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A Matrix Approach to Solving Hyperbolic Partial Differential Equations Using Bernoulli Polynomials

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Abstract. The present study considers the solutions of hyperbolic partial differential equations. For this, an approximate method based on Bernoulli polynomials is developed. This method transforms the equation into the matrix equation and the unknown of this equation is a Bernoulli coefficients matrix. To demostrate the validity and applicability of the method, an error analysis developed based on residual function. Also examples are presented to illustrate the accuracy of the method.

1. Introduction

Hyperbolic partial differential equations are important for a variety of reasons. The defining properties of hyperbolic problems include well posed Cauchy problems, finite speed of propagation, and the existence of wave like structures with infinitely varied form. The infinite variety of wave forms make hyperbolic equations the preferred mode for sending information for example hearing, sight, television and radio. Well posed Cauchy problems with finite speed lead to hyperbolic equations. Since the fundamental laws of physics must respect the principles of relavity, finite speed is required. This together with causality require hyperbolicity. Thus there are many equations from Physics. Those which are most fundamental tend to have close relationship with Lorentzian geometry. A source of countless mathematical and technological problems of hyperbolic type are equations of fluid Dynamics [1].

Also there are many models problems of wave equation such as Vibrating string, fixed at both the ends, electromagnetic waves, longitudinal vibration in a bar.

Hyperbolic partial differential equations have attracted attention and investigation of new methods to solve these equations. Characteristics method [2] is used commonly to solve these equations. Also several numerical methods is used such as Homotopy perturbation method, Adomian decomposition, Taylor matrix method, Spline methods, Parameters spline methods, Bernstein Ritz-Galerkin Method, etc. [3]-[9]. In this study, we try to solve this problem using Bernoulli matrix method. Also this method has been used to solve high-order linear differential-difference equations, linear delay difference equations with variable coefficients and mixed linear Fredholm integro-differential-difference equations [10]-[12]. Also there are so many studies about Bernoulli polynomials and its properties [13]-[15].

In this paper, we will consider the second-order linear hyperbolic partial differential-equation [16]-[17]

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$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} \tag{1}$$

$$+E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y), \;\; 0 \le x, y \le 1,$$

with the Dirichlet boundary conditions

$$u(x,0) = f_1(x), \quad 0 \le x \le 1, \tag{2}$$

$$u(0,y) = g_1(y), \quad 0 \le y \le 1, \tag{3}$$

with the Neumann boundary conditions

$$u_y(x,0) = f_2(x), \quad 0 \le x \le 1, \tag{4}$$

and with the nonlinear integral conditions

$$\int_0^1 u(x,y)dx = g_2(y), \quad 0 \le y \le 1. \tag{5}$$

If $B^2 - 4AC > 0$, then (1) is called a hyperbolic equation. Our purpose is to obtain an approximate solution of (1) with the conditions (2), (3), (4) and (5) in the following Bernoulli polynomial form [18]

$$u(x,y) = \sum_{r=0}^{N}\sum_{s=0}^{N} a_{r,s}B_{r,s}(x,y), \tag{6}$$

where $a_{r,s}$, unknown Bernoulli coefficients and $B_{r,s}(x,y) = B_r(x)B_s(y)$. Here $B_r(x)$ and $B_s(y)$ are r-th and s-th degree Bernoulli polynomials, respectively.

For two variables functions, Bernoulli collocation points can be defined by

$$x_i = \frac{i}{N},\; y_j = \frac{j}{N},\; i = 0,1,\cdots,N,\; j = 0,1,\cdots,N. \tag{7}$$

on the $0 \le x, y \le 1$.

2. Fundamental Matrix Relations Second Order Linear Partial Differential Equations

The approximate solution of (1) in terms of Bernoulli polynomials can be expressed as

$$u(x,y) = \sum_{r=0}^{N}\sum_{s=0}^{N} a_{r,s}B_{r,s}(x,y).$$

The matrix equation for this expression can be written as

$$u(x,y) = \boldsymbol{B}(x)\boldsymbol{Q}(y)\overline{\boldsymbol{A}} \tag{8}$$

so that

$$\boldsymbol{B} = \begin{bmatrix} B_0(x) & B_1(x) & \cdots & B_N(x) \end{bmatrix}_{1\times(N+1)},$$

$$Q=\begin{bmatrix} B_0(y) & \cdots & B_N(y) & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & B_0(y) & \cdots & B_N(y) & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & & \ddots & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & B_0(y) & \cdots & B_N(y) \end{bmatrix}_{(N+1)\times(N+1)^2}$$

and unknown Bernoulli coeffients

$$\overline{\boldsymbol{A}}=\begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,N} & a_{1,0} & a_{1,1} & \cdots & a_{1,N} & \cdots & a_{N,0} & a_{N,1} & \cdots & a_{N,N} \end{bmatrix}^T_{(N+1)^2\times 1}$$

And also we can write the Bernoulli polynomials $B_n(x)$ in the matrix form as follows

$$B^T(x)=\zeta X^T(x) \Leftrightarrow B(x)=X(x)\zeta^T$$

where

$$B(x)=\begin{bmatrix} B_0(x) & B_1(x) & \cdots & B_N(x) \end{bmatrix},\ X(x)=\begin{bmatrix} 1 & x & \cdots & x^N \end{bmatrix}$$

and

$$\zeta=\begin{bmatrix} \binom{0}{0}b_0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}b_1 & \binom{1}{1}b_0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{2}{0}b_2 & \binom{2}{1}b_1 & \binom{2}{2}b_0 & 0 & 0 & 0 & \cdots & 0 \\ \binom{3}{0}b_3 & \binom{3}{1}b_2 & \binom{3}{2}b_1 & \binom{3}{3}b_0 & 0 & 0 & \cdots & 0 \\ \binom{4}{0}b_4 & \binom{4}{1}b_3 & \binom{4}{2}b_2 & \binom{4}{3}b_1 & \binom{4}{4}b_0 & 0 & \cdots & 0 \\ \binom{5}{0}b_5 & \binom{5}{1}b_4 & \binom{5}{2}b_3 & \binom{5}{3}b_2 & \binom{5}{4}b_1 & \binom{5}{5}b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{N}{0}b_N & \binom{N}{1}b_{N-1} & \binom{N}{2}b_{N-2} & \binom{N}{3}b_{N-3} & \binom{N}{4}b_{N-4} & \binom{N}{5}b_{N-5} & \cdots & \binom{N}{N}b_0 \end{bmatrix}$$

where b_k are the Bernoulli numbers.

There is a matrix relation between $\boldsymbol{B}(x)$ and $\boldsymbol{B}^{(1)}(x)$ as

$$\boldsymbol{B}^{(1)}(x)=\boldsymbol{B}(x)T^T$$

where

$$T=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & N & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

Similarly the matrix relation between $\boldsymbol{B}(x)$ and its second derivative can be expressed as

$$\boldsymbol{B}^{(2)}(x) = \boldsymbol{B}^{(1)}(x)T^T = \boldsymbol{B}(x)(T^T)^2.$$

If we continue to differentiate consecutively, the i-th derivative of $\boldsymbol{B}(x)$ is

$$\boldsymbol{B}^{(i)}(x) = \boldsymbol{B}^{(i-1)}(x)T^T = \boldsymbol{B}(x)(T^T)^i. \tag{9}$$

Accordingly the j-th derivative of $\boldsymbol{Q}(y)$ is obtained

$$\begin{aligned} \boldsymbol{Q}^{(1)}(y) &= \boldsymbol{Q}(y)\overline{T} \\ \boldsymbol{Q}^{(2)}(y) &= \boldsymbol{Q}^{(1)}(y)\overline{T} = \boldsymbol{Q}(y)(\overline{T})^2 \\ &\vdots \\ \boldsymbol{Q}^{(j)}(y) &= \boldsymbol{Q}^{(j-1)}(y)\overline{T} = \boldsymbol{Q}(y)(\overline{T})^j \end{aligned} \tag{10}$$

where

$$\overline{T} = \begin{bmatrix} T^T & 0 & \cdots & 0 \\ 0 & T^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T^T \end{bmatrix}_{(N+1)^2\times(N+1)^2}$$

Here $\overline{T}$ is block diagonale matrix which dimention is $(N+1)^2 \times (N+1)^2$.

Consequently we obtain a matrix relation as

$$u^{(i,j)}(x,y) = \boldsymbol{B}^{(i)}(x)\boldsymbol{Q}^{(j)}(y)\overline{\boldsymbol{A}} = \boldsymbol{B}(x)(T^T)^i\boldsymbol{Q}(y)(\overline{T})^j\overline{\boldsymbol{A}} \tag{11}$$

for the truncated Bernoulli series with two variables $u^{(i,j)}(x,y) = \sum_{r=0}^{N}\sum_{s=0}^{N} a_{r,s}B_{r,s}^{(i,j)}(x,y)$.

On the other hand,with the help of (8) and (11), the unknown function and its partial derivatives in (1) can be written as follows

$$\begin{aligned} u^{(0,0)}(x,y) &= u(x,y) = \boldsymbol{B}(x)\boldsymbol{Q}(y)\overline{\boldsymbol{A}} \\ u^{(1,0)}(x,y) &= \frac{\partial u}{\partial x} = \boldsymbol{B}(x)(T^T)\boldsymbol{Q}(y)\overline{\boldsymbol{A}} \\ u^{(2,0)}(x,y) &= \frac{\partial^2 u}{\partial x^2} = \boldsymbol{B}(x)(T^T)^2\boldsymbol{Q}(y)\overline{\boldsymbol{A}} \\ u^{(0,1)}(x,y) &= \frac{\partial u}{\partial y} = \boldsymbol{B}(x)\boldsymbol{Q}(y)(\overline{T})\overline{\boldsymbol{A}} \\ u^{(0,2)}(x,y) &= \frac{\partial^2 u}{\partial y^2} = \boldsymbol{B}(x)\boldsymbol{Q}(y)(\overline{T})^2\overline{\boldsymbol{A}} \\ u^{(1,1)}(x,y) &= \frac{\partial^2 u}{\partial x\partial y} = \boldsymbol{B}(x)(T)^T\boldsymbol{Q}(y)(\overline{T})\overline{\boldsymbol{A}} \end{aligned} \tag{12}$$

The corresponding matrix form of the conditions (2), (3), (4) and (5) respectively, is obtained as follows

$$u(x,0) = \boldsymbol{B}(x)\boldsymbol{Q}(0)\overline{\boldsymbol{A}} = f_1(x),\ 0 \le x \le 1, \tag{13}$$

$$u(0,y) = \boldsymbol{B}(0)\boldsymbol{Q}(y)\overline{\boldsymbol{A}} = g_1(y),\ 0 \le y \le 1, \tag{14}$$

$$u_y(x,0) = \boldsymbol{B}(x)\boldsymbol{Q}(0)\overline{T\boldsymbol{A}} = f_2(x),\ 0 \le x \le 1, \tag{15}$$

and

$$\int_0^1 \boldsymbol{B}(x)\boldsymbol{Q}(y)\overline{\boldsymbol{A}}dx = \left\{\int_0^1 X(x)dx\right\}\zeta^T\boldsymbol{Q}(y)\overline{\boldsymbol{A}} \tag{16}$$
$$= \boldsymbol{L}\zeta^T\boldsymbol{Q}(y)\overline{\boldsymbol{A}} = g_2(y),\ 0 \le y \le 1$$

where

$$\boldsymbol{L} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N+1} \end{bmatrix}.$$

3. Collocation Method

We substitute the matrix relations (12) into (1) to construct fundamental matrix equation. So we obtain the matrix equation

$$\begin{aligned} &A(x,y)\boldsymbol{B}(x)(T^T)^2\boldsymbol{Q}(y)\overline{\boldsymbol{A}} + B(x,y)\boldsymbol{B}(x)T^T\boldsymbol{Q}(y)(\overline{T})\overline{\boldsymbol{A}}+ \\ &C(x,y)\boldsymbol{B}(x)\boldsymbol{Q}(y)(\overline{T})^2\overline{\boldsymbol{A}} + D(x,y)\boldsymbol{B}(x)(T^T)\boldsymbol{Q}(y)\overline{\boldsymbol{A}}+ \\ &E(x,y)\boldsymbol{B}(x)\boldsymbol{Q}(y)(\overline{T})\overline{\boldsymbol{A}} + F(x,y)\boldsymbol{B}(x)\boldsymbol{Q}(y)\overline{\boldsymbol{A}} = G(x,y). \end{aligned} \tag{17}$$

By using in (17) collocation points defined by (7), the system of the matrix equation is obtained as

$$\left.\begin{array}{c} A(x_i,y_j)\boldsymbol{B}(x_i)(T^T)^2\boldsymbol{Q}(y_j)\overline{\boldsymbol{A}} + B(x_i,y_j)\boldsymbol{B}(x_i)T^T\boldsymbol{Q}(y_j)(\overline{T})\overline{\boldsymbol{A}}+ \\ C(x_i,y_j)\boldsymbol{B}(x_i)\boldsymbol{Q}(y_j)(\overline{T})^2\overline{\boldsymbol{A}} + D(x_i,y_j)\boldsymbol{B}(x_i)(T^T)\boldsymbol{Q}(y_j)\overline{\boldsymbol{A}}+ \\ E(x_i,y_j)\boldsymbol{B}(x_i)\boldsymbol{Q}(y_j)(\overline{T})\overline{\boldsymbol{A}} + F(x_i,y_j)\boldsymbol{B}(x_i)\boldsymbol{Q}(y_j)\overline{\boldsymbol{A}} = G(x_i,y_j). \\ i = 0,1,\cdots,N,\ \ j = 0,1,\cdots,N \end{array}\right\} \tag{18}$$

or briefly the fundamental matrix equation becomes

$$\begin{array}{c} A\boldsymbol{B}(T^T)^2\boldsymbol{Q}\overline{\boldsymbol{A}} + B\boldsymbol{B}T^T\boldsymbol{Q}(\overline{T})\overline{\boldsymbol{A}} + C\boldsymbol{B}\boldsymbol{Q}(\overline{T})^2\overline{\boldsymbol{A}}+ \\ D\boldsymbol{B}(T^T)\boldsymbol{Q}\overline{\boldsymbol{A}} + E\boldsymbol{B}\boldsymbol{Q}(\overline{T})\overline{\boldsymbol{A}} + F\boldsymbol{B}\boldsymbol{Q}\overline{\boldsymbol{A}} = G. \end{array} \tag{19}$$

(19) corresponds to a system of $(N+1)^2$ linear algebraic equations with unknown Bernoulli coefficients $a_{0,0}, a_{0,1}, \cdots a_{0,N}, a_{1,0}, a_{1,1}, \cdots, a_{1,N}, \cdots, a_{N,0}, a_{N,1}, \cdots, a_{N,N}$.

Therefore, we can write the fundamental matrix equation (19) corresponding to (1) as

$$\boldsymbol{W}\overline{\boldsymbol{A}} = \boldsymbol{G} \tag{20}$$

where

$$\begin{aligned} \boldsymbol{W} = {} & A\boldsymbol{B}(T^T)^2\boldsymbol{Q} + B\boldsymbol{B}T^T\boldsymbol{Q}(\overline{T}) + C\boldsymbol{B}\boldsymbol{Q}(\overline{T})^2+ \\ & D\boldsymbol{B}(T^T)\boldsymbol{Q} + E\boldsymbol{B}\boldsymbol{Q}(\overline{T}) + F\boldsymbol{B}\boldsymbol{Q}. \end{aligned}$$

Also we express the matrix forms of conditions (13), (14), (15) and (16) by using collocation points as

$$\boldsymbol{U}\overline{\boldsymbol{A}} = [\boldsymbol{F}_1], \tag{21}$$

$$\boldsymbol{V}\overline{\boldsymbol{A}} = [\boldsymbol{G}_1], \tag{22}$$

$$\overline{\boldsymbol{U}\boldsymbol{A}} = [\boldsymbol{F}_2], \tag{23}$$

$$\overline{\boldsymbol{V}\boldsymbol{A}} = [\boldsymbol{G}_2]. \tag{24}$$

Here

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{U}_0 & \boldsymbol{U}_1 & \cdots & \boldsymbol{U}_N \end{bmatrix}^T, \boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_0 & \boldsymbol{V}_1 & \cdots & \boldsymbol{V}_N \end{bmatrix}^T,$$

$$\overline{\boldsymbol{U}} = \begin{bmatrix} \overline{\boldsymbol{U}}_0 & \overline{\boldsymbol{U}}_1 & \cdots & \overline{\boldsymbol{U}}_N \end{bmatrix}^T, \overline{\boldsymbol{V}} = \begin{bmatrix} \overline{\boldsymbol{V}}_0 & \overline{\boldsymbol{V}}_1 & \cdots & \overline{\boldsymbol{V}}_N \end{bmatrix}^T,$$

$$\boldsymbol{F}_1 = \begin{bmatrix} f_1(x_0) & f_1(x_1) & \cdots & f_1(x_N) \end{bmatrix}^T, \boldsymbol{G}_1 = \begin{bmatrix} g_1(y_0) & g_1(y_1) & \cdots & g_1(y_N) \end{bmatrix}^T,$$

$$\boldsymbol{F}_2 = \begin{bmatrix} f_2(x_0) & f_2(x_1) & \cdots & f_2(x_N) \end{bmatrix}^T, \boldsymbol{G}_2 = \begin{bmatrix} g_2(y_0) & g_2(y_1) & \cdots & g_2(y_N) \end{bmatrix}^T,$$

$$i = 0, 1, \cdots, N \quad j = 0, 1, \cdots, N$$

$$\boldsymbol{U}_i = \boldsymbol{B}(x_i)\boldsymbol{Q}(0) = \begin{bmatrix} u_{i1} & u_{i2} & \cdots & u_{i(N+1)^2} \end{bmatrix}, \boldsymbol{V}_j = \boldsymbol{B}(0)\boldsymbol{Q}(y_j) = \begin{bmatrix} v_{j1} & v_{j2} & \cdots & v_{j(N+1)^2} \end{bmatrix},$$

$$\overline{\boldsymbol{U}_i} = \boldsymbol{B}(x_i)\boldsymbol{Q}(0)\overline{\boldsymbol{T}} = \begin{bmatrix} \overline{u}_{i1} & \overline{u}_{i2} & \cdots & \overline{u}_{i(N+1)^2} \end{bmatrix}, \overline{\boldsymbol{V}}_j = \boldsymbol{L}\boldsymbol{\zeta}^T\boldsymbol{Q}(y_j) = \begin{bmatrix} \overline{v}_{j1} & \overline{v}_{j2} & \cdots & \overline{v}_{j(N+1)^2} \end{bmatrix}.$$

To obtain the approximate solution of (1) under conditions (2), (3), (4) and (5), we form the augmented matrix

$$\left[\widetilde{\boldsymbol{W}}, \widetilde{\boldsymbol{G}}\right] = \begin{bmatrix} \boldsymbol{U} & ; & \boldsymbol{F}_1 \\ \boldsymbol{V} & ; & \boldsymbol{G}_1 \\ \overline{\boldsymbol{U}} & ; & \boldsymbol{F}_2 \\ \overline{\boldsymbol{V}} & ; & \boldsymbol{G}_2 \\ \boldsymbol{W} & ; & \boldsymbol{G} \end{bmatrix}. \tag{25}$$

The unknown Bernoulli coefficients are obtained as

$$\overline{\boldsymbol{A}} = (\widetilde{\widetilde{\boldsymbol{W}}})^{-1}\widetilde{\widetilde{\boldsymbol{G}}}$$

where $\left[\widetilde{\widetilde{\boldsymbol{W}}}; \widetilde{\widetilde{\boldsymbol{G}}}\right]$ is generated by using the Gauss elimination method and then removing zero rows of gauss eliminated matrix.

By substituting the determined coefficients into (6), we obtain the Bernoulli polynomial solution

$$u(x, y) = \sum_{r=0}^{N} \sum_{s=0}^{N} a_{r,s} B_{r,s}(x, y). \tag{26}$$

4. Residual Correction and Error Estimation

In this section, the residual correction method [19]-[20], an efficient error estimation will be given for the Bernoulli collocation method. For our purpose, we can define the residual function with two variables of the Bernoulli collocation method as

$$R(x, y) = Lu_{N,N} - f \tag{27}$$

where $u_{N,N}$, which is the Bernoulli polynomial solution defined by (6), is the approximate solution of the problem (1)-(5), hence $u_{N,N}$ satisfies the problem

$$\begin{aligned} L\left[u_N(x, y)\right] = A(x, y)\tfrac{\partial^2 u}{\partial x^2} + B(x, y)\tfrac{\partial^2 u}{\partial x \partial y} + C(x, y)\tfrac{\partial^2 u}{\partial y^2} + D(x, y)\tfrac{\partial u}{\partial x} \\ + E(x, y)\tfrac{\partial u}{\partial y} + F(x, y)u = G(x, y) + R_N(x, y). \end{aligned} \tag{28}$$

Also, the error function $e_N(x, y)$ can be defined as

$$e_N(x, y) = u(x, y) - u_N(x, y) \tag{29}$$

where $u(x,y)$ is the exact solution of the problem (1)-(5). Substituting (29) into (1)-(5) and using (27)-(28), we have the error differential equation

$$L\left[e_N(x,y)\right] = L\left[u(x,y)\right] - L\left[u_N(x,y)\right] = -R_N(x,y)$$

with the homogenous conditions

$$\begin{aligned} u(x,0) &= 0,\ 0 \le x \le 1, \\ u(0,y) &= 0,\ 0 \le y \le 1, \\ u_y(x,0) &= 0,\ 0 \le x \le 1, \\ \int_0^1 u(x,y)dx &= 0,\ 0 \le y \le 1. \end{aligned}$$

or clearly, the problem

$$\begin{aligned} A(x,y)\tfrac{\partial^2 e_N}{\partial x^2} + B(x,y)\tfrac{\partial^2 e_N}{\partial x \partial y} + C(x,y)\tfrac{\partial^2 e_N}{\partial y^2} + D(x,y)\tfrac{\partial e_N}{\partial x} \\ +E(x,y)\tfrac{\partial e_N}{\partial y} + F(x,y)e_N = -R_N(x,y). \end{aligned} \tag{30}$$

Solving the problem (30) in the same way as Section 3, we get the approximation $e_{N,M}(x,y)$ to $e_N(x,y)$, $(M \ge N)$ which is the error function based on the Residual function $R_N(x,y)$.

Consequently, by means of the polynomials $u_N(x,y)$ and $e_{N,M}(x,y)$, $(M \ge N)$ we obtain the corrected Bernoulli polynomial solution $u_{N,M}(x,y) = u_N(x,y) + e_{N,M}(x,y)$. Also, we construct the Bernoulli error function $e_N(x,y) = u(x,y) - u_N(x,y)$, the corrected Bernoulli error function $E_{N,M}(x,y) = e_N(x,y) - e_{N,M}(x,y) = u(x,y) - u_{N,M}(x,y)$ and the estimated error function $e_{N,M}(x,y)$.

5. Numerical Example

Consider the second order linear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -2(x-t)e^{-x-t},\ 0 \le x \le 1,\ 0 \le t \le 1,$$

with Dirichlet boundary conditions given by

$$\begin{aligned} u(x,0) &= 0,\ 0 \le x \le 1, \\ u(0,t) &= 0,\ 0 \le t \le 1, \end{aligned}$$

Neumann boundary condition

$$u_t(x,0) = xe^{-x},\ 0 \le x \le 1,$$

and the nonlinear integral condition

$$\int_0^1 u(x,t)dx = -2te^{-t-1} + te^{-t},\ 0 \le t \le T.$$

The exact solution of the this problem is $u(x,t) = xte^{-x-t}$ [9]. By using Bernoulli Collocation Method, absolute error functions and corrected Bernoulli error functions are compared in Table for the various values of M and N. As shown in the table the better results may be obtained by increasing tha values of N and M.

Table The Comparison of absolute error function $|e_N|$ and corrected Bernoulli error function $\left|E_{N,M}\right|$.

(x_i, t_i)	Absolute error function $\lvert e_N\rvert$	Corrected Bernoulli error function $\lvert E_{N,M}\rvert$	Absolute error function $\lvert e_N\rvert$	Corrected Bernoulli error function $\lvert E_{N,M}\rvert$
	$N = 3, M = 4$		$N = 7, M = 10$	
$(0, 0)$	$3.7147e-4$	$7.0844e-4$	$1.6642e-9$	$1.4642e-9$
$(0, \frac{1}{2})$	$6.4603e-4$	$8.1347e-4$	$2.7762e-8$	$2.7708e-8$
$(0, 1)$	$2.9759e-4$	$7.4609e-4$	$1.7049e-8$	$1.7038e-8$
$(\frac{1}{2}, 0)$	$2.1300e-3$	$2.4480e-3$	$6.7822e-8$	$6.8070e-8$
$(\frac{1}{2}, \frac{1}{2})$	$1.9052e-3$	$1.3789e-3$	$4.3160e-8$	$3.7241e-8$
$(\frac{1}{2}, 1)$	$5.3472e-3$	$2.3655e-3$	$5.3314e-7$	$5.2575e-7$
$(1, 0)$	$1.6690e-3$	$2.3616e-3$	$1.0216e-7$	$1.0295e-7$
$(1, \frac{1}{2})$	$2.5654e-3$	$8.9822e-4$	$4.9311e-7$	$4.8538e-7$
$(1, 1)$	$3.5809e-2$	$1.5865e-2$	$7.7803e-6$	$7.8498e-6$

6. Conclusion

In the present article, Bernoulli matrix method is presented to solve numerically second order linear hyperbolic equation. Also an error analysis based on Residual functions has been descriped. To illustrate the accuracy and efficiency of the new method, an example has been analyzed.The obtained numerical results show that this method can solve the problem effectively.

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