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Generating Functions For Two-Variable Polynomials Related To a Family of Fibonacci Type Polynomials and Numbers

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Abstract. The purpose of this paper is to construct generating functions for the family of the Fibonacci and Jacobsthal polynomials. Using these generating functions and their functional equations, we investigate some properties of these polynomials. We also give relationships between the Fibonacci, Jacobsthal, Chebyshev polynomials and the other well known polynomials. Finally, we give some infinite series applications related to these polynomials and their generating functions.

1. Introduction

The Fibonacci numbers, F_n and the Jacobsthal numbers, J_n are very famous in the array of integer sequences. These sequences have an enormous amount of information in the mathematical literature (*cf.* [1]-[12]; see also the references cited in each of these earlier works). Similarly, the famous Fibonacci and Jacobsthal polynomials have also recently investigated and studied by many authors in different methods. In this paper, therefore we give some generating functions related to these sequences and polynomials. By using these generating functions, we derive various identities, formulas and relations, some old and some new, for the Fibonacci and the Jacobsthal sequences and polynomials. It is well-known that the classes of the Fibonacci and the Jacobsthal polynomials are related to the Chebyshev polynomials (*cf.* [1], [7]). We also construct new generating functions related to new classes of polynomials which include the Fibonacci polynomials, the Jacobsthal polynomials, the generalized Chebyshev polynomials, the Vieta-Fibonacci and the Vieta-Lucas polynomials, the Humbert polynomials, the Geganbauer polynomials, etc.

The well-known Fibonacci polynomials and Jacobsthal polynomials are defined by the following definition:

Definition 1.1. *It is well known that the Fibonacci polynomials, $\{F_n(x)\}_{n=0}^{\infty}$ and the Jacobsthal polynomials, $\{J_n(x)\}_{n=0}^{\infty}$ are series of polynomials that satisfy the recurrence relation, respectively:*

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

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for all $n \geq 3$, *with initials* $F_1(x) = 1$ *and* $F_2(x) = x$ *and*

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x)$$

for all $n \geq 3$, *with initials* $J_1(x) = J_2(x) = 1$*(cf. [1]-[11]).*

Throughout of this paper for generating functions, we assume that $|t| < 1$. The Fibonacci and Jacobsthal polynomials are given by means of the following generating functions, respectively:

$$G_F(x,t) = \frac{t}{1 - xt - t^2} = \sum_{n=0}^{\infty} F_n(x)\, t^n \tag{1}$$

and

$$G_J(x,t) = \frac{t}{1 - t - xt^2} = \sum_{n=0}^{\infty} J_n(x)\, t^n \tag{2}$$

(*cf.* [1]-[11]).

Remark 1.2. *By (1), we easily see that the following results:*

$$F_{2n+1}(0) = 1, \quad F_{2n}(0) = 0, \quad F_n(1) = F_n, \quad F_n(2) = P_n,$$

where P_n *denotes the Pell numbers (cf. [1]-[11]).*

2. Generating Functions For The Family of Fibonacci and Jacobsthal Polynomials

In this section, we construct novel generating functions for various kind of the well-known polynomials. We investigate some properties of these functions. By using derivative operators and other algebraic manipulations, we give some functional equations and PDEs of these generating functions. By using these functions, we also derive some identities and relations associated with these polynomials.

We define a new family of two variable polynomials, $\mathcal{G}_j(x,y;k,m,n)$ by means of the following generating functions:

$$H(t;x,y;k,m,n) = \sum_{j=0}^{\infty} \mathcal{G}_j(x,y;k,m,n)\, t^j = \frac{1}{1 - x^k t - y^m t^{n+m}}, \tag{3}$$

where $k, m, n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Note that there is one generating function for each value of k, m and n.

Theorem 2.1. *([10, P. 69 Lemma 10-11], [12]) For* $m \in \mathbb{N}$, *we have*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{m}\right]} A(k, n - mk)$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{m}\right]} B(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n + mk).$$

By using (3) and Theorem 2.1, we give explicit formula for the polynomials, $\mathcal{G}_j(x,y;k,m,n)$ by the following theorem:

Theorem 2.2.

$$\mathcal{G}_j(x,y;k,m,n)=\sum_{c=0}^{\left[\frac{j}{m+n}\right]}\binom{j-c(m+n-1)}{c}y^{mc}x^{jk-mck-nck}, \tag{4}$$

where $[a]$ *is the largest integer* $\leq a$.

Proof. By using (3), we get

$$H(t;x,y;k,m,n)=\sum_{j=0}^{\infty}\sum_{c=0}^{\infty}\binom{j+c}{c}\left(x^kt\right)^j\left(y^mt^{m+n}\right)^c,$$

where $\left|x^kt\right|<1$ and $\left|y^mt^{m+n}\right|<1$. By using Theorem 2.1, after some elementary calculations in the above equation, we get

$$H(t;x,y;k,m,n)=\sum_{j=0}^{\infty}\sum_{c=0}^{\left[\frac{j}{m+n}\right]}\binom{j-mc-nc+c}{c}\left(x^kt\right)^{j-mc-nc}\left(y^mt^{m+n}\right)^c.$$

Therefore

$$\sum_{j=0}^{\infty}\mathcal{G}_j(x,y;k,m,n)t^j=\sum_{j=0}^{\infty}\left(\sum_{c=0}^{\left[\frac{j}{m+n}\right]}\binom{j-mc-nc+c}{c}x^{jk-mck-nck}y^{mc}\right)t^j.$$

Comparing the coefficients of t^j on both sides of the above equation, we arrive at the desired result. □

A relation between the polynomials $\mathcal{G}_j(x,y;k,m,n)$ and the Legendre polynomials $P_j(x)$ is given by the following theorem:

Theorem 2.3.

$$\mathcal{G}_j(2x,-1;1,1,1)=\sum_{c=0}^{j}P_{j-c}(x)P_c(x).$$

Proof. In [11, P. 83, Eq-(8)], the Legendre polynomials are given by

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}}=\sum_{j=0}^{\infty}P_j(x)t^j.$$

From the above equation and (3), we get

$$\sum_{j=0}^{\infty}\mathcal{G}_j(2x,-1;1,1,1)t^j=\sum_{j=0}^{\infty}\left(\sum_{c=0}^{j}P_{j-c}(x)P_c(x)\right)t^j.$$

Comparing the coefficients of t^j on both sides of the above equation, we arrive at the desired result. □

Two variable Fibonacci type polynomials, $W_j(x, y; k, m, n)$ are defined by means of the following generating functions:

$$\begin{aligned} R(t; x, y; k, m, n) &= H(t; x, y; k, m, n)t^n \\ &= \sum_{j=0}^{\infty} W_j(x, y; k, m, n)\, t^j. \end{aligned} \tag{5}$$

Observe that if we substitute $k = m = n = 1$ and $y = 1$ into (5), we arrive at (1). That is

$$F_j(x) = W_j(x, 1; 1, 1, 1).$$

Also, if we substitute $k = m = n = 1$ and $x = 1$ into (5), we arrive at (2). That is

$$J_j(y) = W_j(1, y; 1, 1, 1).$$

We can give a relation between the polynomials $W_j(x, y; k, m, n)$ and $\mathcal{G}_j(x, y; k, m, n)$ as follows:

$$\sum_{j=n}^{\infty} \mathcal{G}_{j-n}(x, y; k, m, n)\, t^j = \sum_{j=0}^{\infty} W_j(x, y; k, m, n)\, t^j.$$

Remark 2.4. *Substituting $k = m = 1$, $y = -1$ and $n = l - 1$ ($l \in \mathbb{N}$) into (3), we get the well-known first and second kind generalized Chebyshev polynomials, respectively:*

$$V_{j,l}(x) = \mathcal{G}_j(x, -1; 1, 1, l - 1)$$

and also

$$H(t; x, -1; 1, 1, l - 1) - tR(t; x, -1; 1, 1, l - 1) = \sum_{j=1}^{\infty} \Omega_{j,l}(x)\, t^j.$$

(see for details cf. [1]).

Remark 2.5. *If we substitute $k = m = n = 1$ and $y = -1$ into (5), we obtain the following well-known Vieta-Fibonacci polynomials, $V_j(x)$ and Vieta-Lucas polynomials, $v_j(x)$, respectively (cf. [6]):*

$$W_j(x, -1; 1, 1, 1) = V_j(x)$$

and

$$v_j(x) = 2\mathcal{G}_j(x, -1; 1, 1, 1) - 2xW_j(x, -1; 1, 1, 1).$$

Remark 2.6. *By setting $k = m = 1$, $y = -1$ and $n = a - 1$, a is an integer with $a > 1$, into (5), we obtain the Humbert polynomials, $h^{(1)}_{n,a}(x)$ (cf. [11, P. 86, Eq-(26)]):*

$$h^{(1)}_{n,a}(x) = \mathcal{G}_j(ax, -1; k, 1, a - 1).$$

For $a = 2$, we easily see that the Humbert polynomials, $h^{(v)}_{n,2}(x)$ reduce to the Gegenbauer polynomials $C^{v}_n(x)$ (cf. [11, P. 86, Eq-(26)]).

3. Partial Derivatives For The Generating Functions

In this section, by applying $\frac{\partial}{\partial x}H(t;x,y;k,m,n)$, $\frac{\partial}{\partial y}H(t;x,y;k,m,n)$ and $\frac{\partial}{\partial t}H(t;x,y;k,m,n)$ partial derivative operators, we give some partial derivative equations. By using these equations with generating functions, we derive two derivative formulas and a recurrence formula for our polynomials $\mathcal{G}_j(x,y;k,m,n)$.

Using partial derivative, with respect to x, we obtain the following higher-order partial differential equation:

$$\frac{\partial}{\partial x}H(t;x,y;k,m,n) = kx^{k-1}tH^2(t;x,y;k,m,n). \tag{6}$$

Using partial derivative, with respect to y, we obtain the following higher-order partial differential equation:

$$\frac{\partial}{\partial y}H(t;x,y;k,m,n) = my^{m-1}t^{m+n}H^2(t;x,y;k,m,n). \tag{7}$$

Using partial derivative, with respect to t, we obtain the following higher-order partial differential equation:

$$\frac{\partial}{\partial t}H(t;x,y;k,m,n) = \left(x^k + (m+n)\,y^m t^{m+n-1}\right)H^2(t;x,y;k,m,n). \tag{8}$$

3.1. *Derivative Formulas*

By applying PDEs of the generating functions, we give two derivative formulas for the polynomials $\mathcal{G}_j(x,y;k,m,n)$.

By combining (6) with (3), since $\mathcal{G}_j(x,y;k,m,n) = 1$, we get

$$\sum_{j=1}^{\infty}\frac{\partial}{\partial x}\mathcal{G}_j(x,y;k,m,n)\,t^j = \sum_{j=1}^{\infty}\sum_{l=0}^{j-1}kx^{k-1}\mathcal{G}_l(x,y;k,m,n)\,\mathcal{G}_{j-1-l}(x,y;k,m,n)\,t^j.$$

Comparing the coefficients of t^j on both sides of the above equation, we arrive at the following theorem:

Theorem 3.1. *Let j be a positive integer. Then we have*

$$\frac{\partial}{\partial x}\mathcal{G}_j(x,y;k,m,n) = \sum_{l=0}^{j-1}kx^{k-1}\mathcal{G}_l(x,y;k,m,n)\,\mathcal{G}_{j-1-l}(x,y;k,m,n).$$

By combining (7) and (3) with the following evaluation:
If $j < m+n$, then $\left[\frac{j}{m+n}\right] = 0$. Thus, the polynomials $\mathcal{G}_j(x,y;k,m,n)$ don't include the term y. Therefore, $\frac{\partial}{\partial y}\mathcal{G}_j(x,y;k,m,n) = 0$. We obtain

$$\sum_{j=m+n}^{\infty}\frac{\partial}{\partial y}\mathcal{G}_j(x,y;k,m,n)\,t^j = \sum_{j=m+n}^{\infty}\sum_{l=0}^{j-m-n}my^{m-1}\mathcal{G}_l(x,y;k,m,n)\,\mathcal{G}_{j-m-n-l}(x,y;k,m,n)\,t^j.$$

Comparing the coefficients of t^j on both sides of the above equation, we arrive at the following theorem:

Theorem 3.2. *Let $j \geq m+n$. Then we have*

$$\frac{\partial}{\partial y}\mathcal{G}_j(x,y;k,m,n) = \sum_{l=0}^{j-m-n}my^{m-1}\mathcal{G}_l(x,y;k,m,n)\,\mathcal{G}_{j-m-n-l}(x,y;k,m,n).$$

3.2. Recurrence Formula

Here, using PDE with generating functions, we derive a recurrence formula for our polynomials $\mathcal{G}_j(x, y; k, m, n)$.

By combining (8) with (3), we get

$$\sum_{j=0}^{\infty}(j+1)\mathcal{G}_{j+1}(x, y; k, m, n)t^j - x^k\sum_{j=0}^{\infty}\sum_{l=0}^{j}\mathcal{G}_l(x, y; k, m, n)\mathcal{G}_{j-l}(x, y; k, m, n)t^j$$

$$= \sum_{j=m+n-1}^{\infty}\sum_{l=0}^{j-m-n+1}(m+n)y^m\mathcal{G}_l(x, y; k, m, n)\mathcal{G}_{j-m-n-l+1}(x, y; k, m, n)t^j.$$

Comparing the coefficients of t^j on both sides of the above equation, we arrive at the following theorem:

Theorem 3.3. *If $j < m + n - 1$, then we have*

$$(j+1)\mathcal{G}_{j+1}(x, y; k, m, n) - x^k\sum_{l=0}^{j}\mathcal{G}_l(x, y; k, m, n)\mathcal{G}_{j-l}(x, y; k, m, n) = 0.$$

If $j \geq m + n - 1$, then we have

$$\begin{aligned} &(j+1)\mathcal{G}_{j+1}(x, y; k, m, n) - x^k\sum_{l=0}^{j}\mathcal{G}_l(x, y; k, m, n)\mathcal{G}_{j-l}(x, y; k, m, n) \\ = \quad &(m+n)y^m\sum_{l=0}^{j-m-n+1}\mathcal{G}_l(x, y; k, m, n)\mathcal{G}_{j-m-n-l+1}(x, y; k, m, n). \end{aligned}$$

4. Applications of The Generating Functions, $R(t; x, y; k, m, n)$

In this section, we give some series including our polynomials and also the Fibonacci type polynomials and numbers. In 1953, F. Stancliff found the following sum including the Fibonacci numbers:

$$\sum_{n=0}^{\infty}\frac{F_n}{10^{n+1}} = \frac{1}{F_{11}} \tag{9}$$

($cf.$ [7, P. 424]).

Setting $t = \frac{1}{c}$ with $c > 1$ in (5), we get the following series:

$$\sum_{j=0}^{\infty}\frac{W_j(x, y; k, m, n)}{c^j} = \frac{c^m}{c^{m+n} - x^k c^{n+m-1} - y^m}. \tag{10}$$

Remark 4.1. *If we set $c = 10$, $x = y = 1$, $k = m = n = 1$ in (10), we get (9).*

And also, by substituting $c = 2$, $x = y = 1$, $k = m = n = 1$ into in (10), we get

$$\sum_{j=0}^{\infty}\frac{F_j}{2^j} = 2.$$

($cf.$ [7, P. 437]).

Remark 4.2. *For $c = 2$, $y = 1$, $k = m = n = 1$, (10) reduces to the following sum:*

$$\sum_{j=0}^{\infty} \frac{F_j(x)}{2^j} = \frac{2}{3-2x}.$$

Similarly, setting $c = 3$, $x = 1$, $k = m = n = 1$ in (10), we get the following sum:

$$\sum_{d=0}^{\infty} \frac{J_d(y)}{3^d} = \frac{3}{6-y}.$$

Remark 4.3. *For $t = \frac{1}{5}$, $y = -1$, $k = 1$, $m = 1$, $n = l - 1$, (3) reduces to the following sum reletad to the generalized Chebyshev polynomials of the first kind:*

$$\sum_{j=0}^{\infty} \frac{V_{j,l}(x)}{5^j} = \frac{5^l}{5^l - 5^{l-1}x + 1}.$$

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