Filomat 30:4 (2016), 985–992
DOI 10.2298/FIL1604985K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on Generating Functions for the Unification of the Bernstein Type Basis Functions

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Abstract. In [3], Simsek unified generating function of the Bernstein basis functions. In this paper, by using knot sequence, we rewrite generating functions for the unification of the Bernstein type basis functions. By using these generating functions, we also find generating function for the Hermite type numbers. We investigate some properties of this functions and these basis. Finally, we simulate these polynomials with their plots for some selected numerical values.

1. Introduction

Recently many authors have study many kind of the Bernstein basis functions and their generating functions with various different methods. Because these basis functions have many applications almost all mathematics, engineering, medicine and also the others. Therefore, we give another aspect of the generating functions for the unification of the Bernstein type basis functions.

In [3], Simsek constructed unification of the Bernstein type basis functions by means of the following generating functions:

$$\begin{aligned} \mathcal{F}\left(t,b,s:x\right) &= \frac{2^b x^{bs}\left(\frac{t}{2}\right)^{bs}\exp(t\left(1-x\right))}{(bs)!} \\ &= \sum_{n=0}^{\infty}\mathcal{G}_n\left(b,s,x\right)\frac{t^n}{n!}, \end{aligned} \tag{1}$$

where $b, s \in N_0 := \{0, 1, 2, 3, \ldots\}$, $t \in \mathbb{C}$ and $x \in [0, 1]$, $\exp(z) = e^z$.

By using (1), unification of the Bernstein type basis functions are defined by

$$\mathcal{G}_n\left(b,s,x\right) = \binom{n}{bs}\frac{x^{bs}\left(1-x\right)^{n-bs}}{2^{b(s-1)}} \tag{2}$$

where $n \geq bs$ and $x \in [0, 1]$ (*cf.* [3]).

2010 *Mathematics Subject Classification*. 65D07, 11B83, 40A30, 65D17, 65D18, 68U07, 11Y16, 14F10, 26C05

Keywords. Bernstein type polynomials; Generating function; Knot sequence; Laplace transform; Binomial coefficients, Gamma and Beta functions; Hermite numbers.

Received: 02 July 2015; Accepted: 24 August 2015

Communicated by Gradimir Milovanović

The present investigation was supported by the Scientific Research Project Administration of Akdeniz University

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2. The Polynomials $\mathcal{G}_n(b,s,x)$ on the Knot Sequence $\{t_1,t_2,\ldots,t_{m+1}\}$

In this section, we rewrite generating functions for the unification of the Bernstein type basis functions $\mathcal{G}_n(b,s,x)$. We investigate some properties of these functions. Simsek [3] gave the formulas and identities for the interval $[0,1]$. Here, we extended (2) to arbitrary intervals $\left[t_j,t_{j+1}\right]$, $j\in\{1,2,\ldots,m\}$. Replacing x by $\frac{x-t_j}{t_{j+1}-t_j}$, (2) yields the corresponding results concerning the unification Bernstein type basis functions $\mathcal{G}_n\left(b_j,s,x;t_j,t_{j+1}\right)$:

$$\mathcal{G}_n\left(b_j,s,x;t_j,t_{j+1}\right)=\binom{n}{b_js}\frac{\left(\frac{x-t_j}{t_{j+1}-t_j}\right)^{b_js}\left(\frac{t_{j+1}-x}{t_{j+1}-t_j}\right)^{n-b_js}}{2^{b_j(s-1)}}, \tag{3}$$

where b_j, n and s be nonnegative integers and $x\in\left[t_j,t_{j+1}\right]$, $t_j\neq t_{j+1}$. Therefore, the generating functions are written as

$$\begin{aligned}\mathcal{F}\left(t,b_j,s,x;t_j,t_{j+1}\right) &= \frac{2^{b_j}\left(\frac{x-t_j}{t_{j+1}-t_j}\right)^{b_js}\left(\frac{t}{2}\right)^{b_js}\exp\left(t\frac{t_{j+1}-x}{t_{j+1}-t_j}\right)}{\left(b_js\right)!}\\ &= \sum_{n=0}^{\infty}\mathcal{G}_n\left(b_j,s,x;t_j,t_{j+1}\right)\frac{t^n}{n!}.\end{aligned} \tag{4}$$

We redefine the above generating functions on the knot sequence $\{t_1,t_2,\ldots,t_{m+1}\}$ where $t_1<t_2<\ldots<t_{m+1}$. Thus we set

$$\begin{aligned}h\left(t,\overrightarrow{b},s,x;\overrightarrow{t};m\right) &= \sum_{j=1}^{m}\mathcal{F}\left(t,b_j,s,x;t_j,t_{j+1}\right)\\ &= \sum_{n=0}^{\infty}\mathcal{S}_n\left(\overrightarrow{b},s,x;\overrightarrow{t};m\right)\frac{t^n}{n!},\end{aligned} \tag{5}$$

where $\overrightarrow{b}=(b_1,b_2,\ldots,b_m)$ and $\overrightarrow{t}=(t_1,t_2,\ldots,t_{m+1})$. By using the above equation and (4), we get

$$\mathcal{S}_n\left(\overrightarrow{b},s,x;\overrightarrow{t};m\right)=\sum_{j=1}^{m}\mathcal{G}_n\left(b_j,s,x;t_j,t_{j+1}\right).$$

By using (4), we also set

$$\begin{aligned}g\left(t,\vec{b},s,x;\vec{t};m\right) &= \prod_{j=1}^{m}\mathcal{F}\left(t,b_j,s,x;t_j,t_{j+1}\right)\\ &= \sum_{n=0}^{\infty}\mathcal{T}_n\left(\vec{b},s,x;\vec{t};m\right)\frac{t^n}{n!}\end{aligned}$$

or

$$g\left(t,\overrightarrow{b},s,x;\overrightarrow{t};m\right)=\frac{2^{b_1+\ldots+b_m}\prod_{j=1}^{m}\left(\frac{t}{2}\frac{x-t_j}{t_{j+1}-t_j}\right)^{sb_j}}{\prod_{j=1}^{m}\left(b_js\right)!}\exp\left(t\sum_{j=1}^{m}\frac{t_{j+1}-x}{t_{j+1}-t_j}\right). \tag{6}$$

Alternative form of the generating functions for (6):

$$g\left(t,\vec{b},s,x;\vec{t};m\right)\exp\left(-t\sum_{j=1}^{m}\frac{t_{j+1}}{t_{j+1}-t_j}\right)=\frac{2^{b_1+\ldots+b_m}\prod_{j=1}^{m}\left(\frac{t}{2}\frac{x-t_j}{t_{j+1}-t_j}\right)^{sb_j}}{\prod_{j=1}^{m}\left(b_js\right)!}\exp\left(-tx\sum_{j=1}^{m}\frac{1}{t_{j+1}-t_j}\right).$$

By applying the Laplace transformation to the above equation, we get

$$\sum_{n=0}^{\infty}\frac{\mathcal{T}_n\left(\overrightarrow{b},s,x;\overrightarrow{t};m\right)}{n!}\int_0^{\infty}t^n\exp\left(-t\sum_{j=1}^{m}\frac{t_{j+1}}{t_{j+1}-t_j}\right)dt$$
$$=\frac{2^{(b_1+\ldots+b_m)(1-s)}\prod_{j=1}^{m}\left(\frac{x-t_j}{t_{j+1}-t_j}\right)^{sb_j}}{\prod_{j=1}^{m}\left(b_js\right)!}\int_0^{\infty}t^{(b_1+\ldots+b_m)s}\exp\left(-tx\sum_{j=1}^{m}\frac{1}{t_{j+1}-t_j}\right).$$

By using the above equation, we arrive at the following Theorem:

Theorem 2.1. *Let $\left|t_{j+1}-x\right|<\left|t_{j+1}-t_j\right|$. Then we have*

$$\sum_{n=0}^{\infty}\frac{1}{\left(\sum_{j=1}^{m}\frac{t_{j+1}}{t_{j+1}-t_j}\right)^{n+1}}\mathcal{T}_n\left(\vec{b},s,x;\vec{t};m\right)=\frac{2^{(b_1+\ldots+b_m)(1-s)}\prod_{j=1}^{m}\left(\frac{x-t_j}{t_{j+1}-t_j}\right)^{sb_j}\left(\left(b_1+\ldots+b_m\right)s\right)!}{\prod_{j=1}^{m}\left(b_js\right)!\left(x\sum_{j=1}^{m}\frac{1}{t_{j+1}-t_j}\right)^{(b_1+\ldots+b_m)s+1}}.$$

Remark 2.2. *If we substitute $m=1$, $t_1=0$ and $t_2=1$ into the Theorem 2.1, we arrive at Theorem 4.1 in [5, p.6].*

Remark 2.3. *If we substitute $m=1$, $s=1$, $t_1=0$ and $t_2=1$ into the Theorem 2.1, we arrive at Theorem 5.1 in [4, p.12].*

2.1. Integral Representations

In this section, we derive integral representations of extensions of the polynomials $\mathcal{G}_n(b,s,x)$. Furthermore, we give an identity which connects the binomial coefficients, gamma and beta functions.

The *beta function* $B(\alpha,\beta)$ is defined by

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \tag{7}$$

where $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$ (*cf.* [9, p.9, Eq. (60)]). The beta function satisfies the following well-known relation:

$$B(n,m)=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}=\frac{(n-1)!(m-1)!}{(n+m-1)!},$$

where n and m are positive integers and $\Gamma(z)$ denotes the gamma function (*cf.* [9, p.9, Eq. (62)]).

Replacing t by $\frac{x-t_j}{t_{j+1}-t_j}$ and taking $\alpha = b_j s + 1, \beta = n - b_j s + 1$ with $n \geq b_j s$, $j \in \{1, 2, \ldots, m\}$ in (7) using same method as in [5] and [9], we have

$$\int_{t_j}^{t_{j+1}} \left(x - t_j\right)^{b_j s} \left(t_{j+1} - x\right)^{n-b_j s} dx = \left(t_{j+1} - t_j\right)^{n+1} B\left(b_j s + 1, n - b_j s + 1\right).$$

Multiplying of the above equation both sides by $2^{b_j(1-s)}\binom{n}{b_j s}$ and using (3) and (5), we easily arrive at the following result:

$$\int_{t_j}^{t_{j+1}} \mathcal{S}_n\left(\overrightarrow{b}, s, x; \overrightarrow{t}; m\right) dx = \frac{1}{n+1} \sum_{j=1}^{m} 2^{b_j(1-s)} \left(t_{j+1} - t_j\right). \tag{8}$$

Remark 2.4. *If we substitute $m = 1$, $s = 1$, $t_1 = a$ and $t_2 = b$ into the (8), we arrive at Theorem 6.1 in [6, p.483].*

3. Applications of the Generating Functions for the Bernstein Type Polynomials

In this section, we derive generating functions for the Hermite type numbers from (1). Throughout this section, we need the following notations and definitions. The two-variable Hermite polynomials are defined by means of the following generating functions:

$$\exp\left(xt + yt^j\right) = \sum_{n=0}^{\infty} H_n^{(j)}(x, y) \frac{t^n}{n!} \tag{9}$$

and

$$H_n^{(j)}(x, y) = n! \sum_{r=0}^{\left[\frac{n}{j}\right]} \frac{x^{n-jr} y^r}{r!\,(n - jr)!}, \tag{10}$$

where $j \geq 2$ is an integer (*cf.* [1], [2], [8]).

Replacing t by x in (1), we get

$$\sum_{n=0}^{\infty} \mathcal{G}_n(b, s, x) \frac{x^n}{n!} = \frac{2^{b(1-s)} x^{2bs}}{(bs)!} \exp(x - x^2). \tag{11}$$

By setting $x = 1$, $y = -1$ and $j = 2$ in (9), we have the following the Hermite type numbers:

$$\exp(t - t^2) = \sum_{n=0}^{\infty} H_n^{(2)}(1, -1) \frac{t^n}{n!}.$$

Combining the above equation with (11), we get

$$(bs)! 2^{b(s-1)} \mathcal{F}(x, b, s : x) = \sum_{n=0}^{\infty} (n)_{2bs} H_n^{(2)}(1, -1) \frac{x^n}{n!},$$

where $(n)_k = n(n-1)\ldots(n-k+1)$ and $(n)_0 = 1$.

Consequently, we obtain generating function for the Hermite type numbers $(n)_{2bs} H_n^{(2)}(1, -1)$ as follows:

$$(bs)! 2^{b(s-1)} \mathcal{F}(x, b, s : x).$$

If we substitute $x = 1$, $y = -1$ into the 10, we compute the following numbers:

$H_0^{(2)}(1, -1) = 1, H_1^{(2)}(1, -1) = 1, H_2^{(2)}(1, -1) = -1, \ldots$

If we substitute $x = \frac{1}{2}$ and $t = \frac{1}{2}$ into (1), and using (2), we get the following sum:

$$\sum_{n=0}^{\infty} \mathcal{G}_n\left(b, s, \frac{1}{2}\right)\left(\frac{1}{2}\right)^n = \frac{2^{b-3bs}\sqrt[4]{e}}{(bs)!}.$$

If we set $x = d \in (0, 1)$ and $t = \frac{1}{c}$; $c > 1$ into (1), and using (2), we have the following sum:

$$\sum_{n=0}^{\infty} \frac{\mathcal{G}_n(b, s, d)}{c^n n!} = \frac{2^{b-bs}}{(bs)!}\left(\frac{d}{c}\right)^{bs} \sqrt[c]{\exp(1-d)}.$$

In the special case when $d = \frac{1}{3}$, $c = 2$, $b = 3$ and $s = 1$, after some elementary calculations in the above, we obtain the following numerical result:

$$\sum_{n=0}^{\infty} \frac{\mathcal{G}_n\left(3, 1, \frac{1}{3}\right)}{2^n n!} = 6^{-4}\sqrt[3]{e}.$$

4. Simulation of the Polynomials $\mathcal{G}_n(b, s, x)$

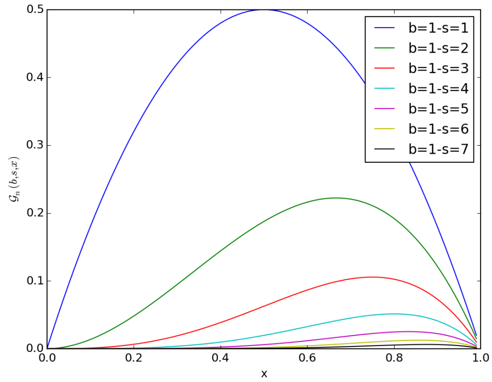
In this section, some curves and surfaces are plotted using the polynomials $\mathcal{G}_n(b, s, x)$, $\mathcal{S}_n\left(\overrightarrow{b}, s, x; \overrightarrow{t}; m\right)$, $\mathcal{T}_n\left(\overrightarrow{b}, s, x; \overrightarrow{t}; m\right)$ and their generating functions. We simulate these polynomials with their plots for some selected numerical values. The effects of b, s and n on the shape of the curve are demonstrated for the given range. These graphics may not only be used in Computer Aided Geometric Design (CAGD) but also in other areas (*cf.* [7]).

The figures below are obtained by varying b and s values using (2) for $x \in [0, 1]$. Since $n \geq bs$, we set

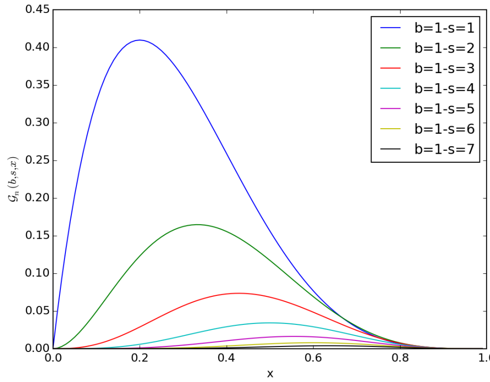
$$n = bs + offset$$

where offset is valid between 0 and $n - 1$.

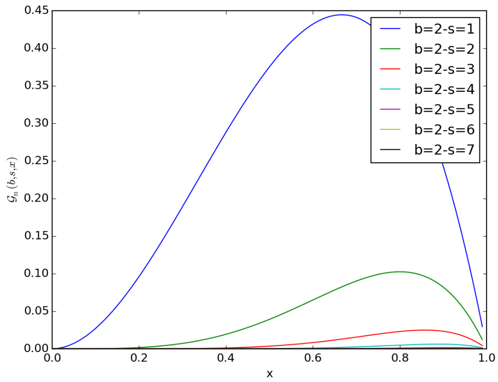
Figure 1 is obtained by $s \in \{1, \ldots, 7\}$, $b = 1$ or $b = 2$ and $offset = 1$ or $offset = 4$ using (2) for $x \in [0, 1]$. By the following figures we see that the unification of the Bernstein type polynomials affect the shape of the curves related to the b, s and $offset$.



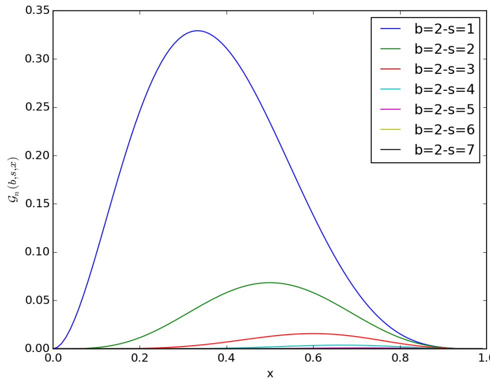
(a) Varying s values for $b = 1$ and $n = bs + 1$.



(b) Varying s values for $b = 1$ and $n = bs + 4$.



(c) Varying s values for $b = 2$ and $n = bs + 1$.



(d) Varying s values for $b = 2$ and $n = bs + 4$.

Figure 1: $\mathcal{G}_n\,(b, s, x)$ obtained by varying s, b and n values

Figure 2 is obtained by $s \in \{1, \ldots, 7\}$, $m = 4$, $\overrightarrow{t} = (-2, -1, 0, 1, 2)$, $\overrightarrow{b} = (1, 1, 1, 1)$ and $offset = 1$ using $\mathcal{S}_n\left(\overrightarrow{b}, s, x; \overrightarrow{t}; m\right)$ for $x \in [-2, 2]$. In this case, we set $n = \max\left\{b_j\right\} s + offset$, $j \in \{1, 2, \ldots, m\}$

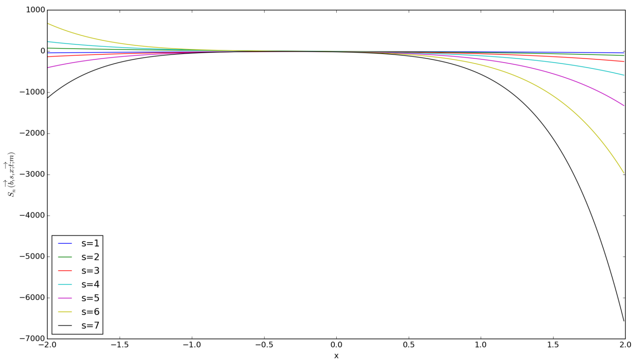


Figure 2: The polynomials $\mathcal{S}_n\left(\vec{b}, s, x; \vec{t}; m\right)$ obtained by varying s values.

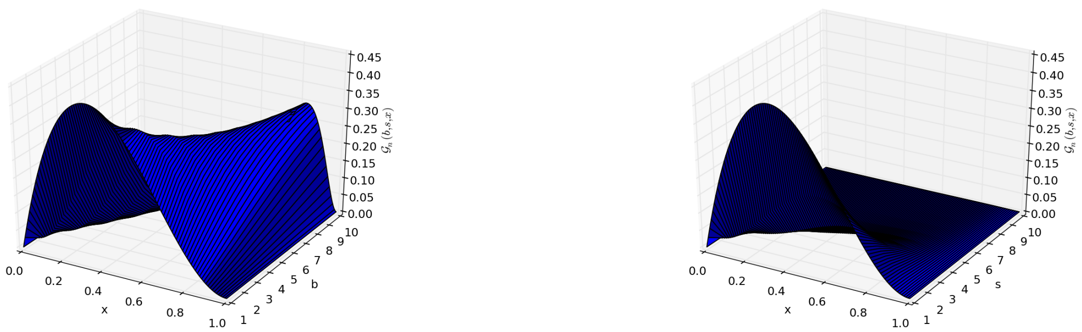
Figure 3 is obtained by $m = 4$, $\overrightarrow{t} = (-2, -1, 0, 1, 2)$, $\overrightarrow{b} = (1, 1, 1, 1)$ and $offset = 1$ using (5) and (6) for $x \in [-2, 2]$ and $t \in [0, 1]$. In this case, figure of the functions $g\left(t, \overrightarrow{b}, s, x; \overrightarrow{t}; m\right)$ and $h\left(t, \overrightarrow{b}, s, x; \overrightarrow{t}; m\right)$, that is, generating functions of $\mathcal{S}_n\left(\overrightarrow{b}, s, x; \overrightarrow{t}; m\right)$ and $\mathcal{T}_n\left(\overrightarrow{b}, s, x; \overrightarrow{t}; m\right)$, respectively are given as follows:

(a) Generating function of $\mathcal{S}_n\left(\vec{b}, s, x; \vec{t}; m\right)$ (i.e. $h\left(t, \vec{b}, s, x; \vec{t}; m\right)$) for $x \in [-2, 2]$ and $t \in [0, 1]$.

(b) Generating function of $\mathcal{T}_n\left(\vec{b}, s, x; \vec{t}; m\right)$ (i.e. $g\left(t, \vec{b}, s, x; \vec{t}; m\right)$) for $x \in [-2, 2]$ and $t \in [0, 1]$.

Figure 3: Generating functions for $x \in [-2, 2]$ and $t \in [0, 1]$.

We also plot the surfaces obtained by $s = 1$, $b \in \{1, \ldots, 10\}$ and $b = 1$, $s \in \{1, \ldots, 10\}$, $offset = 2$ using (2) for $x \in [0, 1]$. In this case, surface figures of the polynomials $\mathcal{G}_n(b, s, x)$ is given by Figure 4.



(a) Surface obtained by varying b values for $s = 1$ and $n = bs + 2$.

(b) Surface obtained by varying s values for $b = 1$ and $n = bs + 2$.

Figure 4: Surface figures of the polynomials $\mathcal{G}_n(b, s, x)$.

References

[1] G. Bretti and P. E. Ricci, Multidimensional Extensions of the Bernoulli and Appell Polynomials, *Taiwanese J. Math.* **8**, 415-428 (2004).

[2] R. Dere and Y. Simsek, Hermite base Bernoulli type polynomials on the umbral algebra, *Russ. J. Math. Phys.* **22**(1), 1-5 (2015).

[3] Y. Simsek, Construction a new generating function of Bernstein type polynomials, *Appl. Math. Comput.* **218**, 1072-1076 (2011).

[4] Y. Simsek, Functional equations from generating functions: a novel approach to deriving identities for the Bernstein basis functions, *Fixed Point Theory Appl.* **2013**, 80 (2013).

[5] Y. Simsek, Unification of the Bernstein-type polynomials and their applications, *Bound. Value Probl.* **2013**, 2013:56.

[6] Y. Simsek, Deriving Novel Formulas and Identities for the Bernstein Basis Functions and Their Generating Functions, *Lecture Notes in Comput. Sci.* **8177** (2014), 471490.

[7] Y. Simsek and M. Gunay, On Bernstein type polynomials and their applications, *Adv. Differ. Equ.* **2015**, 2015:79.
[8] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1984.
[9] H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic, Dordrecht (2001).