bmatrix}_q \mathcal{P}_{l,q,\beta}(x,y,k,a,b)\,q^l\mathcal{P}_{n+k-l,q,\beta}(1,0,k,a,b)\right\}\frac{t^n}{[n]_q!}.$$

Using Cauchy product and by equating the coefficients of $\frac{t^n}{n!}$ on both sides of the resulting equation, we obtain the desired result. □

References

[1] Carlitz L., q-Bernoulli numbers and polynomials, Duke Math. J., **15**(1948), 987-1050.
[2] Carlitz L., q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc., **76**(1954), 332-350.
[3] Choi J., Anderson P. J. and Srivastava H. M.; Some q-extensions of the Apostol-Bernoulli and Apostol-Euler polynomials of order n and the multiple Hurwitz zeta functions, App. Math. and Comp., **199** (2008) 723-727.
[4] Dere R., Simsek Y. and Srivastava H. M.; Unified presentation of three families of generalized Apostol-type polynomials based upon the theory of the umbral calculus and the umbral algebra, J. of Number Theory, **13** (2013), 3245-3265.
[5] El-Desouky B. S. and Gamma R. S.; A new unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, App. Math. and Comp., **247** (2014), 695-702.
[6] Karande B. K. and Thakare N. K.; On the unification of the Bernoulli and Euler polynomials, Indian J. of Pure App. Math., **6**(1)(1975), 98-107.
[7] Kac V. and Cheung P., Quantum Calculus, Springer (2002).
[8] Kurt V.; New identities and relations derived from the generalized Bernoulli polynomials, Euler polynomials and Genocchi polynomials, Advances in Difference Equations, 2014.
[9] Luo Q.-M. and Srivastava H. M., Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, Comp. Math. App., **51**(2006), 631-642.
[10] Luo Q.-M. and Srivastava H. M., q-Extensions of some relationships between the Bernoulli and Euler polynomials, Taiwanese Journal of Mathematics, **15** (2011), 241-257.
[11] Luo Q.-M., Some results for the q-Bernoulli and q-Euler polynomials, J. Math. Anal. Appl., **363** (2010), 7-18.
[12] Mahmudov N. I., q-analogues of the Bernoulli and Genocchi polynomials and the Srivastava-Pintér addition theorems, Discrete Dynamics in Nature and Soc. Article number 169348, 2012. doi:10.1155/2012/169348.
[13] Mahmudov N. I.; On a class of q-Bernoulli and q-Euler polynomials, Adv. in Differ. Equ.,(2013), doi:10.1186/1687-1847-2013-103.
[14] Ozarslan M. A.; Unified Apostol-Bernoulli,Euler and Genocchi polynomials, Comp. Math. App., **62**(2011), 2452-2462.
[15] Ozden H., Simsek Y. and Srivastava H. M.; A unified presentation of the generating function of the generalized Bernoulli, Euler and Genocchi polynomials, Comp. and Math. with Appl, **60**(2010), 2779-2789.
[16] Srivastava H. M., Kurt B. and Simsek Y., Some families of Genocchi type polynomials and their interpolation function, Integral Trans. and Special func., **23**(2012).
[17] Srivastava H. M. and Choi J.; Zeta and q-zeta functions and associated series and integers, Elsevier Sciences Publ., Amsterdam, London and N. York, 2012.
[18] Srivastava H. M., Kim T and Simsek Y., q-Bernoulli numbers and polynomials associated with multiple q-Zeta functions and basic L-Series, Russian J. Math., **12** (2005), 241-268.
[19] Srivastava H. M., Some generalization and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, Appl. Mah. Inform. Sci., **5**(2011), 390-444.
[20] Srivastava H. M., Some generalization and basic ($or-q$) extension of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. and Infor. Sci., **5**(3) (2011) 390-444.