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A Formula for Generating Weakly Modular Forms with Weight 12

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Abstract. In this short paper, generally, we define a family of functions f_k depends on the Eisenstein series with weight $2k$, for $k \in \mathbb{N}$. More detail, by considering the function f_k, we define a derivative formula for generating weakly modular forms with weight 12. As a result for this, we claim that this formula gives an advantage to find the special solutions of some differential equations.

1. Introduction and Main Result

Let $\mathbb{H} = \{z = u + iv : v > 0\}$ for $u, v \in \mathbb{R}$. The set of all Möbius transformations of the form $Mz = z' = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, is called the "modular group" and given by

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

(cf. [2]). Let $k \in \mathbb{R}$. A weakly modular form with weight $2k$ is a meromorphic function f on $\mathbb{H}$ such that, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$ and all $z \in \mathbb{H}$, we have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z)$$

(cf. [2], [3]). Throughout this paper, the space of weakly modular forms with weight $2k$ is denoted by M_{2k}. For $2 \leq k \in \mathbb{N}$, the Eisenstein series $G_{2k}(z)$ ([4], [5]) is defined by

$$G_{2k}(z) = \sum_{0 \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(mz+n)^{2k}}.$$

The normalized Eisenstein series ([6]) is defined by

$$E_{2k}(z) = \frac{1}{2\zeta(2k)} G_{2k}(z)$$

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where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

for Re $s > 1$ (cf. [2]).

$E_{2k}(z)$ is a holomorphic modular function with weight $2k$ in M_{2k}.

In [1], Aygunes showed that there exist a family of functions f_k with weight $2k$ as follows:

$$f_k\left(\frac{az+b}{cz+d}\right) = 2kc(cz+d) + (cz+d)^2 f_k(z) \tag{1.1}$$

where $ad - bc = 1$ for $a, b, c, d \in \mathbb{Z}$ and also he choose

$$f_k(\tau) = \frac{E'_{2k}(\tau)}{E_{2k}(\tau)}.$$

Then he shows the existence of the family of the functions f_k. Therefore, we conclude that the function f_k depends on the Eisenstein series with weight $2k$.

Let H_{2k} denote the family of meromorphic functions f_k on $\mathbb{H}$ satisfying (1.1), for $ad - bc = 1$, which are also meromorphic at $i\infty$.

In this paper, by using the family of functions f_k, we give a formula to derive the weakly modular forms with weight 12.

Suppose that $ad - bc = 1$ for $a, b, c, d \in \mathbb{Z}$. Then,

$$\frac{d}{dz} f_k\left(\frac{az+b}{cz+d}\right) = \frac{1}{(cz+d)^2} f'_k\left(\frac{az+b}{cz+d}\right), \tag{1.2}$$

$$\frac{d^2}{dz^2} f_k\left(\frac{az+b}{cz+d}\right) = -\frac{2c}{(cz+d)^3} f'_k\left(\frac{az+b}{cz+d}\right) + \frac{1}{(cz+d)^4} f''_k\left(\frac{az+b}{cz+d}\right) \tag{1.3}$$

and

$$\frac{d^3}{dz^3} f_k\left(\frac{az+b}{cz+d}\right) = \frac{6c^2}{(cz+d)^4} f'_k\left(\frac{az+b}{cz+d}\right) - \frac{6c}{(cz+d)^5} f''_k\left(\frac{az+b}{cz+d}\right) + \frac{1}{(cz+d)^6} f'''_k\left(\frac{az+b}{cz+d}\right). \tag{1.4}$$

On the other hand, by using the equation (1.1), we have

$$\frac{d}{dz} f_k\left(\frac{az+b}{cz+d}\right) = 2kc^2 + 2c(cz+d) f_k(z) + (cz+d)^2 f'_k(z), \tag{1.5}$$

$$\frac{d^2}{dz^2} f_k\left(\frac{az+b}{cz+d}\right) = 2c^2 f_k(z) + 4c(cz+d) f'_k(z) + (cz+d)^2 f''_k(z) \tag{1.6}$$

and

$$\frac{d^3}{dz^3} f_k\left(\frac{az+b}{cz+d}\right) = 6c^2 f'_k(z) + 6c(cz+d) f''_k(z) + (cz+d)^2 f'''_k(z). \tag{1.7}$$

We use the equations (1.2), (1.3) and (1.4) to arrange the equations (1.5), (1.6) and (1.7). Then,

$$f'_k\left(\frac{az+b}{cz+d}\right) = 2kc^2(cz+d)^2 + 2c(cz+d)^3 f_k(z) + (cz+d)^4 f'_k(z), \tag{1.8}$$

$$- 2c(cz+d)f_k'\left(\frac{az+b}{cz+d}\right) + f_k''\left(\frac{az+b}{cz+d}\right)$$
$$= 2c^2(cz+d)^4 f_k(z) + 4c(cz+d)^5 f_k'(z) + (cz+d)^6 f_k''(z) \tag{1.9}$$

and

$$6c^2(cz+d)^2 f_k'\left(\frac{az+b}{cz+d}\right) - 6c(cz+d)f_k''\left(\frac{az+b}{cz+d}\right) + f_k'''\left(\frac{az+b}{cz+d}\right)$$
$$= 6c^2(cz+d)^6 f_k'(z) + 6c(cz+d)^7 f_k''(z) + (cz+d)^8 f_k'''(z) \tag{1.10}$$

By using the equation (1.8), we arrange the equation (1.9) as follows:

$$f_k''\left(\frac{az+b}{cz+d}\right) = 4kc^3(cz+d)^3 + 6c^2(cz+d)^4 f_k(z) + 6c(cz+d)^5 f_k'(z) + (cz+d)^6 f_k''(z).$$

By using the equation (1.8) and (1.9), we arrange the equation (1.10) as follows:

$$f_k'''\left(\frac{az+b}{cz+d}\right) = 12kc^4(cz+d)^4 + 24c^3(cz+d)^5 f_k(z) + 36c^2(cz+d)^6 f_k'(z)$$
$$+ 12c(cz+d)^7 f_k''(z) + (cz+d)^8 f_k'''(z).$$

Then, we set

$$\left\{f_k'\left(\frac{az+b}{cz+d}\right)\right\}^3 = (cz+d)^6 \times \left\{2kc^2 + 2c(cz+d)f_k(z) + (cz+d)^2 f_k'(z)\right\}^3,$$

$$\left\{f_k''\left(\frac{az+b}{cz+d}\right)\right\}^2 = (cz+d)^6 \times \left\{4kc^3 + 6c^2(cz+d)f_k(z) + 6c(cz+d)^2 f_k'(z) + (cz+d)^3 f_k''(z)\right\}^2,$$

$$f_k'\left(\frac{az+b}{cz+d}\right) f_k'''\left(\frac{az+b}{cz+d}\right) = (cz+d)^6 \times \left(2kc^2 + 2c(cz+d)f_k(z) + (cz+d)^2 f_k'(z)\right)$$
$$\times \left\{ \begin{array}{c} 12kc^4 + 24c^3(cz+d)f_k(z) + 36c^2(cz+d)^2 f_k'(z) \\ +12c(cz+d)^3 f_k''(z) + (cz+d)^4 f_k'''(z) \end{array} \right\},$$

$$f_k\left(\frac{az+b}{cz+d}\right) f_k'\left(\frac{az+b}{cz+d}\right) f_k''\left(\frac{az+b}{cz+d}\right) = (cz+d)^6 \times (2kc + (cz+d)f_k(z))$$
$$\times \left(2kc^2 + 2c(cz+d)f_k(z) + (cz+d)^2 f_k'(z)\right)$$
$$\times (4kc^3 + 6c^2(cz+d)f_k(z) + 6c(cz+d)^2 f_k'(z) + (cz+d)^3 f_k''(z)),$$

$$\left\{f_k\left(\frac{az+b}{cz+d}\right)\right\}^2 \left\{f_k'\left(\frac{az+b}{cz+d}\right)\right\}^2 = (cz+d)^6 \times (2kc + (cz+d)f_k(z))^2$$
$$\times \left(2kc^2 + 2c(cz+d)f_k(z) + (cz+d)^2 f_k'(z)\right)^2,$$

$$\left\{f_k\left(\frac{az+b}{cz+d}\right)\right\}^2 f_k'''\left(\frac{az+b}{cz+d}\right) = (cz+d)^6 \times (2kc + (cz+d)f_k(z))^2$$
$$\times \left\{ \begin{array}{c} 12kc^4 + 24c^3(cz+d)f_k(z) + 36c^2(cz+d)^2 f_k'(z) \\ +12c(cz+d)^3 f_k''(z) + (cz+d)^4 f_k'''(z) \end{array} \right\},$$

$$\left\{f_k\left(\frac{az+b}{cz+d}\right)\right\}^3 f_k''\left(\frac{az+b}{cz+d}\right) = (cz+d)^6 \times (2kc + (cz+d)f_k(z))^3$$
$$\times (4kc^3 + 6c^2(cz+d)f_k(z) + 6c(cz+d)^2 f_k'(z) + (cz+d)^3 f_k''(z)).$$

By using the above equations, we get

$$\left\{\begin{array}{c} 4k\left\{f_k'\left(\frac{az+b}{cz+d}\right)\right\}^3 + 5k^2\left\{f_k''\left(\frac{az+b}{cz+d}\right)\right\}^2 - 4k^2 f_k'\left(\frac{az+b}{cz+d}\right) f_k'''\left(\frac{az+b}{cz+d}\right) - 6k f_k\left(\frac{az+b}{cz+d}\right) f_k'\left(\frac{az+b}{cz+d}\right) f_k''\left(\frac{az+b}{cz+d}\right) \\ +3\left\{f_k\left(\frac{az+b}{cz+d}\right)\right\}^2\left\{f_k'\left(\frac{az+b}{cz+d}\right)\right\}^2 + 2k\left\{f_k\left(\frac{az+b}{cz+d}\right)\right\}^2 f_k'''\left(\frac{az+b}{cz+d}\right) - 2\left\{f_k\left(\frac{az+b}{cz+d}\right)\right\}^3 f_k''\left(\frac{az+b}{cz+d}\right) \end{array}\right\}$$
$$= (cz+d)^{12}\left\{\begin{array}{c} 4k\left\{f_k'(z)\right\}^3 + 5k^2\left\{f_k''(z)\right\}^2 - 4k^2 f_k'(z) f_k'''(z) - 6k f_k(z) f_k'(z) f_k''(z) \\ +3\left\{f_k(z)\right\}^2\left\{f_k'(z)\right\}^2 + 2k\left\{f_k(z)\right\}^2 f_k'''(z) - 2\left\{f_k(z)\right\}^3 f_k''(z) \end{array}\right\}.$$

Consequently, we arrive at the following theorem:

Theorem 1.1. *Let $\mathcal{D}_k$ be an operator defined on H_{2k}. Let $\tau \in \mathbb{H}$ and k be any integer. If an operator $\mathcal{D}_k$ is defined by*

$$\mathcal{D}_k(f_k) = 4k\left(f_k'\right)^3 + 5k^2\left(f_k''\right)^2 - 4k^2 f_k' f_k''' - 6k f_k f_k' f_k'' + 3\left(f_k\right)^2\left(f_k'\right)^2 + 2k\left(f_k\right)^2 f_k''' - 2\left(f_k\right)^3 f_k'',$$

then we have

$$\mathcal{D}_k : H_{2k} \longrightarrow M_{12}.$$

Corollary 1.2. *In Theorem 1.1, if we choose*

$$f_k(\tau) = \frac{E_{2k}'(\tau)}{E_{2k}(\tau)},$$

we obtain a weakly modular form g_k in M_{12} such that

$$g_k = 4k\left(f_k'\right)^3 + 5k^2\left(f_k''\right)^2 - 4k^2 f_k' f_k''' - 6k f_k f_k' f_k'' + 3\left(f_k\right)^2\left(f_k'\right)^2 + 2k\left(f_k\right)^2 f_k''' - 2\left(f_k\right)^3 f_k''$$

Therefore we conclude that, for a special case of the function g_k, there exist a solution of the above differential equation.

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