Filomat 30:4 (2016), 1045–1052
DOI 10.2298/FIL1604045A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Application of the Bernstein Polynomials for Solving Volterra Integral Equations with Convolution Kernels

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Abstract. In this article, we consider the second-type linear Volterra integral equations whose kernels based upon the difference of the arguments. The aim is to convert the integral equation to an algebraic one. This is achieved by approximating functions appearing in the integral equation with the Bernstein polynomials. Since the kernel is of convolution type, the integral is represented as a convolution product. Taylor expansion of kernel along with the properties of convolution are used to represent the integral in terms of the Bernstein polynomials so that a set of algebraic equations is obtained. This set of algebraic equations is solved and approximate solution is obtained. We also provide a simple algorithm which depends both on the degree of the Bernstein polynomials and that of monomials. Illustrative examples are provided to show the validity and applicability of the method.

1. Introduction

Volterra integral equations of the second kind is written as

$$u(t) = f(t) + \lambda \int_a^t K(t,x)u(x)\,dx. \tag{1}$$

In this equation non-homogenous term $f(t)$, the kernel $K(x,t)$, and a constant parameter λ are given and the desired function is $u(t)$. Since linear and nonlinear Volterra integral equations appear in many scientific applications with a very wide range from physical sciences to engineering, considerable amount of work has been done on solving them. The literature is very dense on the subject. Many analytical and numerical techniques have been introduced so far and it is still expanding [13], [14],[23].
The kernel $K(t,x)$ plays a vital role in classification of the integral equation and in constructing solution methods. There exist different solution techniques for different types of kernels. Difference kernels form an important class of kernels, some of which arise in neutron transport theory, gas dynamics, etc,[3],[7],[16], [24]. In this article, we aim to investigate the Volterra integral equation of the second kind with a difference kernel. We first focus on the kernel which is of the form $K(t,x) = (t-x)^n, n \in \mathbb{Z}^+$. To be more precise, we first consider

$$u(t) = f(t) + \lambda \int_0^t (t-x)^n u(x)\,dx. \tag{2}$$

2010 *Mathematics Subject Classification*. 44A35, 45A05, 45D05, 45E10.
Keywords. Volterra integral equations; Bernstein polynomials; Difference Kernel.
Received: 15 July 2015; Accepted: 12 October 2015
Communicated by Gradimir Milovanović and Yilmaz Simsek
The present investigation was supported by the Scientific Research Project Administration of Amasya University
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We show that this analysis form the basis for a more general kernels (see section 4). We then investigate (see section 4)

$$u(t) = f(t) + \lambda \int_0^t K(t-x)u(x)\,dx. \tag{3}$$

Approximations with polynomials or in general approximations with orthonormal functions have always been an attractive method for scientists to solve integral equations. In addition to their nice properties for analysis, they are also computer friendly. Easy to use algorithms can be obtained from them [5]. In particular, some efficient methods in which polynomials are used as main tools constructed for solving linear and nonlinear Volterra integral equations. These methods include Taylor polyomials, Chebyshev polynomials, Legendre polynomials, Laguerre polynomials, etc, just to name a few, [8], [10], [15], [17].
The Bernstein polynomials form a useful class of functions of Mathematical Physics. A constructive proof of Weierstrass approximation theorem based on the Bernstein polynomials is given in [4]. They have recently been applied to some classes of integral equations to obtain numerical solutions [1], [2], [11], [21],[22], [25]. In [1], authors use them to solve the Abel's integral equation. It is known that Abel's integral equation is a singular Volterra integral equation. In this article, we make an attempt to extend the application of the Bernstein polynomials. In particular, we use them to find approximate solutions for (2) and (3).
The rest of this paper is organised as follows. In section 2, we give basic definitions and theorems required for subsequent sections. In particular, analytic functions, the Bernstein polynomials, and their properties are reviewed. In section 3, approximation properties of the Bernstein polynomials are introduced. In section 4, the Bernstein polynomials are applied to solve the integral equation (2) and (3). Furthermore, we summarize the method and give a simple procedure about how to apply it. In subsection 4.1, illustrative examples are given. The graphs and error tables are also provided. In section 5, we conclude and discuss the work done in this paper.

2. Review of Basic Concepts

Definition 2.1. *The Bernstein polynomials of m^{th}-degree are defined by*

$$B_{k,m}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \quad k = 0, 1, \dots, m,$$

where the binomial coefficients are given by $\binom{m}{k} = \frac{m!}{k!(m-k)!}$.

They have many useful properties [12]. What follows is a list of some of these properties, especially those that will be used throughout the paper.

- The Bernstein polynomials of m^{th}-degree, $\{B_{k,m}(x) : 0 \leq k \leq m,\ m \geq 0\}$, form a complete system in $L^2[0, 1]$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$, and the associated norm $\|f\| = \langle f, f\rangle^{1/2}$.
- They form a partition of unity. That is, $\sum_{k=0}^{m} B_{k,m}(x) = \sum_{k=0}^{m-1} B_{k,m-1}(x)$.
- They can be written in terms of power basis, $\{1, x, x^2, \dots, x^m\}$,

$$B_{k,m}(x) = \sum_{i=k}^{m} (-1)^{i-k} \binom{m}{i}\binom{i}{k} x^i.$$

- Power basis functions can be written in terms of the Bernstein polynomials. That is,

$$x^k = \sum_{i=k}^{m} \frac{\binom{i}{k}}{\binom{m}{k}} B_{i,m}(x).$$

This list of properties and many others are known properties of the Bernstein polynomials. For a more detailed treatment we refer the reader to the articles and books published on this subject [5], [6], [12],[18] - [20].
We let

$$\mathbf{\Psi}_m(x) = [B_{0,m}, B_{1,m}, \ldots, B_{m,m}]^T \text{ and } T_m(x) = [1, x, x^2, \ldots, x^m]^T \tag{4}$$

so that we could write $\mathbf{\Psi}_m(x) = AT_m(x)$, where A is a $(m+1)\times(m+1)$ nonsingular, upper triangular matrix with entries

$$A(i+1, j+1) = \begin{cases} (-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i} & \text{if } i \le j, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $i, j = 0, 1, 2, \ldots, m$, [1].

3. Approximation

Theorem 3.1. *If H is a Hilbert space and if S is a closed subspace of H, then for any $f \in H$, the best approximation exists and unique.*

Proof: The proof can be found in [9].
We assume that $H = L^2[0,1]$ and $S = \{B_{0,m}, B_{1,m}, \ldots, B_{m,m}\}$.
Then for any $f \in H$,

$$f(x) \approx f_0(x) = \sum_{i=0}^{m} \alpha_i B_{i,m}(x) = \boldsymbol{\alpha}^T \Psi_m(x),$$

where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]^T$.
In order to calculate $\boldsymbol{\alpha}$, we define

$$\langle f, \mathbf{\Psi}_m\rangle = \int_0^1 f(x)\mathbf{\Psi}_m^T(x)\,dx, \quad \langle \mathbf{\Psi}_m, \mathbf{\Psi}_m\rangle = \int_0^1 \mathbf{\Psi}_m(x)\mathbf{\Psi}_m^T(x)\,dx, \tag{6}$$

and let $\langle \mathbf{\Psi}_m, \mathbf{\Psi}_m\rangle = Q_m$. We note that Q_m is a $(m+1)\times(m+1)$ matrix with entires

$$Q_m(i+1, j+1) = \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}}, \tag{7}$$

where $i, j = 0, 1, 2, \ldots, m$, [1].

Lemma 3.2. *[1] Suppose that $f \in C^{m+1}([0,1])$ and $S = span\{B_{0,m}, B_{1,m}, \ldots, B_{m,m}\}$. If $\boldsymbol{\alpha}^T B$ is the best approximation of f in S, then*

$$\| f - \boldsymbol{\alpha}^T B \|_{L^2[0,1]} \le \frac{max|f^{(m+1)}(x)|}{(m+1)!(\sqrt{2m+3})}.$$

4. Solution of Volterra Integral Equation

In this section, we consider the Volterra integral equation of type (2) and (3), respectively.
Case1 :$K(t,x) = (t-x)^n, n \in \mathbb{Z}^+$
We first appoximate the function $u(t)$ and $f(t)$ by the Bernstein polynomials as follows.

$$u(t) \approx \boldsymbol{\alpha}^T \mathbf{\Psi}_m(t) \text{ and } f(t) \approx \boldsymbol{\beta}^T \mathbf{\Psi}_m(t). \tag{8}$$

By substituting (8) into (2) the integral equation turns out to have the following form:

$$\boldsymbol{\alpha}^T \boldsymbol{\Psi}_m(t) = \boldsymbol{\beta}^T \boldsymbol{\Psi}_m(t) + \lambda \int_0^t (t-x)^n \boldsymbol{\alpha}^T \boldsymbol{\Psi}_m(x)\, dx. \tag{9}$$

In order to transform (9) into an algebraic equation, we need to find the operational matrix F such that

$$\int_0^t (t-x)^n \boldsymbol{\Psi}_m(x)\, dx \approx F \boldsymbol{\Psi}_m(t). \tag{10}$$

To find out F we interpret the integral in (10) as a convolution product and write it as

$$\int_0^t (t-x)^n \boldsymbol{\Psi}_m(x)\, dx = t^n * \boldsymbol{\Psi}_m(t),$$

where $*$ denotes the convolution product and

$$t^n * \boldsymbol{\Psi}_m(t) = \left[t^n * B_{0,m}, t^n * B_{1,m}, \ldots t^n * B_{m,m} \right]^T = t^n * (A T_m(t)) = A(t^n * T_m(t)), \tag{11}$$

where T_m and A are as defined in (4), (5), respectively. If we expand the convolution product in (11), we obtain

$$t^n * T_m(t) = \left[t^n * 1, t^n * t, \ldots t^n * t^m \right]^T = D_{n.m} \overline{T}_{n,m}, \tag{12}$$

where $D_{n,m}$ is an $(m+1) \times (m+1)$ with entries

$$D_{n,m}(i+1, j+1) = \begin{cases} \frac{n!}{(i+1)(i+2)\ldots(i+n+1)} & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases} \tag{13}$$

for $i, j = 0, 1, 2, \ldots, m$ and $\overline{T}_{n,m} = \left[t^{n+1}, t^{n+2}, \ldots t^{n+m+1} \right]^T$.
We now want to approximate $\overline{T}_{n,m}$ and write it as $\overline{T}_{n,m} \approx E \boldsymbol{\Psi}_m(t)$, where E is a $(m+1) \times (m+1)$ matrix. It turns out that when we approximate each entries of $\overline{T}_{n,m}$ by the Bernstein polynomials, we obtain

$$t^{n+1} = E_1^T \boldsymbol{\Psi}_m(t), \quad t^{n+2} = E_2^T \boldsymbol{\Psi}_m(t), \quad \ldots, \quad t^{n+m+1} = E_{m+1}^T \boldsymbol{\Psi}_m(t), \tag{14}$$

where $E_i^T = \langle t^{n+i}, \boldsymbol{\Psi}_m(t) \rangle (\langle \boldsymbol{\Psi}_m(t), \boldsymbol{\Psi}_m(t) \rangle)^{-1}$ for $i = 1, 2, \ldots, m+1$. From (6) and (7) E_i^T can simply be written as

$$E_i^T = \langle t^{n+i}, \boldsymbol{\Psi}_m(t) \rangle Q^{-1} = \int_0^1 t^{n+i} \boldsymbol{\Psi}_m^T(t)\, dt. \tag{15}$$

We now let $E = [E_1, E_2, \ldots, E_n]^T$ so that $\overline{T}_{n,m} \approx E \boldsymbol{\Psi}_m(t)$.
The last calculation allows us to write

$$F = A D_{n,m} E. \tag{16}$$

We finally ready to obtain the algebraic equation corresponding to the integal equation (2). Combining equations from (8) to (16), we have

$$\begin{aligned} \boldsymbol{\alpha}^T \boldsymbol{\Psi}_m(t) &= \boldsymbol{\beta}^T \boldsymbol{\Psi}_m(t) + \lambda \int_0^t (t-x)^n \boldsymbol{\alpha}^T \boldsymbol{\Psi}_m(x)\, dx, \\ \boldsymbol{\alpha}^T \boldsymbol{\Psi}_m(t) &= \boldsymbol{\beta}^T \boldsymbol{\Psi}_m(t) + \lambda \boldsymbol{\alpha}^T F \boldsymbol{\Psi}_m(t), \\ \boldsymbol{\alpha}^T (\mathbf{I} - \lambda F) \boldsymbol{\Psi}_m(t) &= \boldsymbol{\beta}^T \boldsymbol{\Psi}_m(t). \end{aligned}$$

Thus,

$$\boldsymbol{\alpha}^T = \boldsymbol{\beta}^T(\mathbf{I} - \lambda F)^{-1}. \tag{17}$$

Then the approximate solution is

$$u(t) \approx \boldsymbol{\alpha}^T \boldsymbol{\Psi}_m(t). \tag{18}$$

We now summarize the above calculation and give step by step directions for how to obtain an approximate solution.

- Determine the degree m of the Bernstein polynomials.
- Since n is the degree of the kernel, given n and m, compute (5), (7), (13), and (15).
- Calculate $F = AD_{n,m}E$ and $\boldsymbol{\beta}$.
- From (17), evaluate $\boldsymbol{\alpha}^T$. Plug this into (18).

Case2 :$K(t, x) = K(t - x)$
In this section, we consider the Volterra integral equation of type (3). We assume that the kernel is analytic at $a = 0$. The reason for this assumption is that we are going to use truncated Taylor expansion of the kernel. We follow similar steps as in Case1. For the sake of simplicity, we only explain the steps which did not show up in the previous case.
We again use the following approximations.

$$u(t) \approx \boldsymbol{\alpha}^T \boldsymbol{\Psi}_m(t) \text{ and } f(t) \approx \boldsymbol{\beta}^T \boldsymbol{\Psi}_m(t).$$

We replace the convolution product in (11) by $K(t) * \boldsymbol{\Psi}_m(t)$. Instead of $K(t)$ we use its truncated Taylor expansion. Convolution has the distributivity property over summation. This takes us to the previous case. We note that the more terms we use in truncated Taylor series, the better approximation result we obtain. In the following, we consider two examples whose exact solutions are known. In order to focus on to a particular point, the graphs and error tables are provided only for example 2.

4.1. Illustrative Examples

Example 1: Consider the following Volterra integral equation of the second kind [23]:

$$u(t) = 1 - \int_0^t (t - x)u(x)dx.$$

Here $f(t) = 1$, $\lambda = 1$, and since $k(t, x) = (t - x)$, $n = 1$.
We first let $m = 3$, so we use the 3^{rd} Bernstein polynomials to approximate the functions. We let

$$u(t) \approx \boldsymbol{\alpha}^T \boldsymbol{\Psi}_3(t) \text{ and } 1 \approx \boldsymbol{\beta}^T \boldsymbol{\Psi}_3(t).$$

We obtain $\boldsymbol{\alpha}^T \approx [0.9995, 1.0030, 0.8236, 0.5398]$, $\boldsymbol{\beta}^T = [1, 1, 1, 1]$, and

$$F = \begin{pmatrix} -1/504 & 1/84 & 1/8 & 25/126 \\ 1/420 & -13/840 & 13/210 & 127/840 \\ 1/840 & -1/210 & 1/840 & 43/420 \\ -1/630 & 1/120 & -3/140 & 121/2520 \end{pmatrix}$$

The approximate solution becomes $u(t) \approx 0.0788t^3 - 0.5492t^2 + 0.0107t + 0.9995$
We now let $m = 5$, so we use the 5^{th} Bernstein polynomials to approximate the functions. We let

$$u(t) \approx \boldsymbol{\alpha}^T \boldsymbol{\Psi}_5(t) \text{ and } 1 \approx \boldsymbol{\beta}^T \boldsymbol{\Psi}_5(t).$$

We obtain $\boldsymbol{\alpha}^T \approx [1, 1, 0.95, 0.85, 0.71, 0.54]$, $\boldsymbol{\beta}^T = [1, 1, 1, 1, 1, 1]$, and

$$F = \begin{pmatrix} -1/10296 & 19/25740 & 811/17160 & 191/2574 & 5669/51480 & 245/1716 \\ 1/3276 & -283/120120 & 404/45045 & 20753/360360 & 153/1820 & 8597/72072 \\ -1/4004 & 92/45045 & -139/16380 & 829/30030 & 11267/180180 & 857/9009 \\ -1/9009 & 113/180180 & -29/30030 & -61/16380 & 1808/45045 & 285/4004 \\ 17/72072 & -3/1820 & 1877/360360 & -454/45045 & 1433/120120 & 157/3276 \\ -1/12012 & 43/72072 & -179/90090 & 529/120120 & -1583/180180 & 1709/72072 \end{pmatrix}$$

The approximate solution becomes $u(t) \approx -0.0040t^5 + 0.0460t^4 - 0.0023t^3 - 0.4994t^2 - 0.0001t + 1$.

Example 2: Consider the following Volterra integral equation of the second kind [23]:

$$u(t) = \sin(t) + \cos(t) + 2\int_0^t \sin(t-x)u(x)dx.$$

Here $f(t) = \sin(t) + \cos(t)$, $\lambda = 2$ and since $k(t, x) = \sin(t - x)$. Since $k(t, x) = \sin(t - x)$ so we take $k(t) = \sin(t)$. The Taylor series expansion of $k(t)$ and the truncated series are given, respectively, by

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \sin(x) \approx \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

We first let $m = 3$ and $N = 0$ so we use the 3^{rd} Bernstein polynomials to approximate the functions.

$$u(t) \approx \boldsymbol{\alpha}^T \boldsymbol{\Psi}_3(t) \text{ and } f(t) = \sin(t) + \cos(t) \approx \boldsymbol{\beta}^T \boldsymbol{\Psi}_3(t).$$

We obtain $\boldsymbol{\alpha}^T \approx [0.9970, 1.3497, 1.7865, 2.8215]$, $\boldsymbol{\beta}^T \approx [0.9992, 1.3377, 1.4868, 1.3810]$, and

$$F = \begin{pmatrix} -1/504 & 1/84 & 1/8 & 25/126 \\ 1/420 & -13/840 & 13/210 & 127/840 \\ 1/840 & -1/210 & 1/840 & 43/420 \\ -1/630 & 1/120 & -3/140 & 121/2520 \end{pmatrix}$$

The approximate solution becomes

$$u(t) \approx 0.5142t^3 + 0.2522t^2 + 1.0581t + 0.9970.$$

Let $m = 3$ and $N = 1$. We again assume that

$$u(t) \approx \boldsymbol{\alpha}^T \boldsymbol{\Psi}_3(t) \text{ and } f(t) = \sin(t) + \cos(t) \approx \boldsymbol{\beta}^T \boldsymbol{\Psi}_3(t).$$

We obtain
$\boldsymbol{\alpha}^T \approx [0.9992, 1.3377, 1.8202, 2.7142]$, $\boldsymbol{\beta}^T \approx [0.9992, 1.3377, 1.4868, 1.3810]$

$$F = \begin{pmatrix} -103/55440 & 23/2079 & 191/1485 & 269/1540 \\ 1/385 & -103/6160 & 911/13860 & 1933/13860 \\ 19/13860 & -79/13860 & 13/3696 & 113/1155 \\ -1/660 & 83/10395 & -43/2079 & 2603/55440 \end{pmatrix}$$

The approximate solution becomes

$$u(t) \approx 0.2674t^3 + 0.4321t^2 + 1.0154t + 0.9992.$$

We now let $m = 5$ and $N = 0$ so we use the 5^{th} Bernstein polynomials to approximate the functions. We let

$$u(t) \approx \boldsymbol{\alpha}^T \boldsymbol{\Psi}_5(t) \text{ and } f(t) = \sin(t) + \cos(t) \approx \boldsymbol{\beta}^T \boldsymbol{\Psi}_5(t).$$

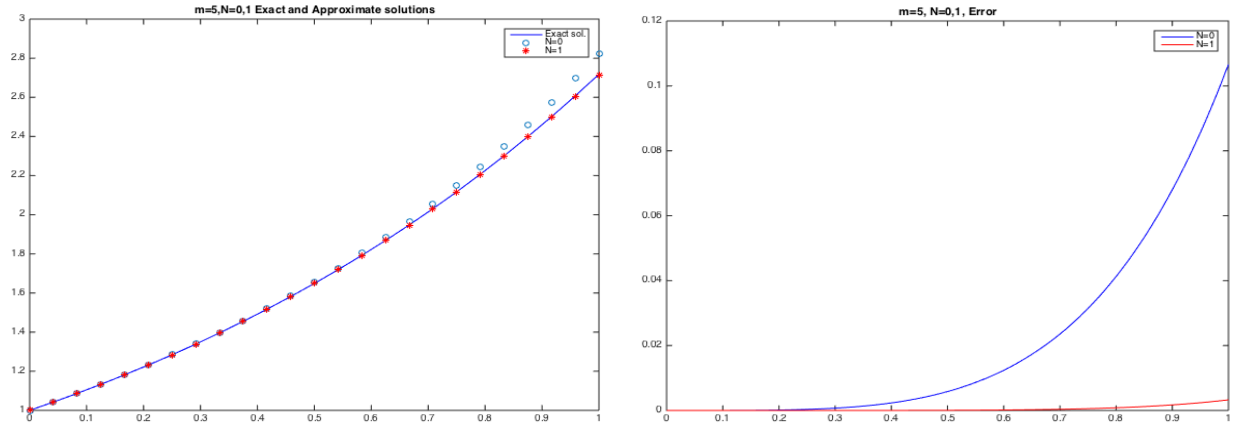

Figure 1: Exact and approximate solutions (left figure); Comparison of errors (right figure).

We obtain $\boldsymbol{\alpha}^T \approx [1, 1.2001, 1.4497, 1.7675, 2.1898, 2.8249]$, $\boldsymbol{\beta}^T \approx [1, 1.2, 1.3501, 1.4332, 1.4420, 1.3818]$, and

$$F = \begin{pmatrix} -1/10296 & 19/25740 & 811/17160 & 191/2574 & 5669/51480 & 245/1716 \\ 1/3276 & -283/120120 & 404/45045 & 20753/360360 & 153/1820 & 8597/72072 \\ -1/4004 & 92/45045 & -139/16380 & 829/30030 & 11267/180180 & 857/9009 \\ -1/9009 & 113/180180 & -29/30030 & -61/16380 & 1808/45045 & 285/4004 \\ 17/72072 & -3/1820 & 1877/360360 & -454/45045 & 1433/120120 & 157/3276 \\ -1/12012 & 43/72072 & -179/90090 & 529/120120 & -1583/180180 & 1709/72072 \end{pmatrix}$$

The approximate solution becomes $u(t) \approx 0.542t^5 + 0.0880t^4 + 0.1875t^3 + 0.4946t^2 + 1.0006t + 1$.
We now let $m = 5$ and $N = 1$ so we use the 5^{th} Bernstein polynomials to approximate the functions. We consider

$$u(t) \approx \boldsymbol{\alpha}^T \boldsymbol{\Psi}_5(t) \text{ and } f(t) = \sin(t) + \cos(t) \approx \boldsymbol{\beta}^T \boldsymbol{\Psi}_5(t).$$

We obtain $\boldsymbol{\alpha}^T \approx [1, 1.2, 1.4501, 1.765, 2.1754, 2.7150]$, $\boldsymbol{\beta}^T \approx [1, 1.2, 1.3501, 1.4332, 1.4420, 1.3818]$, and

$$F = \begin{pmatrix} -1/10920 & 25/36036 & 2137/45045 & 379/5148 & 577/5544 & 7463/60060 \\ 1/3276 & -11/4680 & 107/12012 & 1161/20020 & 4889/60060 & 1109/10296 \\ -37/144144 & 151/72072 & -481/55440 & 337/12012 & 4957/80080 & 6379/72072 \\ -25/216216 & 203/308880 & -113/108108 & -607/166320 & 8747/216216 & 29347/432432 \\ 103/432432 & -901/540540 & 1631/308880 & -2221/216216 & 2437/196560 & 229/4914 \\ -43/540540 & 19/33264 & -2063/1081080 & 9197/2162160 & -463/54054 & 4597/196560 \end{pmatrix}$$

The approximate solution becomes $u(t) \approx 0.0023t^5 + 0.0494t^4 + 0.1623t^3 + 0.5011t^2 + t + 1$.

5. Conclusion

In this article we give a simple and efficient method for solving Volterra integral equations with special type kernels using the Bernstein polynomials. We provide a simple formulation which depends both the power of the kernel and the order of Bernstein polynomials. We make use of this formula to find approximate solutions for the integral equations with convolution type kernels. Two examples with known exact solutions are considered and approximate solutions by using the Bernstein polynomials are found out. The graphs of exact and approximate solutions for example 2 are provided along with the error tables.

Table 1: Error table for m=5

x	Exact sol.	N=0 & App.sol.	N=1 & App.sol.	N=0 & Abs.error	N=1 & Abs.error
0	1.0	0.999986617462127	1.000002844157279	0.000013382537873	0.000002844157279
0.1	1.105170918075648	1.105184755739150	1.105169779818933	0.000013837663502	0.000001138256715
0.2	1.221402758160170	1.221539622730921	1.221403033251516	0.000136864570751	0.000000275091346
0.3	1.349858807576003	1.350574448272398	1.349857566625894	0.000715640696394	0.000001240950109
0.4	1.491824697641270	1.494153830383556	1.491812414952816	0.002329132742285	0.000012282688455
0.5	1.648721270700128	1.654548823255265	1.648673428670091	0.005827552555137	0.000047842030038
0.6	1.822118800390509	1.834502025235175	1.821976016229770	0.012383224844666	0.000142784160739
0.7	2.013752707470477	2.037292666813593	2.013387886685321	0.023539959343116	0.000364820785156
0.8	2.225540928492468	2.266801698609366	2.224711792278812	0.041260770116898	0.000829136213655
0.9	2.459603111156950	2.527576879355765	2.457888271028087	0.067973768198816	0.001714840128863
1	2.718281828459046	2.824897863886363	2.714998389313944	0.106616035427317	0.003283439145102

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