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Approximation Properties of Jain-Stancu Operators

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Abstract. In the present paper, we introduce the Stancu type Jain operators, which generalize the well-known Szász–Mirakyan operators via Lagrange expansion. We investigate their weighted approximation properties and compute the error of approximation by using the modulus of continuity. We also give an asymptotic expansion of Voronovskaya type. Finally, we introduce a modified form of our operators, which preserves linear functions, provides a better error estimation than the Jain operators and allows us to give global results in a certain subclass of $C[0,\infty)$. Note that the usual Jain operators do not preserve linear functions and the global results in a certain subspace of $C[0,\infty)$ can not be given for them.

1. Introduction

Positive linear operators are one of the main tools in Approximation Theory. These operators attract many researhers, since the three simple test functions $1, y$ and y^2 determine the convergence to a function on the whole space (see [2],[4],[5],[6],[7],[8],[10],[15],[19],[20],[21],[22], [23],[28] and [29]). One of the main representatives for approximating a function on $[0,\infty)$ are the Szász-Mirakjan operators.

As usual, let $C[0,\infty)$ denote the space of all continuous functions on $[0,\infty)$. The classical Szász-Mirakjan operators [24] are given by

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1}$$

where f belongs to an appropriate subspace of $C[0,\infty)$ for which this series converges.

In 1970, Jain [14] used the Lagrange expansion

$$e^{\alpha z} = \sum_{k=0}^{\infty}\alpha\left(\alpha+k\beta\right)^{k-1}\frac{u^k}{k!}, \quad u = ze^{-\beta z}, \left|\beta u\right| < e^{-1}, \left|\beta z\right| < 1$$

to introduce the operators

$$B_n^{\beta}(f,x) = \sum_{k=0}^{\infty} I_{n,k}^{(\beta)}(x) f\left(\frac{k}{n}\right), \ x\in[0,\infty),$$

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where

$$I_{n,k}^{(\beta)}(x) = \frac{nx(nx+k\beta)^{k-1}}{k!}e^{-\left(nx+k\beta\right)}.$$

Clearly $\sum_{k=0}^{\infty} I_{n,k}^{(\beta)}(x) = 1$ and the special case $\beta = 0$ reduces to the original Szasz-Mirakjan operators. For the convergence of these operators, the condition $\beta \to 0$ is needed. Thus throughout this paper, we take $\beta := \beta_n = n^{-1} (n \in \mathbb{N})$.

Farcas proved a Voronovskaja type asymptotic formula for the Jain operators [9]. However, as mentioned in [12], there are some minor errors, especially in Lemma 2.1, which affect the main result. Note that the authors state the corrected version of Lemma 2.1 in [12]. The Kantorovich variant of the Jain operators was studied by Umar and Razi [27]. Different Durrmeyer type modifications of the Jain operators were investigated by Tarabie [25], Agratini [1] and Gupta et al. [11] and [13]. Finally, very recently Olgun et al. investigated some approximation properties of ρ-type generalization of Jain's operator [18].

In the present investigation, we propose

$$B_n^{(n^{-1};\alpha,\gamma)}(f,x) = \sum_{k=0}^{\infty} I_{n,k}^{(n^{-1})}(x) f\left(\frac{k+\alpha}{n+\gamma}\right),\ x \in [0,\infty), \tag{2}$$

where $\alpha,\gamma \in [0,\infty)$ with $0 \le \alpha \le \gamma$.

In section 2, we prove a weighted Korovkin theorem by obtaining rate of approximation in terms of modulus of continuity. In section 3, we give a Voronovskaya type asymptotic formula.The main advantages of the operators are given in section 4. In this section we modify the operators given in (2). This modification has the following advantages comparing with usual Jain operators. The modified operators preserve linear functions, while the Jain operators do not. These operators provide a better error estimation than the Jain operators. Finally these operators allow us to give global results in a certain subclass of $C[0,\infty)$, while it is not possible to give such a result for the usual Jain operators.

Lemma 1.1. *For the first few moments of the operators $B_n^{n^{-1}}(f,x)$, we have (see [13] and [14])*

$$B_n^{n^{-1}}(1,x) = 1,\ B_n^{n^{-1}}(y,x) = \frac{nx}{n-1},$$

$$B_n^{n^{-1}}(y^2,x) = \frac{(nx)^2}{(n-1)^2} + \frac{n^2x}{(n-1)^3},$$

$$B_n^{n^{-1}}(y^3,x) = \frac{(nx)^3}{(n-1)^3} + \frac{3n^3x^2}{(n-1)^4} + \frac{x(n^3+2n^2)}{(n-1)^5},$$

$$B_n^{n^{-1}}(y^4,x) = \frac{(nx)^4}{(n-1)^4} + \frac{6n^4x^3}{(n-1)^5} + \frac{x^2(7n^4+8n^3)}{(n-1)^6} + \frac{x\left(n^4+8n^3+6n^2\right)}{(n-1)^7}.$$

Lemma 1.2. *For the moments of the operators $B_n^{(n^{-1};\alpha,\gamma)}(f,x)$, we have*

$$B_n^{(n^{-1};\alpha,\gamma)}(1,x) = 1,\ B_n^{(n^{-1};\alpha,\gamma)}(y,x) = \frac{n^2x}{(n-1)(n+\gamma)} + \frac{\alpha}{n+\gamma},$$

$$B_n^{(n^{-1};\alpha,\gamma)}(y^2,x) = \frac{n^4x^2}{(n+\gamma)^2(n-1)^2} + \frac{2\alpha n^2 x}{(n+\gamma)^2(n-1)} + \frac{n^4x}{(n+\gamma)^2(n-1)^3} + \left(\frac{\alpha}{n+\gamma}\right)^2,$$

$$B_n^{(n^{-1};\alpha,\gamma)}(y^3,x) = \left(\frac{\alpha}{n+\gamma}\right)^3 + \frac{3\alpha^2n^2x}{(n+\gamma)^3(n-1)} + \frac{3\alpha n^4x^2}{(n+\gamma)^3(n-1)^2} + \frac{3\alpha n^4x}{(n+\gamma)^3(n-1)^3}$$
$$+ \frac{n^6x^3}{(n+\gamma)^3(n-1)^3} + \frac{3n^6x^2}{(n+\gamma)^3(n-1)^4} + \frac{x(n^6+2n^5)}{(n+\gamma)^3(n-1)^5},$$

$$B_n^{(n^{-1};\alpha,\gamma)}(y^4,x)=\left(\frac{\alpha}{n+\gamma}\right)^4+\frac{4\alpha^3n^2x}{(n+\gamma)^4(n-1)}+\frac{6\alpha^2n^2}{(n+\gamma)^4}\left(\frac{(nx)^2}{(n-1)^2}+\frac{n^2x}{(n-1)^3}\right)$$
$$+\frac{4\alpha n^3}{(n+\gamma)^4}\left(\frac{(nx)^3}{(n-1)^3}+\frac{3n^3x^2}{(n-1)^4}+\frac{x(n^3+2n^2)}{(n-1)^5}\right)$$
$$+\frac{n^8x^4}{(n+\gamma)^4(n-1)^4}+\frac{6n^8x^3}{(n+\gamma)^4(n-1)^5}+\frac{x^2(7n^8+8n^7)}{(n+\gamma)^4(n-1)^6}+\frac{x\left(n^8+8n^7+6n^6\right)}{(n+\gamma)^4(n-1)^7}$$

and

$$B_n^{(n^{-1};\alpha,\gamma)}((y-x)^2,x)$$
$$=\frac{(n+\gamma-n\gamma)^2}{(n+\gamma)^2(n-1)^2}x^2+\left(\frac{2\alpha n^2}{(n+\gamma)^2(n-1)}+\frac{n^4}{(n+\gamma)^2(n-1)^3}-\frac{2\alpha}{n+\gamma}\right)x+\left(\frac{\alpha}{n+\gamma}\right)^2,$$

$$B_n^{(n^{-1};\alpha,\gamma)}((y-x)^4,x)$$
$$=\left(\frac{\alpha}{n+\gamma}\right)^4+\frac{4\alpha^3n^2x}{(n+\gamma)^4(n-1)}+\frac{6\alpha^2n^2}{(n+\gamma)^4}\left(\frac{(nx)^2}{(n-1)^2}+\frac{n^2x}{(n-1)^3}\right)$$
$$+\frac{4\alpha n^3}{(n+\gamma)^4}\left(\frac{(nx)^3}{(n-1)^3}+\frac{3n^3x^2}{(n-1)^4}+\frac{x(n^3+2n^2)}{(n-1)^5}\right)+\frac{n^8x^4}{(n+\gamma)^4(n-1)^4}$$
$$+\frac{6n^8x^3}{(n+\gamma)^4(n-1)^5}+\frac{x^2(7n^8+8n^7)}{(n+\gamma)^4(n-1)^6}+\frac{x\left(n^8+8n^7+6n^6\right)}{(n+\gamma)^4(n-1)^7}$$
$$-4x\left(\left(\frac{\alpha}{n+\gamma}\right)^3+\frac{3\alpha^2n^2x}{(n+\gamma)^3(n-1)}+\frac{3\alpha n^4x^2}{(n+\gamma)^3(n-1)^2}+\frac{3\alpha n^4x}{(n+\gamma)^3(n-1)^3}\right.$$
$$\left.+\frac{n^6x^3}{(n+\gamma)^3(n-1)^3}+\frac{3n^6x^2}{(n+\gamma)^3(n-1)^4}+\frac{x(n^6+2n^5)}{(n+\gamma)^3(n-1)^5}\right)$$
$$+6x^2\left(\frac{n^4x^2}{(n+\gamma)^2(n-1)^2}+\frac{2\alpha n^2x}{(n+\gamma)^2(n-1)}+\frac{n^4x}{(n+\gamma)^2(n-1)^3}+\left(\frac{\alpha}{n+\gamma}\right)^2\right)$$
$$-4x^3\left(\frac{n^2x}{(n-1)(n+\gamma)}+\frac{\alpha}{n+\gamma}\right)+x^4.$$

Proof. The proof follows from Lemma 1.1 and the linearity of the operator. □

2. Korovkin Type Theorem

Let $B_2[0,\infty)$ denotes the set of functions f satisfying the condition $\left|f(x)\right|\leq K_f(1+x^2)$, where $x\in[0,\infty)$ and K_f is a positive constant depending only on f. Let $C_2[0,\infty):=B_2[0,\infty)\cap C[0,\infty)$. In this section, we prove a Korovkin type theorem in the space

$$E:=\left\{f\in C_2[0,\infty):\exists\lim_{x\to\infty}\frac{\left|f(x)\right|}{1+x^2}<\infty\right\},$$

endowed with the norm $\left\|f\right\|_*:=\sup_{x\in[0,\infty)}\frac{\left|f(x)\right|}{1+x^2}$.

Let $A > 0$. The usual modulus of continuity of f on the closed interval $[0, A]$ is defined by

$$\omega_A(f,\delta) = \sup_{\substack{|t-x|\leq\delta \\ x,t\in[0,A]}} \left|f(t) - f(x)\right|.$$

It is well known that, for a function $f \in E$, $\lim_{\delta\to\infty} \omega_A(f,\delta) = 0$.

The next theorem gives the rate of convergence of the operators $B_n^{(n^{-1};\alpha,\gamma)}(f,x)$ to $f(x)$, for all $f \in E$.

Theorem 2.1. *Let $f \in E$ and let $\omega_{A+1}(f,\delta)$ $(A > 0)$ be its modulus of continuity on the finite interval $[0, A+1] \subset [0,\infty)$. Then,*

$$\left\|B_n^{(n^{-1};\alpha,\gamma)}(f,\cdot) - f\right\|_{C[0,A]} \leq N_f\left(1 + A^2\right)\delta_n^2 + 2\omega_{A+1}(f,\delta_n)$$

where $\delta_n = \left[\frac{(n+\gamma-n\gamma)^2}{(n+\gamma)^2(n-1)^2}A^2 + \left|\frac{2\alpha n^2}{(n+\gamma)^2(n-1)} + \frac{n^4}{(n+\gamma)^2(n-1)^3} - \frac{2\alpha}{n+\gamma}\right|A + \left(\frac{\alpha}{n+\gamma}\right)^2\right]^{1/2}$ *and N_f is a positive constant depending on f.*

Proof. Let $x \in [0, A]$ and $y \leq A+1$. It is clear that

$$\left|f(y) - f(x)\right| \leq \omega_{A+1}(f, \left|y-x\right|) \leq \left(1 + \frac{\left|y-x\right|}{\delta}\right)\omega_{A+1}(f,\delta) \tag{3}$$

where $\delta > 0$. On the other hand, for $x \in [0, A]$ and $y \geq A+1$, using the fact that $y - x \geq 1$, we have

$$\left|f(y) - f(x)\right| \leq K_f(1 + x^2 + y^2) \leq K_f(2 + 3x^2 + 2(y-x)^2) \leq N_f\left(1 + A^2\right)(y-x)^2 \tag{4}$$

where $N_f = 6K_f$. Combining (3) and (4), we get for all $x \in [0, A]$ and $y \geq 0$ that

$$\left|f(y) - f(x)\right| \leq N_f\left(1 + A^2\right)(y-x)^2 + \left(1 + \frac{\left|y-x\right|}{\delta}\right)\omega_{A+1}(f,\delta)$$

and therefore

$$\left|B_n^{(n^{-1};\alpha,\gamma)}(f;x) - f(x)\right| \leq N_f\left(1 + A^2\right)B_n^{(n^{-1};\alpha,\gamma)}((y-x)^2;x) + \left(1 + \frac{B_n^{(n^{-1};\alpha,\gamma)}(\left|y-x\right|;x)}{\delta}\right)\omega_{A+1}(f,\delta).$$

By Cauchy-Schwarz inequality, we obtain

$$\left|B_n^{(n^{-1};\alpha,\gamma)}(f;x) - f(x)\right| \leq N_f\left(1 + A^2\right)B_n^{(n^{-1};\alpha,\gamma)}((y-x)^2;x) + \left(1 + \frac{\left[B_n^{(n^{-1};\alpha,\gamma)}((y-x)^2;x)\right]^{1/2}}{\delta}\right)\omega_{A+1}(f,\delta).$$

Using Lemma 1.2 and then taking supremum over $[0, A]$ on both sides of the final inequality, the proof is completed. □

Corollary 2.2. *For all $f \in E$, the sequence $\left\{B_n^{(n^{-1};\alpha,\gamma)}(f;x)\right\}$ converges uniformly to f on $[0, A]$ $(A > 0)$.*

3. A Voronovskaya-Type Theorem

In this section, we prove a Voronovskaya-type theorem for the operators $B_n^{(n^{-1};\alpha,\gamma)}$ given by (2). We first need the following lemma.

Lemma 3.1. $\lim_{n\to\infty} n^2 B_n^{(n^{-1};\alpha,\gamma)}\left((y-x)^4, x\right) = 3x^2$ *uniformly with respect to* $x \in [0, A]$ *with* $A > 0$.

Proof. The proof follows from Lemma 1.2. □

Theorem 3.2. *For every* $f \in E$ *such that* $f', f'' \in E$,

$$\lim_{n\to\infty} n\left\{B_n^{(n^{-1};\alpha,\gamma)}(f;x) - f(x)\right\} = \left[\alpha - x(\gamma-1)\right] f'(x) + \frac{x}{2} f''(x).$$

uniformly with respect to $x \in [0, A]$, $(A > 0)$.

Proof. Let $f, f', f'' \in E$. Define

$$\Omega(y,x) = \begin{cases} \dfrac{f(y) - f(x) - (y-x)f'(x) - \frac{1}{2}(y-x)^2 f''(x)}{(y-x)^2}, & \text{if } y \neq x \\ 0, & \text{if } y = x. \end{cases}$$

Then by assumption $\Omega(x,x) = 0$ and the function $\Omega(\cdot, x)$ belongs to E. Hence, by Taylor's theorem

$$f(y) = f(x) + (y-x)f'(x) + \frac{(y-x)^2}{2} f''(x) + (y-x)^2 \Omega(y,x).$$

Applying the operators $B_n^{(n^{-1};\alpha,\gamma)}$ on both sides of this equality, we get

$$\begin{aligned} n\left\{B_n^{(n^{-1};\alpha,\gamma)}(f;x) - f(x)\right\} = {} & nB_n^{(n^{-1};\alpha,\gamma)}((y-x);x)f'(x) + nB_n^{(n^{-1};\alpha,\gamma)}((y-x)^2;x)\tfrac{f''(x)}{2} \\ & + n\, B_n^{(n^{-1};\alpha,\gamma)}((y-x)^2\Omega(y,x);x). \end{aligned} \tag{5}$$

By the Cauchy-Schwarz inequality, we get for the second term on the right-hand side of (5) that

$$n\left|B_n^{(n^{-1};\alpha,\gamma)}((y-x)^2\Omega(y,x);x)\right| \leq \left(n^2 B_n^{(n^{-1};\alpha,\gamma)}\left((y-x)^4, x\right)\right)^{\frac{1}{2}} \left(B_n^{(n^{-1};\alpha,\gamma)}(\Omega^2(y,x);x)\right)^{\frac{1}{2}}. \tag{6}$$

Now, observe that $\Omega^2(x,x) = 0$ and $\Omega^2(\cdot, x) \in E$. Therefore it follows from Corollary 2.2 that

$$\lim_{n\to\infty} B_n^{(n^{-1};\alpha,\gamma)}(\Omega^2(y,x);x) = \Omega^2(x,x) = 0$$

uniformly with respect to $x \in [0, A]$, $(A > 0)$. Now by Lemma 3.1, we see immediately that

$$\lim_{n\to\infty} n\, B_n^{(n^{-1};\alpha,\gamma)}((y-x)^2\Omega(y,x);x) = 0 \tag{7}$$

On the other hand, observe that

$$\lim_{n\to\infty} nB_n^{(n^{-1};\alpha,\gamma)}(y-x,x) = \lim_{n\to\infty} n\left(\frac{n^2 x}{(n-1)(n+\gamma)} + \frac{\alpha}{n+\gamma} - x\right) = \alpha - x(\gamma-1) \tag{8}$$

and

$$\lim_{n\to\infty} nB_n^{(n^{-1};\alpha,\gamma)}((y-x)^2,x)$$
$$= \lim_{n\to\infty} n\left(\frac{(n+\gamma-n\gamma)^2}{(n+\gamma)^2(n-1)^2}x^2+\left(\frac{2\alpha n^2}{(n+\gamma)^2(n-1)}+\frac{n^4}{(n+\gamma)^2(n-1)^3}-\frac{2\alpha}{n+\gamma}\right)x+\left(\frac{\alpha}{n+\gamma}\right)^2\right)$$
$$= x \tag{9}$$

uniformly with respect to $x \in [0,A]$. Then, taking the limit as $n \to \infty$ in (5) and using (7), (8) and (9), we have

$$\lim_{n\to\infty} n\left\{B_n^{(n^{-1};\alpha,\gamma)}(f;x)-f(x)\right\} = [\alpha - x(\gamma-1)]\, f'(x) + \frac{x}{2}f''(x).$$

uniformly with respect to $x \in [0,A]$. Hence, the proof is completed. □

4. Global Results for Modified operators

In this section, we choose $\alpha = 0$ and $\gamma := \gamma_n = \frac{n}{n-1}$ $(n = 2,3,\cdots)$ and modify our operators as

$$\mathcal{B}_n(f,x) := B_n^{(n^{-1};0,\gamma_n)}(f,x) = \sum_{k=0}^{\infty} I_{n,k}^{(n^{-1})}(x) f\left(\frac{k(n-1)}{n^2}\right)$$
$$= \sum_{k=0}^{\infty} \frac{nx(nx+k/n)^{k-1}}{k!} e^{-(nx+k/n)} f\left(\frac{k(n-1)}{n^2}\right),\ x \in [0,\infty),$$

where $n = 2,3,\cdots$. Then it is clear from Lemma 1.2 that the moments of the operators $\mathcal{B}_n(f,x)$ are as follows:

$$\mathcal{B}_n(1,x) = 1,$$
$$\mathcal{B}_n(y,x) = x$$
$$\mathcal{B}_n(y^2,x) = x^2 + \frac{x}{n-1}$$
$$\mathcal{B}_n((y-x)^2,x) = \frac{x}{n-1}. \tag{10}$$

Therefore, this modification preserves linear functions. For each fixed $x \in [0,\infty)$ and $n \in 2,3,4,...$ it is clear that

$$\mathcal{B}_n((y-x)^2,x) = \frac{x}{n-1} < \frac{n^2 x}{(n-1)^3} + \frac{x^2}{(n-1)^2} = \mathcal{B}_n^{n^{-1}}((y-x)^2,x).$$

This inequality shows that the modified operators provide a better error estimation than the usual Jain operators.
Totik [26], investigated the problem of determining the subclasses of continuous functions for which the operators L_n converge uniformly to f on the whole interval $[0,\infty)$ as $n \to \infty$. A similar problem was investigated by de la Cal and Cárcamo [3]. This problem was further studied by Mahmudov in [16] and [17]. In the next theorem, we investigate a similar problem for the operators $\mathcal{B}_n(f,x)$.

Theorem 4.1. *Let $f \in C_B[0,\infty)$ and let $f^*(y) = f(y^2)$ is uniformly continuous on $[0,\infty)$. Then, for all $x \in [0,\infty)$,*

$$\left|\mathcal{B}_n(f,x) - f(x)\right| \le 2\omega(f^*; \frac{1}{\sqrt{n-1}}).$$

Therefore, $\mathcal{B}_n(f,x)$ converges uniformly to f as $n \to \infty$, provided that f^* is uniformly continuous on $[0,\infty)$.

Proof. For $x = 0$, the statement is obvious. Now, let's prove the statement for $x > 0$. Using the definition of the modulus of continuity and its well known property

$$\omega(f;\delta_1\delta_2) \leq (1+\delta_1)\,\omega(f;\delta_2);\ \ \delta_1,\delta_2 \geq 0,$$

we get

$$\begin{aligned}\left|\mathcal{B}_n(f,x)-f(x)\right| &= \left|\mathcal{B}_n(f^*\circ\sqrt{y},x)-f^*\circ\sqrt{x}\right|\\ &\leq \sum_{k=0}^{\infty}\frac{nx(nx+k/n)^{k-1}}{k!}e^{-(nx+k/n)}\left|f^*\left(\frac{\sqrt{k(n-1)}}{n}\right)-f^*(\sqrt{x})\right|\\ &\leq \sum_{k=0}^{\infty}\frac{nx(nx+k/n)^{k-1}}{k!}e^{-(nx+k/n)}\omega(f^*;\left|\frac{\sqrt{k(n-1)}}{n}-\sqrt{x}\right|)\\ &\leq 2\omega(f^*;\mathcal{B}_n(\left|\sqrt{y}-\sqrt{x}\right|,x)).\end{aligned} \tag{11}$$

On the other hand, using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\mathcal{B}_n(\left|\sqrt{y}-\sqrt{x}\right|,x) &= \sum_{k=0}^{\infty}\frac{nx(nx+k/n)^{k-1}}{k!}e^{-(nx+k/n)}\left|\frac{\sqrt{k(n-1)}}{n}-\sqrt{x}\right|\\ &= \sum_{k=0}^{\infty}\frac{nx(nx+k/n)^{k-1}}{k!}e^{-(nx+k/n)}\frac{\left|\frac{k(n-1)}{n^2}-x\right|}{\frac{\sqrt{k(n-1)}}{n}+\sqrt{x}}\\ &\leq \frac{1}{\sqrt{x}}\sum_{k=0}^{\infty}\frac{nx(nx+k/n)^{k-1}}{k!}e^{-(nx+k/n)}\left|\frac{k(n-1)}{n^2}-x\right|\\ &\leq \frac{1}{\sqrt{x}}\sqrt{\sum_{k=0}^{\infty}\frac{nx(nx+k/n)^{k-1}}{k!}e^{-(nx+k/n)}\left(\frac{k(n-1)}{n^2}-x\right)^2}\\ &= \frac{1}{\sqrt{x}}\sqrt{\mathcal{B}_n((y-x)^2,x)}.\end{aligned} \tag{12}$$

The theorem follows from (10), (11) and (12). □

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