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Some Hermite Base Polynomials on q-Umbral Algebra

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Abstract. The aim of this paper is to investigate the q-Hermite type polynomials by using umbral calculus methods. Using this method, we derive new type polynomials which are related to the q-Bernoulli polynomials and the q-Hermite type polynomials. Furthermore, we also derive some new identities of those polynomials which are derived from q-umbral calculus.

1. Introduction, Definitions and Preliminaries

Throughout this paper, we use the following definitions and notations which are given by Roman [11]:

Let $\boldsymbol{P}$ be the algebra of polynomials in the single variable x over the field complex numbers. Let $\boldsymbol{P}^*$ be the vector space of all linear functionals on $\boldsymbol{P}$. Let $\langle L \mid p(x)\rangle$ be the action of a linear functional L on a polynomial $p(x)$. Let $\mathfrak{F}$ denotes the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k.$$

This kind of algebra is called an *umbral algebra*. Each $f(t) \in \mathfrak{F}$ defines a linear functional on $\boldsymbol{P}$ given by for all $k \geqslant 0$, $a_k = \left\langle f(t) \mid x^k \right\rangle$.

In particular,

$$\left\langle t^k \mid x^n \right\rangle = n!\delta_{n,k},$$

where

$$\delta_{n,k} = \begin{cases} 0 \text{ if } n \neq k \\ 1 \text{ if } n = k. \end{cases}$$

Let c_n be a fixed sequence of nonzero constants. Then we may map the algebra $\mathfrak{F}$ of all formal power series in t isomorphically onto the vector space $\boldsymbol{P}^*$ of all linear functionals on $\boldsymbol{P}$ by setting

$$\left\langle t^k \mid x^n \right\rangle = c_n\delta_{n,k}.$$

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If

$$f(t)=\sum_{k=0}^{\infty}\frac{a_k}{c_k}t^k$$

then we have

$$\langle f(t)\mid x^n\rangle=a_n.$$

q-umbral calculus is defined by setting

$$c_n=\frac{(1-q)\left(1-q^2\right)\cdots(1-q^n)}{(1-q)^n},$$

where $0<q<1$ when $q\in\mathbb{R}$ and $\left|q\right|<1$ when $q\in\mathbb{C}$.

Then, we have

$$\frac{c_n}{c_{n-1}}=\frac{1-q^n}{1-q}.$$

In this paper, we use the following notation:

$$[x]_q=\begin{cases}\frac{1-q^x}{1-q},q\neq 1\\ x,q=1,\end{cases}$$

[6],[11].

Derivative operator t is defined by

$$tx^n=[n]_q\,x^{n-1}.$$

So, for all $p(x)\in\boldsymbol{P}$

$$tp(x)=\frac{p(x)-p(qx)}{x-qx}. \tag{1}$$

q-binomial coefficient is

$$\binom{n}{k}_q=\frac{c_n}{c_kc_{n-k}}=\frac{[n]_q!}{[k]_q!\,[n-k]_q!}. \tag{2}$$

The evaluation functional is

$$\varepsilon_q(yt)=\sum_{k=0}^{\infty}\frac{(yt)^k}{[k]_q!}$$

which is the analog of the exponential series for the q-umbral calculus [11].

Remark 1.1. *The notational difference between*

$$\langle f(t)\mid p(x)\rangle \quad \textit{and} \quad f(t)\,p(x)$$

will make the particular role of $f(t)$ clear. We use $\langle f(t)\mid p(x)\rangle$ notation for functionals and $f(t)\,p(x)$ notation for operators.

For example, the functional $\varepsilon_q(yt)$ satisfies

$$\left\langle \varepsilon_q(yt) \mid x^n \right\rangle = \left\langle \sum_{k=0}^{\infty} \frac{(yt)^k}{[k]_q!} \mid x^n \right\rangle = y^n.$$

Then

$$\left\langle \varepsilon_q(yt) \mid p(x) \right\rangle = p(y), \tag{3}$$

for all polynomials $p(x) \in \boldsymbol{P}$.

The operator $\varepsilon_q(yt)$ satisfies

$$\varepsilon_q(yt)\, x^n = \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k}.$$

Theorem 1.2. *Let $f(t)$, $g(t)$ be in $\mathfrak{F}$, we have*

$$\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle. \tag{4}$$

The *order* $o(f(t))$ of a nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ for which $o(f(t)) = 1$ is called a *delta series*. And a series $f(t)$ for which $o(f(t)) = 0$ is called a *invertible series*.

Theorem 1.3. *Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exist a unique sequence $S_n(x)$ of polynomials satisfying the orthogonality conditions*

$$\left\langle g(t)f(t)^k \mid S_n(x) \right\rangle = [n]_q!\delta_{n,k} \tag{5}$$

for all $n, k \geq 0$.

Remark 1.4. *The sequence $S_n(x)$ in (5) is the q-Sheffer polynomials for pair $(g(t), f(t))$, where $g(t)$ must be invertible and $f(t)$ must be delta series. The q-Sheffer polynomials for pair $(g(t), t)$ is the q-Appell polynomials or the q-Appell sequences for $g(t)$.*

2. q-Appell Polynomials

Roman gave some principal results for nonclassical Sheffer polynomials in [11, p. 163-165]. All theorems in this section can be proved by Roman's method. Thus we omit them.

The q-Appell polynomials satisfy the following properties:

Theorem 2.1. *Let $S_n(x)$ be the q-Appell polynomial for $g(t)$. Then for any $h(t)$ in $\mathfrak{F}$,*

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid S_k(x) \rangle}{[k]_q!} g(t)\, t^k. \tag{6}$$

Setting $h(t) = \varepsilon_q(yt)$ in (6) and using (3), one can obtain the following corollary:

Corollary 2.2.

$$\varepsilon_q(yt) = \sum_{k=0}^{\infty} \frac{\left\langle \varepsilon_q(yt) \mid S_k(x) \right\rangle}{[k]_q!} g(t)\, t^k = \sum_{k=0}^{\infty} \frac{S_k(y)}{[k]_q!} g(t)\, t^k.$$

Theorem 2.3. *Let $y \in \mathbb{C}$. The polynomial $S_n(x)$ is the q-Appell polynomials for $g(t)$ if and only if*

$$\frac{1}{g(t)}\varepsilon_q(yt) = \sum_{k=0}^{\infty} \frac{S_k(y)}{[k]_q!} t^k \tag{7}$$

for all constants $y \in \mathbb{C}$.

Theorem 2.4. *The polynomial $S_n(x)$ is the q-Appell polynomials for $g(t)$ if and only if*

$$S_n(x) = g(t)^{-1} x^n \tag{8}$$

Theorem 2.5. *The polynomial $S_n(x)$ is the q-Appell polynomials for $g(t)$ if and only if*

$$tS_n(x) = [n]_q S_{n-1}(x). \tag{9}$$

Theorem 2.6. *A polynomial $S_n(x)$ is the q-Appell polynomials for $g(t)$ if and only if*

$$\varepsilon_q(yt) S_n(x) = \sum_{k=0}^{n} \binom{n}{k}_q y^k S_{n-k}(x). \tag{10}$$

2.1. q-Hermite Type Polynomials

The q-Hermite type polynomials are defined by means of the following generating function:

$$\sum_{k=0}^{\infty} H_{k,q}^{(v)}(x) \frac{t^k}{[k]_q!} = \varepsilon_q^{-1}\left(\frac{vt^2}{2}\right)\varepsilon_q(xt).$$

The q-Hermite polynomials are the q-Appell polynomials for

$$g(t) = \varepsilon_q\left(\frac{vt^2}{2}\right). \tag{11}$$

From (8) and (11), we get the following lemma:

Lemma 2.7. *The following relationship holds true:*

$$H_{n,q}^{(v)}(x) = \varepsilon_q^{-1}\left(\frac{vt^2}{2}\right)x^n. \tag{12}$$

By using (9), we arrive at the following theorem:

Theorem 2.8. *The following operator identity holds true:*

$$tH_{n,q}^{(v)}(x) = [n]_q H_{n-1,q}^{(v)}(x). \tag{13}$$

The following theorem gives us action of linear operators $\varepsilon_q(yt)$ and $\varepsilon_q\left(\frac{vt^2}{2}\right)$ on the polynomials $H_{n,q}^{(v)}(x)$:

Theorem 2.9.

$$\varepsilon_q(yt) H_{n,q}^{(v)}(x) = \sum_{k=0}^{n} \binom{n}{k}_q y^k H_{n-k,q}^{(v)}(x) \tag{14}$$

and

$$\varepsilon_q\left(\frac{vt^2}{2}\right) H_{n,q}^{(v)}(x) = x^n. \tag{15}$$

Proof. The first part of proof is completed by using (10). We use (8) for proving (15). Therefore

$$\begin{aligned} \varepsilon_q\left(\frac{vt^2}{2}\right)H_{n,q}^{(v)}(x) &= \varepsilon_q\left(\frac{vt^2}{2}\right)\varepsilon_q^{-1}\left(\frac{vt^2}{2}\right)x^n, \\ &= x^n. \end{aligned}$$

□

2.2. q-Hermite Base Bernoulli Polynomials

In this section, we construct new generating functions for the Hermite base Bernoulli type polynomials which are generalized Milne-Thomson polynomials (see for details: [9],[4],[2]).

Our generating function is defined as follows:

$$\sum_{k=0}^{\infty} B_{H,k,q}^{(\alpha)}(x,v)\frac{t^k}{[k]_q!} = \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha}\varepsilon_q^{-1}\left(\frac{vt^2}{2}\right)\varepsilon_q(xt).$$

The q-Hermite type polynomials are the q-Appell polynomials for

$$g(t) = \left(\frac{\varepsilon_q(t)-1}{t}\right)^{\alpha}\varepsilon_q\left(\frac{vt^2}{2}\right). \tag{16}$$

From (8) and (16), we arrive at the following lemma:

Lemma 2.10. *The following relationship holds true:*

$$B_{H,n,q}^{(\alpha)}(x,v) = \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha}\varepsilon_q^{-1}\left(\frac{vt^2}{2}\right)x^n. \tag{17}$$

By using (9), we give the action of linear operator t on the polynomials $B_{H,k,q}^{(\alpha)}(x,v)$:

Theorem 2.11.

$$tB_{H,n,q}^{(\alpha)}(x,v) = [n]_q B_{H,n-1,q}^{(\alpha)}(x,v). \tag{18}$$

Theorem 2.12. *The following identities hold true:*

$$\left(\varepsilon_q(t)-1\right)B_{H,n,q}^{(\alpha)}(x,v) = [n]_q B_{H,n-1,q}^{(\alpha-1)}(x,v) \tag{19}$$

and

$$\left\langle \left(\varepsilon_q(t)-1\right) \mid B_{H,n,q}^{(\alpha)}(x,v)\right\rangle = [n]_q\left\langle \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha-1} \mid H_{n-1,q}^{(v)}(x)\right\rangle. \tag{20}$$

Proof. We prove assertion (19) by using (17):

$$\left(\varepsilon_q(t)-1\right)B_{H,n,q}^{(\alpha)}(x,v) = \left(\varepsilon_q(t)-1\right)\left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha}\varepsilon_q^{-1}\left(\frac{vt^2}{2}\right)x^n.$$

After some calculations, it yields

$$\left(\varepsilon_q(t)-1\right)B_{H,n,q}^{(\alpha)}(x,v) = tB_{H,n,q}^{(\alpha-1)}(x,v).$$

Using (18) in the above equation, we get the desired result.
We now prove (19):
From (17), we obtain

$$\left\langle \left(\varepsilon_q(t)-1\right) \mid B_{H,n,q}^{(\alpha)}(x,v)\right\rangle = \left\langle \left(\varepsilon_q(t)-1\right) \mid \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha} \varepsilon_q^{-1}\left(\frac{vt^2}{2}\right)x^n\right\rangle.$$

By using (4) and (12), we get

$$\left\langle \left(\varepsilon_q(t)-1\right) \mid B_{H,n,q}^{(\alpha)}(x,v)\right\rangle = \left\langle \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha-1} \mid tH_{n,q}^{(v)}(x)\right\rangle. \tag{21}$$

Using (13) in the above equation, we complete the proof. □

Theorem 2.13. *The following identity holds true:*

$$\left(\frac{t}{\varepsilon_q(t)-1}\right)^{\beta} B_{H,n,q}^{(a)}(x,v) = B_{H,n,q}^{(\alpha+\beta)}(x,v).$$

Proof. As a consequence of (17), we have

$$\left(\frac{t}{\varepsilon_q(t)-1}\right)^{\beta} B_{H,n,q}^{(a)}(x,v) = \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\beta}\left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha} \varepsilon_q^{-1}\left(\frac{vt^2}{2}\right)x^n.$$

Hence, we obtain the result by using (17). □

Let

$$\sum_{k=0}^{\infty} B_{k,q}^{(\alpha)}(x)\frac{t^k}{[k]_q!} = \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha}\varepsilon_q(xt).$$

The action of linear operator $\varepsilon_q\left(\frac{vt^2}{2}\right)$ on $B_{H,n,q}^{(\alpha)}(x,v)$ gives us the q-Bernoulli polynomials of higher order $B_{n,q}^{(\alpha)}(x)$, as follows:

Theorem 2.14.

$$\varepsilon_q\left(\frac{vt^2}{2}\right)B_{H,n,q}^{(\alpha)}(x,v) = B_{n,q}^{(\alpha)}(x).$$

Proof. By using (17), we have

$$\varepsilon_q\left(\frac{vt^2}{2}\right)B_{H,n,q}^{(\alpha)}(x,v) = \varepsilon_q\left(\frac{vt^2}{2}\right)\left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha}\varepsilon_q^{-1}\left(\frac{vt^2}{2}\right)x^n.$$

Therefore

$$\varepsilon_q\left(\frac{vt^2}{2}\right)B_{H,n,q}^{(\alpha)}(x,v) = \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha}x^n,$$

Combining this equation with the following result which was proved by Kim et al. [8], we complete the proof:

$$B_{n,q}^{(\alpha)}(x) = \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha}x^n.$$

Hence, we completed the proof. □

Theorem 2.15. *Relationship between $H_{n,q}^{(v)}(x)$ and $B_{H,n,q}^{(\alpha)}(x,v)$ is given by following equation:*

$$\left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha} H_{n,q}^{(v)}(x) = B_{H,n,q}^{(\alpha)}(x,v),$$

Proof. By using (12) and (17), we get,

$$\left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha} H_{n,q}^{(v)}(x) = \left(\frac{t}{\varepsilon_q(t)-1}\right)^{\alpha} \varepsilon_q^{-1}\left(\frac{vt^2}{2}\right) x^n = B_{H,n,q}^{(\alpha)}(x,v).$$

□

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