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A Note on Hardy type Sums and Dedekind Sums

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Abstract. In [9], Cetin et al. defined a new special finite sum which is denoted by $C_1(h,k)$. In this paper, with the help of two-term polynomial relation, we will give the explicit values of the sum $C_1(h,k)$. We will see that for the odd values of h and k, this sum only depends on one variable. After that we will give many properties of this sum and connections with other well-known finite sums such as the Dedekind sums, the Hardy sums and the Simsek sums $Y(h,k)$. By using the Fibonacci numbers and two-term polynomial relation, we will also give some relations for these sums.

1. Introduction

Finite arithmetic sums have great importance in analytic number theory, analysis and many other areas of mathematics. These sums including the greatest integer function $[x]$, became more popular in the nineteenth century. So mathematicians need to know more about $[x]$ and also related functions like the sawtooth function $((x))$, where

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{if } x \text{ is not an integer} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

With the help of these functions, many finite arithmetic sums have been defined and studied. During this paper, the set of positive integers will be shown by $\mathbb{N}^+$and the set of integers will be shown by $\mathbb{Z}$.

In the nineteenth century, Richard Dedekind defined the sums called Dedekind sums as follows:

$$s(h,k) = \sum_{j=1}^{k-1} \left(\left(\frac{hj}{k}\right)\right)\left(\left(\frac{j}{k}\right)\right),$$

where h is an integer, k is a positive integer. The basic introduction to the arithmetic properties of the Dedekind sum is [17]. Dedekind defined these sums with the help of the famous Dedekind eta function. Although Dedekind sums arise in the transformation formula for the eta function, they can be defined independently of the eta function. Dedekind sums have many interesting properties but most important

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one is the reciprocity theorem: When h and k are coprime positive integers, the following reciprocity law holds [10]:

$$s(h,k)+s(k,h)=-\frac{1}{4}+\frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}+\frac{1}{hk}\right). \tag{1}$$

The first proof of (1) was given by Richard Dedekind in 1892 [10]. After R. Dedekind, Apostol [1] and many authors have given many different proofs [17]. By using contour integration, in 1905, Hardy, [12], gave another proof of the reciprocity theorem. In that work, Hardy also gave some finite arithmetical sums which are called Hardy sums. These Hardy sums are also related to the Dedekind sums and have many useful properties.

We are ready to recall the Hardy sums which are needed in the further sections: If h and $k \in \mathbb{Z}$ with $k>0$, the Hardy sums are defined by

$$\begin{aligned}
S(h,k) &= \sum_{j \bmod k} (-1)^{j+1+[\frac{jh}{k}]}, \\
s_1(h,k) &= \sum_{j \bmod k} (-1)^{[\frac{jh}{k}]}\left(\left(\frac{j}{k}\right)\right), \\
s_2(h,k) &= \sum_{j \bmod k} (-1)^{j}\left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{jh}{k}\right)\right), \\
s_3(h,k) &= \sum_{j \bmod k} (-1)^{j}\left(\left(\frac{jh}{k}\right)\right), \\
s_4(h,k) &= \sum_{j \bmod k} (-1)^{[\frac{jh}{k}]}, \\
s_5(h,k) &= \sum_{j \bmod k} (-1)^{j+[\frac{hj}{k}]}\left(\left(\frac{j}{k}\right)\right).
\end{aligned} \tag{2}$$

We also note that some authors have called Hardy sums as Hardy-Berndt sums. For $s_5(h,k)$, the below equality also holds true:

$$s_5(h,k)=\frac{1}{k}\sum_{j=1}^{k-1} j(-1)^{j+\left[\frac{hj}{k}\right]} \tag{3}$$

when h and k are odd integers, [7]. Further, following equations will be necessary in the next section, [15]:

$$\begin{aligned}
\sum_{j=1}^{b-1}(-1)^{j+\left[\frac{cj}{b}\right]}\left(\frac{j}{b}\right) &= s_5(c,b)-\frac{1}{2}S(c,b), \\
\sum_{j=1}^{c-1}(-1)^{j+\left[\frac{bj}{c}\right]}\left(\frac{j}{c}\right) &= s_5(b,c)-\frac{1}{2}S(b,c).
\end{aligned} \tag{4}$$

Reciprocity law for the $s_5(h,k)$ is given by the following theorem:

Theorem 1.1. *Let h and k be coprime positive integers. If h and k are odd, then*

$$s_5(h,k)+s_5(k,h)=\frac{1}{2}-\frac{1}{2hk}, \tag{5}$$

(*cf.* [2], [4], [7], [11], [12], [24] and the references cited in each of these works).

The proof of the next reciprocity theorem was given by Hardy [12] and Berndt [6]:

Theorem 1.2. *Let h and k be coprime positive integers. Then*

$$S(h,k) + S(k,h) = 1 \quad \text{if} \quad h + k \text{ is odd.} \tag{6}$$

In the light of equation (6), Apostol [2] gave the below result:

Theorem 1.3. *If both h and k are odd and $(h,k) = 1$, then*

$$S(h,k) = S(k,h) = 0. \tag{7}$$

In [21], Simsek gave the following new sums: Let h is an integer and k is a positive integer with $(h,k) = 1$. Then

$$Y(h,k) = 4k \sum_{j \bmod k} (-1)^{j+[\frac{hj}{k}]} \left(\left(\frac{j}{k}\right)\right).$$

We observe that $Y(h,k)$ sums are also related to the Hardy sums $s_5(h,k)$. That is

$$Y(h,k) = 4ks_5(h,k).$$

The reciprocity law for this sum was given by Simsek in [21, p. 5, Theorem 4] as below:

$$hY(h,k) + kY(k,h) = 2hk - 2. \tag{8}$$

Simsek gave two different proofs of this reciprocity law. Another proof of (8) was also given in [9].

The below theorem was given by Sitaramachandrarao in [24]:

Theorem 1.4. *Let h and k be coprime positive integers. If k is an odd integer, then*

$$2s_3(h,k) - s_4(k,h) = 1 - \frac{h}{k}.$$

In this paper we study the Hardy sums, the Simsek sums $Y(h,k)$ and the Dedekind sums $s(h,k)$ which are related to the symmetric pairs, [13], and the Fibonacci numbers. Before starting our results, we need some properties of the Fibonacci numbers which are given as follows: The Fibonacci numbers are defined by means of the following generating function [14]:

$$F(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n. \tag{9}$$

One can easily derive the following recurrence relation from (9):

$$F_{n+1} = F_n + F_{n-1}.$$

From (9), we also easily compute the first few Fibonacci numbers as follows: $0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots$

In [13], Meyer studied a special case of the Dedekind sums. In that paper, Meyer investigated the pairs of integers $\{h,k\}$ for which $s(h,k) = s(k,h)$. Meyer defined that $\{h,k\}$ is a symmetric pair if this property holds and he showed that $\{h,k\}$ is a symmetric pair if and only if $h = F_{2n+1}$ and $k = F_{2n+3}$ for $n \in \mathbb{N}$ where F_m is the $m-$th Fibonacci number. In [13], Meyer proved the following theorem:

Theorem 1.5. *If $(h,k) = 1$ and $\{h,k\}$ is a symmetric pair, then $s(h,k) = 0$.*

In [9], Cetin et al. defined the sum $C_1(h,k)$ as follows:

$$C_1(h,k) = \sum_{j=1}^{k-1} \left(\left(\frac{hj}{k}\right)\right) (-1)^{j+\left[\frac{hj}{k}\right]}$$

where h, k are positive integers with $(h,k) = 1$.

2. A New Special Finite Sum and Its Properties

As we mentioned above, in [9], Cetin et al. defined the sum $C_1(h,k)$. Now, in this paper, we will give its explicit values when h and k are odd numbers with $k > 0$. We will see that the explicit values of the sum $C_1(h,k)$ depend on only one variable. After that, we will give the reciprocity law for the sum $C_1(h,k)$. The sum $C_1(h,k)$ has relations with the reciprocity laws of the Hardy sums $s_3(h,k)$, $s_4(h,k)$ and $s_5(h,k)$, the Dedekind sum $s(h,k)$ and the Simsek sum $Y(h,k)$. So we will give those relations. Berndt and Dieter [5] showed that

$$\sum_{j=1}^{\frac{1}{2}(k-1)}\left[\frac{hj}{k}\right]+\sum_{j=1}^{\frac{1}{2}(h-1)}\left[\frac{kj}{h}\right]=(h-1)(k-1), \tag{10}$$

where h and k are odd, distinct primes. This relation plays very important role in the proof of Gauss' law of quadratic reciprocity. We will give the equation 10 in terms of the sum $C_1(h,k)$. Finally, we will give relations between $s(h,k)$ and $C_1(h,k)$ with the help of the Fibonacci numbers and in the light of that, we will give a new equality for $C_1(h,k)$ which depends on the Fibonacci numbers. Similarly we will show how can we write the reciprocity laws of the Hardy sum $s_5(h,k)$ and the Simsek sum $Y(h,k)$ when h and k are special Fibonacci numbers.

Many authors in many papers studied two and three term relations because they are related to the Dedekind sums, the Hardy sums and many other finite sums. In [15], [5], [3] and [19] new theorems on three-term relations for the Hardy sums were found by applying derivative operator to the three-term polynomial relation. In [9], we also deeply study about two and three term relations and we gave a new proof of the reciprocity law of the sum $Y(h,k)$. Now in this paper, we will use two-term polynomial relation again. So we remind it as a corollary below:

Corollary 2.1. *(Two-term polynomial relation) If a, b, and c are pairwise coprime positive integers, then*

$$(u-1)\sum_{x=1}^{a-1}u^{x-1}v^{\left[\frac{bx}{a}\right]}+(v-1)\sum_{y=1}^{b-1}v^{y-1}u^{\left[\frac{ay}{b}\right]}=u^{a-1}v^{b-1}-1. \tag{11}$$

Equation (11) is originally due to Berndt and Dieter [5]. We now give explicit values of the sum $C_1(h,k)$ by the following theorem:

Theorem 2.2. *If $(h,k)=1$, h and k are odd integers with $k>0$, then we have*

$$C_1(h,k)=\frac{1}{2}-\frac{1}{2k}. \tag{12}$$

Proof. We are motivated by the two-term polynomial relation for this theorem. We consider the identity (11). We know that when we take the partial derivative of (11) with respect to u, and substitute $u=v=-1$, then we have

$$\sum_{x=1}^{h-1}(-1)^{x+\left[\frac{kx}{h}\right]}-2\sum_{x=1}^{h-1}x(-1)^{x+\left[\frac{kx}{h}\right]}-2\sum_{y=1}^{k-1}\left[\frac{hy}{k}\right](-1)^{y+\left[\frac{hy}{k}\right]}=(h-1)(-1)^{h+k-1}.$$

After some elementary calculations we have,

$$-\sum_{x=1}^{h-1}(-1)^{x+\left[\frac{kx}{h}\right]-1}-2\sum_{x=1}^{h-1}x(-1)^{x+\left[\frac{kx}{h}\right]}-2h\sum_{y=1}^{k-1}\frac{y}{k}(-1)^{y+\left[\frac{hy}{k}\right]}$$

$$+2\sum_{y=1}^{h-1}\left(\left(\frac{hy}{k}\right)\right)(-1)^{y+\left[\frac{hy}{k}\right]}-\sum_{y=1}^{k-1}(-1)^{y+\left[\frac{hy}{k}\right]-1}=1-h.$$

By using (4), (3) and Theorem 1.3, we have

$$-2h\left(s_5(k,h)+s_5(h,k)\right)+2C_1(h,k)=1-h.$$

Finally from (5), we have

$$C_1(h,k)=\frac{1}{2}-\frac{1}{2k}.$$

□

Now we give the reciprocity law for the sum $C_1(h,k)$:

Theorem 2.3 (Reciprocity law for the sum $C_1(h,k)$ for odd case). *If $(h,k)=1$, h and k are positive odd integers, then we have*

$$C_1(h,k)+C_1(k,h)=1-\frac{1}{2}\left(\frac{1}{h}+\frac{1}{k}\right).$$

Proof. By using Theorem 2.2, we have

$$\begin{aligned} C_1(h,k)+C_1(k,h) &= \left(\frac{1}{2}-\frac{1}{2k}\right)+\left(\frac{1}{2}-\frac{1}{2h}\right) \\ &= 1-\frac{1}{2}\left(\frac{1}{h}+\frac{1}{k}\right). \end{aligned}$$

Thus we get the desired result. □

Corollary 2.4. *Let a,h,k be odd positive integers with $(a,hk)=1$. Then we have*

$$s_5(h,k)+s_5(k,h)=C_1(a,hk). \tag{13}$$

Proof. It can be obtained easily from (5) and Theorem 2.2. □

Corollary 2.5. *Let a,h,k be odd positive integers with $(a,hk)=1$. Then we have*

$$hY(h,k)+kY(k,h)=4hkC_1(a,hk). \tag{14}$$

Proof. It can be obtained directly from Theorem 2.2 and (8). □

Corollary 2.6. *Let a,h,k be odd positive integers with $(a,hk)=1$. Then we have*

$$s(h,k)+s(k,h)=-\frac{1}{6}+\frac{1}{12}\left(\frac{h}{k}+\frac{k}{h}-2C_1(a,hk)\right).$$

Proof. From Theorem 2.2, we can write

$$1-2C_1(h,k)=\frac{1}{k}.$$

So we can obtain the equalities below:

$$\begin{aligned} \frac{h}{k} &= h-2hC_1(h,k), \\ \frac{k}{h} &= k-2kC_1(k,h), \\ \frac{1}{hk} &= 1-2C_1(a,hk). \end{aligned} \tag{15}$$

If we put (15) into (1) and make some easy calculations, desired result can be obtained. □

Corollary 2.7. *Let h, k be odd positive integers with $(h, k) = 1$, then we have*

$$2s_3(h,k) - s_4(k,h) = 1 - h + 2hC_1(h,k).$$

Proof. It can be directly obtained from Theorem 1.4 and Theorem 2.2. □

Theorem 2.8. *Let h and k be distinct odd primes. Then*

$$\sum_{j=1}^{\frac{1}{2}(k-1)} \left[\frac{hj}{k}\right] + \sum_{j=1}^{\frac{1}{2}(h-1)} \left[\frac{kj}{h}\right] = 4hkC_1(h,k)C_1(k,h).$$

Proof. After some elementary calculations, we get

$$2kC_1(h,k) = k - 1,$$

and

$$2hC_1(k,h) = h - 1$$

from Theorem 2.2. If we put these equations into (10), then we have the desired result. □

Now we can prove the main theorem by means of all the results above:

Theorem 2.9. *Let $a, h, k \in \mathbb{Z}$ with $k > 0$ and $\{h, k\}$ is a symmetric pair. If $(h, k) = 1$, $h = F_{6n-1}$ and $k = F_{6n+1}$ with $n \in \mathbb{N}$, where F_m is the $m-th$ Fibonacci number, then*

$$s(h,k) = -\frac{1}{12} + \frac{1}{24}\left(\frac{h}{k} + \frac{k}{h} - 2C_1(a,hk)\right). \tag{16}$$

Proof. We know from Corollary 2.6 that

$$s(h,k) + s(k,h) = -\frac{1}{6} + \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} - 2C_1(a,hk)\right)$$

where $a, h, k \in \mathbb{Z}$ with $k > 0$, $(h, k) = 1$, h and k are odd numbers. We also know from [13] that if $\{h, k\}$ is a symmetric pair, then

$$s(h,k) = s(k,h).$$

If we add these two equations side by side, then we have the desired result. □

Corollary 2.10. *Let $a, h, k \in \mathbb{Z}$ with $k > 0$ and $\{h, k\}$ is a symmetric pair. If $(h, k) = 1$, $h = F_{6n-1}$ and $k = F_{6n+1}$ with $n \in \mathbb{N}$, where F_m is the $m-th$ Fibonacci number, then*

$$C_1(a,hk) = \frac{h}{2k} + \frac{k}{2h} - 1 \tag{17}$$

Proof. It can be found from (16) and Theorem 1.5. □

Corollary 2.11. *Let $a, h, k \in \mathbb{Z}$ with $k > 0$ and $\{h, k\}$ is a symmetric pair. If $(h, k) = 1$, $h = F_{6n-1}$ and $k = F_{6n+1}$ with $n \in \mathbb{N}$, where F_m is the m-th Fibonacci number, then*

$$s_5(h,k) + s_5(k,h) = \frac{1}{2}\left(\frac{h}{k} + \frac{k}{h} - 2\right),$$

and

$$hY(h,k) + kY(k,h) = 2h^2 + 2k^2 - 4hk.$$

Proof. It can be easily obtained from (17), (13) and (14). □

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