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A Duality Theorem for L-R Crossed Product

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Abstract. In this work, the notion of an L-R crossed product is introduced as a unified approach for L-R smash product and crossed product. Then the duality theorem for L-R crossed product is given. As the applications of the main result, some classical results in some materials can be obtained.

1. Introduction

Classical duality theorems origin in operator algebras, in works of Takesaki and colaborators for describing the duality between actions and coactions of locally compact groups on Von Neumann algebras ([1]). In [2], Cohen and Montgomery considered this duality for actions and coactions of groups on algebras and proved that, given a finite group G acting as linear automorphisms on A, there exists an isomorphism between the smash product $A*G\sharp k[G]^*$ of the skew group ring $A*G$ and the dual group ring $k[G]^* = Hom(kG, k)$ and the full matrix ring $M_n(A)$ over A. This kind of result is important, since coactions of group algebras correspond to group gradings on algebras. The extension of this duality theorem to the context of Hopf algebras was made in the work of Blattner and Montgomery (see [3]). As the generalization of Blattner-Mongomery's result, Koppinen prove the duality theorem for Hopf crossed product which generalized most of duality theorems in [5]. From the perspective of duality, Wang considered the duality theorems of both Hopf comodule coalgebras and crossed coproducts in [6, 7]. Recently, a great deal of work has been done on the duality theorem in [9–11] and [12].

Based on the theory of deformation, the L-R smash product was introduced and studied in [13, 14]. It is defined as follows: if H is a cocommutative bialgebra and A is an H-bimodule algebra, then the L-R smash product $A\sharp H$ is an associative algebra defined on $A \otimes H$ by the multiplication rule

$$(a\sharp h)(b\sharp g) = (a \cdot g_1)(h_1 \cdot b)\sharp h_2 g_2$$

for any $a, b \in A$ and $g, h \in H$. If we replace the above multiplication by

$$(a\sharp h)(b\sharp g) = (a \cdot g_2)(h_1 \cdot b)\sharp h_2 g_1,$$

then this multiplication is associative in [15] without the assumption that H is cocommutative. In [16], the authors introduced and studied the more general version of L-R smash products.

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Following the current trends of further research on this topic and at the angle of unity, the paper will present a general version of duality theorem for L-R crossed product which covers most of the classical product algebras such as smash products, crossed products and L-R smash products etc. It is the motivation of this paper.

The paper is organized as follows.

In Section 2, we recall some useful concepts. In Section 3, the conditions on cocycles are established in order to construct L-R crossed products. Then the duality theorem for L-R crossed product is given in Section 4. In Section 5, we apply our main result to some classical cases.

2. Preliminaries

Throughout the paper, we always work over a fixed field k and follow the Montgomery's book([17]) for terminologies on coalgebras, comodules and bialgebras. Given a vector space M, $\iota : M \to M$ denotes the identity map.

Recall that a left (right) measure of H on an algebra A is a linear map $H \otimes A \to A(A \otimes H \to A)$ given by $h \otimes a \mapsto h \cdot a(a \otimes h) = a \cdot h)$ such that, for any $h \in H, a, b \in A$,

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)(resp.(ab) \cdot h = (a \cdot h_1)(b \cdot h_2)),$$

$$h \cdot 1_A = \varepsilon_H(h)1_A,\ 1_H \cdot a = a\ (resp.\ 1_A \cdot h = \varepsilon_H(h)1_A,\ a \cdot 1_H = a).$$

Given a left (right) measure of H on A, if the measure is module action, then we can get the left (right)-module algebra. If an algebra A is both a left H-module algebra and a right H-module algebra with the compatible module actions , then A is called an H-bimodule algebra.

3. L-R Crossed Products

In this section, we shall introduce the notion of a L-R crossed product.

Assume that H measures on A from the left. Let A be a right H-module algebra with the compatibility with the left measure, and $\sigma : H \otimes H \to A$ a linear map. Define a multiplication on vector space $A \otimes H$ by

$$(a \otimes h)(b \otimes g) = (a \cdot g_3)(h_1 \cdot b)\sigma(h_2, g_1) \otimes h_3 g_2$$

for any $a, b \in A$ and $h, l \in H$.

Definition 3.1. *Let H be a Hopf algebra, A a right H-module algebra and $\sigma : H \otimes H \to A$ a linear map. We say that H is σ-*cocommutative*, if the following relation holds,*

$$\sigma(l, g) \cdot h_1 \otimes h_2 = \sigma(l, g) \cdot h_2 \otimes h_1$$

for all $l, g, h \in H$.

Remark 3.2. *If σ is trivial, i.e., $\sigma(h, g) = \varepsilon_H(h)\varepsilon_H(g)1_A$. Then H is σ-*cocommutative.

The following theorem gives the necessary and sufficient conditions under which $A \otimes H$ is associative and $A \otimes H$ is unital with $1_A \otimes 1_H$ as the identity element.

Theorem 3.3. *Assume that H measures on A from the left. Let A be a right H-module algebra with the compatibility with the left measure, and $\sigma : H \otimes H \to A$ a linear map such that H is σ-cocommutative. Then*

(i) $1_A \otimes 1_H$ is the unit of $A \otimes H$ if and only if, for all $a \in A$,

$$\sigma(h, 1_H) = \varepsilon_H(h)1_A = \sigma(1_H, h), \tag{3.1}$$

(ii) $A \otimes H$ is associative if and only if the following conditions hold:

$$(h_1 \cdot \sigma(l_1, m_1))\sigma(h_2, l_2 m_2) = (\sigma(h_1, l_1) \cdot m_1)\sigma(h_2 l_2, m_2), \tag{3.2}$$

$$(h_1 \cdot (l_1 \cdot a))\sigma(h_2, l_2) = \sigma(h_1, l_1)(h_2 l_2 \cdot a) \tag{3.3}$$

for any $h, l, m \in H$ and $a \in A$.

Proof. The proof of (i) is straightforward, so we omit it. Now, we shall check (ii). Suppose $A \otimes H$ is associative, we have

$$\begin{aligned}
&(1_A \otimes h)[(1_A \otimes l)(a \otimes m)] \\
&= (1_A \otimes h)[(l_1 \cdot a)\sigma(l_2, m_1) \otimes l_3 m_2] \\
&= (h_1 \cdot ((l_1 \cdot a)\sigma(l_2, m_1)))\sigma(h_2, l_3 m_2) \otimes h_3 l_4 m_3
\end{aligned}$$

and

$$\begin{aligned}
&[(1_A \otimes h)(1_A \otimes l)](a \otimes m) \\
&= (\sigma(h_1, l_1) \otimes h_2 l_2)(a \otimes m) \\
&= (\sigma(h_1, l_1) \cdot m_3)(h_2 l_2 \cdot a)\sigma(h_3 l_3, m_1) \otimes h_4 l_4 m_2.
\end{aligned}$$

So it follows that

$$\begin{aligned}
&(h_1 \cdot ((l_1 \cdot a)\sigma(l_2, m_1)))\sigma(h_2, l_3 m_2) \otimes h_3 l_4 m_3 \\
&\qquad = (\sigma(h_1, l_1) \cdot m_3)(h_2 l_2 \cdot a)\sigma(h_3 l_3, m_1) \otimes h_4 l_4 m_2.
\end{aligned}$$

Applying $\iota \otimes \varepsilon_H$ to both side of the above equality, we have

$$(h_1 \cdot ((l_1 \cdot a)\sigma(l_2, m_1)))\sigma(h_2, l_3 m_2) = (\sigma(h_1, l_1) \cdot m_2)(h_2 l_2 \cdot a)\sigma(h_3 l_3, m_1). \tag{3.4}$$

If we take $a = 1_A$ in (3.4) and use that H is *σ-cocommutative*, we get

$$(h_1 \cdot \sigma(l_1, m_1))\sigma(h_2, l_2 m_2) = (\sigma(h_1, l_1) \cdot m_1)\sigma(h_2 l_2, m_2). \tag{3.5}$$

If we take $m = 1_H$ in (3.4), it follows that

$$(h_1 \cdot (l_1 \cdot a))\sigma(h_2, l_2) = \sigma(h_1, l_1)(h_2 l_2 \cdot a). \tag{3.6}$$

Conversely, assume that (3.2) and (3.3) hold. First, we need the following equality

$$(h_1 \cdot (l_1 \cdot a))(\sigma(h_2, l_2) \cdot m) = (\sigma(h_1, l_1) \cdot m)(h_2 l_2 \cdot a)). \tag{3.7}$$

As a matter of fact, for all $h, l, m \in H$ and $a \in A$, we have

$$\begin{aligned}
(h_1 \cdot (l_1 \cdot a))(\sigma(h_2, l_2) \cdot m) &= ((h_1 \cdot (l_1 \cdot a \cdot s(m_1)))\sigma(h_2, l_2)) \cdot m_2 \\
&\overset{(3.3)}{=} (\sigma(h_1, l_1)(h_2 l_2 \cdot (a \cdot s(m_1)))) \cdot m_2 \\
&= (\sigma(h_1, l_1) \cdot m_2)(h_2 l_2 \cdot (a \cdot s(m_1) m_3)) \\
&= (\sigma(h_1, l_1) \cdot m_3)(h_2 l_2 \cdot (a \cdot s(m_1) m_2)) \\
&= (\sigma(h_1, l_1) \cdot m)(h_2 l_2 \cdot a).
\end{aligned}$$

Then, for all $a, b, c \in A$ and $h, l, m \in H$, we have

$$\begin{aligned}
&(a \otimes h)[(b \otimes l)(c \otimes m)] \\
&= (a \otimes h)[(b \cdot m_3)(l_1 \cdot c)\sigma(l_2, m_1) \otimes l_3 m_2] \\
&= (a \cdot l_5 m_4)(h_1 \cdot ((b \cdot m_5)(l_1 \cdot c)\sigma(l_2, m_1)))\sigma(h_2, l_3 m_2) \otimes h_3 l_4 m_3 \\
&= (a \cdot l_5 m_4)(h_1 \cdot (b \cdot m_5))(h_2 \cdot (l_1 \cdot c))(h_3 \cdot \sigma(l_2, m_1))\sigma(h_4, l_3 m_2) \otimes h_5 l_4 m_3 \\
&= (a \cdot l_5 m_4)(h_1 \cdot (b \cdot m_5))(h_2 \cdot (l_1 \cdot c))(\sigma(h_3, l_2) \cdot m_2)\sigma(h_4 l_3, m_1) \otimes h_5 l_4 m_3
\end{aligned}$$

$$\overset{(3.7)}{=} (a\cdot l_5m_4)((h_1\cdot b)\cdot m_5)(\sigma(h_2,l_1)\cdot \underbrace{m_2})(h_3l_2\cdot c)\sigma(h_4l_3,m_1)\otimes h_5l_4\underbrace{m_3}$$
$$= (a\cdot l_5\underbrace{m_4})((h_1\cdot b)\cdot m_5)(\sigma(h_2,l_1)\cdot \underbrace{m_3})(h_3l_2\cdot c)\sigma(h_4l_3,m_1)\otimes h_5l_4m_2$$
$$= (a\cdot l_5m_3)((h_1\cdot b)\cdot \underbrace{m_5})(\sigma(h_2,l_1)\cdot \underbrace{m_4})(h_3l_2\cdot c)\sigma(h_4l_3,m_1)\otimes h_5l_4m_2$$
$$= (a\cdot l_5m_3)((h_1\cdot b)\cdot m_4)(\sigma(h_2,l_1)\cdot m_5)(h_3l_2\cdot c)\sigma(h_4l_3,m_1)\otimes h_5l_4m_2$$
$$= (((a\cdot l_5)(h_1\cdot b)\sigma(h_2,l_1))\cdot m_3)(h_3l_2\cdot c)\sigma(h_4l_3,m_1)\otimes h_5l_4m_2$$
$$= ((a\cdot l_3)(h_1\cdot b)\sigma(h_2,l_1)\otimes h_3l_2)(c\otimes m)$$
$$= [(a\otimes h)(b\otimes l)](c\otimes m).$$

This ends the proof. □

We call the k-algebra $A\otimes H$ an L-R crossed product, denoted by $A\sharp_\sigma H$.

Example 3.4. *Consider the group algebra kZ with the obvious Hopf algebra structure and let g be a generator of Z in multiplication notation. Fix an element $0\neq q\in k$, and define a linear map $\sigma: kZ\otimes kZ\to kZ, g^i\otimes g^j\mapsto q^{ij}1$ and two actions on kZ:*

$$g^t \blacktriangleright g^l = q^{tl}g^l,\ g^t \blacktriangleleft g^l = q^{-tl}g^t.$$

Since

$$(g^t \blacktriangleright g^l)\blacktriangleleft g^k = q^{tl}g^l\blacktriangleleft g^k = q^{tl-kl}g^l$$

and

$$g^t \blacktriangleright (g^l\blacktriangleleft g^k) = q^{-lk}g^t\blacktriangleright g^l = q^{tl-lk}g^l,$$

it follows that $(kZ,\blacktriangleright,\blacktriangleleft)$ is kZ-bimodule. It is not hard to show that $(kZ,\blacktriangleright)$ is a left kZ-module algebra and $(kZ,\blacktriangleleft)$ is a right kZ-module algebra. Straightforward computation can show that σ is a cocycle and conditions (3.2) and (3.3) hold. Thus we have the L-R crossed product $kZ\sharp_\sigma kZ$ with the multiplication via

$$(g^m\sharp g^l)(g^n\sharp g^t) = q^{nl+lt-mt}g^{m+n}\otimes g^{l+t}.$$

Example 3.5. *Consider the polynomial algebra $k[X]$ with the coalgebra structure and the antipode given by*

$$\Delta(X^n) = \sum_{k=0}^{n} C_n^k X^k\otimes X^{n-k},\ \varepsilon(X^n)=0,\ S(X^n)=(-1)^nX^n,\ \forall n>0.$$

Fix an element $0\neq q\in k$, and define a linear map $\sigma: k[X]\otimes k[X]\to k[X]$ via

$$\sigma(X^i,X^j) = \begin{cases} 0, & \text{if } i\neq j;\\ i!q^i1, & \text{if } i=j.\end{cases}$$

Two actions of $k[X]$ on $k[X]$ are given by

$$X^i\blacktriangleright X^j = \begin{cases} 0, & \text{if } i>j;\\ \frac{j!}{(j-i)!}q^iX^{j-i}, & \text{if } i\leq j,\end{cases}\quad X^j\blacktriangleleft X^i = \begin{cases} 0, & \text{if } i>j;\\ (-1)^i\frac{j!}{(j-i)!}q^iX^{j-i}, & \text{if } i\leq j.\end{cases}$$

It is not hard to show that $(k[X],\blacktriangleright,\blacktriangleleft)$ is $k[X]$-bimodule, $(k[X],\blacktriangleright)$ is a left $k[X]$-module algebra and $(k[X],\blacktriangleleft)$ is a right $k[X]$-module algebra. Since

$$\sigma(X^i,1) = \begin{cases} 0, & \text{if } i\neq 0;\\ 1, & \text{if } i=0,\end{cases}$$

it follows that $\sigma(X^i,1)=\varepsilon(X^i)1$. Similarly, we can check that $\sigma(1,X^i)=\varepsilon(X^i)1$. Straightforward computation can show that the conditions (3.2) and (3.3) hold. Thus, we have another L-R crossed product $k[X]\sharp_\sigma k[X]$.

4. The Duality Theorem for L-R-Crossed product

Let A be a right H-module algebra. Assume that there exists a left measure of H on A such that H is σ-cocommutative. If H is a finite dimensional Hopf algebra, the dual vector space H^* has a natural structure of a Hopf algebra.

Now, we will construct the duality theorem for an L-R crossed product. First, we need some lemmas.

Lemma 4.1. *Let H be a finite dimensional Hopf algebra. Then $A\sharp_\sigma H$ is a left H^*-module algebra via*

$$f\cdot(a\sharp_\sigma h)=a\sharp_\sigma h_1f(h_2)$$

for any $a\in A$, $h\in H$ and $f\in H^*$.

Lemma 4.2. *The map*

$$\varphi:(A\sharp_\sigma H)\sharp H^*\to End(A\sharp_\sigma H)_A$$

(here $\sharp$ means smash product and $End(A\sharp_\sigma H)_A$ means the right A-module endomorphisms) defined by

$$\varphi((a\sharp_\sigma h)\sharp f)(b\sharp_\sigma g)=(a\sharp_\sigma h)(b\sharp_\sigma g_1)f(g_2)$$

for any $a,b\in A$, $h,g\in H$ and $f\in H^*$, is a homomorphism of algebras, where $A\sharp_\sigma H$ is a right A-module via

$$(a\sharp_\sigma h)\cdot b=(a\sharp_\sigma h)(b\sharp_\sigma 1_H).$$

Proof. First, we will show that φ commutes with the right action of A on $A\sharp_\sigma H$. Indeed, for any $a,b,d\in A$, $h,g\in H$ and $f\in H^*$, we compute

$$\begin{aligned}
&\varphi((a\sharp_\sigma h)\sharp f)((b\sharp_\sigma g)\cdot d)\\
&=\varphi((a\sharp_\sigma h)\sharp f)(b(g_1\cdot d)\sharp_\sigma g_2)\\
&=(a\cdot g_4)(h_1\cdot(b(g_1\cdot d)))\sigma(h_2,g_2)\sharp_\sigma h_3g_3f(g_5)\\
&=(a\cdot g_4)(h_1\cdot b)(h_2\cdot(g_1\cdot d))\sigma(h_3,g_2)\sharp_\sigma h_4g_3f(g_5)\\
&\overset{(3.3)}{=}(a\cdot g_4)(h_1\cdot b)\sigma(h_2,g_1)(h_3g_2\cdot d)\sharp_\sigma h_4g_3f(g_5)\\
&=((a\cdot g_3)(h_1\cdot b)\sigma(h_2,g_1)\sharp_\sigma h_3g_2f(g_4))\cdot d\\
&=(\varphi((a\sharp_\sigma h)\sharp f)(b\sharp_\sigma g))\cdot d.
\end{aligned}$$

Next, for all $a,b,x\in A$, $h,l,y\in H$ and $f,g\in H^*$, we have

$$\begin{aligned}
&\varphi((a\sharp_\sigma h)\sharp f)\circ\varphi((b\sharp_\sigma l)\sharp g)(x\sharp_\sigma y)\\
&=\varphi((a\sharp_\sigma h)\sharp f)((b\cdot y_3)(l_1\cdot x)\sigma(l_2,y_1)\sharp_\sigma l_3y_2)g(y_4)\\
&=(a\sharp_\sigma h)(b\cdot y_4)(l_1\cdot x)\sigma(l_2,y_1)\sharp_\sigma l_3y_2g(y_5)f(l_4y_3)\\
&=(a\cdot l_5y_4)(h_1\cdot((b\cdot y_6)(l_1\cdot x)\sigma(l_2,y_1)))\sigma(h_2,l_3y_2)\sharp_\sigma h_3l_4y_3g(y_7)f(l_6y_5)\\
&=(a\cdot l_5y_4)(h_1\cdot(b\cdot y_6))(h_2\cdot(l_1\cdot x))\underbrace{(h_3\cdot\sigma(l_2,y_1))\sigma(h_4,l_3y_2)}\sharp_\sigma h_5l_4y_3g(y_7)f(l_6y_5)\\
&\overset{(3.2)}{=}(a\cdot l_5y_4)(h_1\cdot(b\cdot y_6))\underbrace{(h_2\cdot(l_1\cdot x))(\sigma(h_3,l_2)\cdot y_2)}\sigma(h_4l_3,y_1)\sharp_\sigma h_5l_4y_3g(y_7)f(l_6y_5)\\
&\overset{(3.3)}{=}(a\cdot l_5y_4)(h_1\cdot(b\cdot y_6))(\sigma(h_2,l_1)\cdot y_2)(h_3l_2\cdot x)\sigma(h_4l_3,y_1)\sharp_\sigma h_5l_4y_3g(y_7)f(l_6y_5)\\
&=(a\cdot l_5y_4)(h_1\cdot(b\cdot y_5))(\sigma(h_2,l_1)\cdot y_2)(h_3l_2\cdot x)\sigma(h_4l_3,y_1)\sharp_\sigma h_5l_4y_3g(y_7)f(l_6y_6)\\
&=(a\cdot l_5y_4)(h_1\cdot(b\cdot y_5))(\sigma(h_2,l_1)\cdot y_3)(h_3l_2\cdot x)\sigma(h_4l_3,y_1)\sharp_\sigma h_5l_4y_2g(y_7)f(l_6y_6)\\
&=(a\cdot l_5y_3)(h_1\cdot(b\cdot y_5))(\sigma(h_2,l_1)\cdot y_4)(h_3l_2\cdot x)\sigma(h_4l_3,y_1)\sharp_\sigma h_5l_4y_2g(y_7)f(l_6y_6)\\
&=(a\cdot l_5y_3)((h_1\cdot b)\cdot y_4))(\sigma(h_2,l_1)\cdot y_5)(h_3l_2\cdot x)\sigma(h_4l_3,y_1)\sharp_\sigma h_5l_4y_2g(y_7)f(l_6y_6)\\
&=(((a\cdot l_5)(h_1\cdot b)(\sigma(h_2,l_1)))\cdot y_3)(h_3l_2\cdot x)\sigma(h_4l_3,y_1)\sharp_\sigma h_5l_4y_2g(y_5)f(l_6y_4)\\
&=\varphi(((a\sharp_\sigma h)\sharp f)(b\sharp_\sigma l)\sharp g)(x\sharp_\sigma y).
\end{aligned}$$

This ends the proof. □

Lemma 4.3. *Let H be a finite dimensional Hopf algebra and $A\sharp_\sigma H$ be the L-R crossed product with convolution inverse σ. Then*

$$(\sigma^{-1}(h_1,l_1)\cdot m)(h_2\cdot(l_2\cdot a)) = (h_1l_1\cdot a)(\sigma^{-1}(h_2,l_2)\cdot m), \tag{4.1}$$

$$\sigma^{-1}(l,g)\cdot h_1\otimes h_2 = \sigma^{-1}(l,g)\cdot h_2\otimes h_1,$$

$$\sigma(h_1l_1,m_1)\sigma^{-1}(h_2,l_2m_2) = (\sigma^{-1}(h_1,l_1)\cdot m_1)(h_2\cdot\sigma(l_2,m_2)). \tag{4.2}$$

Proof. Here we only check that (4.2) holds. Multiplying convolutively on the right of (3.2) by σ^{-1}, we have

$$(h_1\cdot\sigma(l_1,m_1))\sigma(h_2,l_2m_2)\sigma^{-1}(h_3,l_3m_3) = (\sigma(h_1,l_1)\cdot m_1)\sigma(h_2l_2,m_2)\sigma^{-1}(h_3,l_3m_3).$$

This gives

$$h\cdot\sigma(l,m) = (\sigma(h_1,l_1)\cdot m_1)\sigma(h_2l_2,m_2)\sigma^{-1}(h_3,l_3m_3). \tag{4.3}$$

Since

$$\begin{aligned}
&(\sigma^{-1}(h_1,l_1)\cdot m_1)(h_2\cdot\sigma(l_2,m_2))\\
&\overset{(4.3)}{=} (\sigma^{-1}(h_1,l_1)\cdot m_1)(\sigma(h_2,l_2)\cdot m_2)\sigma(h_3l_3,m_3)\sigma^{-1}(h_4,l_4m_4)\\
&= \sigma(h_1l_1,m_1)\sigma^{-1}(h_2,l_2m_2),
\end{aligned}$$

it follows that (4.2) holds. □

Let $\{e_i\}$ be a basis of H and $\{e_i^*\}$ be the dual basis of H^*, i.e., such that $e_i^*(e_j)=\delta_{ij}$ for all i, j. Then we have the following identities:

$$\sum_i e_i^*(h)e_i = h,\ \sum_i e_i^* f(e_i) = f$$

for all $h\in H$ and $f\in H^*$.

Lemma 4.4. *Let H be a finite dimensional Hopf algebra and $A\sharp_\sigma H$ be the L-R crossed product with convolution inverse σ. Define a linear map*

$$\psi : End(A\sharp_\sigma H)_A \to (A\sharp_\sigma H)\sharp H^*$$

by

$$\psi : T\mapsto \sum_i (T(\sigma^{-1}(e_{i4},S^{-1}(e_{i3}))\cdot e_{i2}\sharp_\sigma e_{i5})(1_A\sharp_\sigma S^{-1}(e_{i1})))\sharp e_i^*.$$

Then the maps φ and ψ are inverse of each other.

Proof. We need to check that

$$\varphi\circ\psi=\iota,\ \psi\circ\varphi=\iota.$$

For all $a\in A$, $h\in H$ and $f\in H^*$, we have

$$\begin{aligned}
&\psi\circ\varphi((a\sharp_\sigma h)\sharp f)\\
&= \sum_i [(a\sharp_\sigma h)(\sigma^{-1}(e_{i4},S^{-1}(e_{i3}))\cdot e_{i2}\sharp_\sigma e_{i5})(1_A\sharp_\sigma S^{-1}(e_{i1}))]\sharp e_i^* f(e_{i6})\\
&= \sum_i [(a\sharp_\sigma h)((\sigma^{-1}(e_{i6},S^{-1}(e_{i5}))\cdot e_{i4}S^{-1}(e_{i1}))\sigma(e_{i7},S^{-1}(e_{i3})))\sharp_\sigma e_{i8}S^{-1}(e_{i2})]\sharp e_i^* f(e_{i9})\\
&= \sum_i [(a\cdot e_{i12}S^{-1}(e_{i2}))(h_1\cdot\underbrace{(\sigma^{-1}(e_{i8},S^{-1}(e_{i7}))\cdot e_{i6}}S^{-1}(e_{i1})\sigma(e_{i9},\underbrace{S^{-1}(e_{i5})})))\\
&\quad \sigma(h_2,e_{i10}S^{-1}(e_{i4}))\sharp_\sigma h_3e_{i11}S^{-1}(e_{i3})]\sharp e_i^* f(e_{i13})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i [(a \cdot e_{i12} S^{-1}(e_{i2}))(h_1 \cdot (\underbrace{(\sigma^{-1}(e_{i8}, S^{-1}(e_{i7})) \cdot e_{i5}} S^{-1}(e_{i1})\sigma(e_{i9}, S^{-1}(e_{i6}))))\\
&\quad \sigma(h_2, e_{i10} \underbrace{S^{-1}(e_{i4})})\sharp_\sigma h_3 e_{i11} S^{-1}(e_{i3})]\sharp e_i^* f(e_{i13})\\
&= \sum_i [(a \cdot e_{i12} S^{-1}(e_{i2}))(h_1 \cdot (\sigma^{-1} \underbrace{(e_{i8}, S^{-1}(e_{i7})) \cdot e_{i4}} S^{-1}(e_{i1})\sigma(e_{i9}, S^{-1}(e_{i6})))\\
&\quad \sigma(h_2, e_{i10} S^{-1}(e_{i5}))\sharp_\sigma h_3 e_{i11} \underbrace{S^{-1}(e_{i3})]} \sharp e_i^* f(e_{i13})\\
&= \sum_i [(\underbrace{a \cdot e_{i12} S^{-1}(e_{i2})})(h_1 \cdot (\sigma^{-1} \underbrace{(e_{i8}, S^{-1}(e_{i7})) \cdot e_{i3}} S^{-1}(e_{i1})\sigma(e_{i9}, S^{-1}(e_{i6}))))\\
&\quad \sigma(h_2, e_{i10} S^{-1}(e_{i5}))\sharp_\sigma h_3 e_{i11} S^{-1}(e_{i4})]\sharp e_i^* f(e_{i13})\\
&= \sum_i [(a \cdot e_{i12} S^{-1}(e_{i3}))(h_1 \cdot (\sigma^{-1}(e_{i8}, S^{-1}(e_{i7})) \cdot \underbrace{e_{i2} S^{-1}(e_{i1})} \sigma(e_{i9}, S^{-1}(e_{i6}))))\\
&\quad \sigma(h_2, e_{i10} S^{-1}(e_{i5}))\sharp_\sigma h_3 e_{i11} S^{-1}(e_{i4})]\sharp e_i^* f(e_{i13})\\
&= \sum_i [(a \cdot e_{i10} S^{-1}(e_{i1}))(h_1 \cdot \underbrace{(\sigma^{-1}(e_{i6}, S^{-1}(e_{i5}))\sigma(e_{i7}, S^{-1}(e_{i4})))}))\\
&\quad \sigma(h_2, e_{i8} S^{-1}(e_{i3}))\sharp_\sigma h_3 e_{i9} S^{-1}(e_{i2})]\sharp e_i^* f(e_{i11})\\
&= \sum_i [(a \cdot e_{i10} S^{-1}(e_{i1}))(h_1 \cdot (\sigma^{-1}(e_{i6}, S^{-1}(e_{i5}))\sigma(e_{i7}, S^{-1}(e_{i4}))))\\
&\quad \sigma(h_2, e_{i8} S^{-1}(e_{i3}))\sharp_\sigma h_3 e_{i9} S^{-1}(e_{i2})]\sharp e_i^* f(e_{i11})\\
&= \sum_i ((a \cdot e_{i6} S^{-1}(e_{i1}))\sigma(h_1, \underbrace{e_{i4} S^{-1}(e_{i3})})\sharp_\sigma h_2 e_{i5} S^{-1}(e_{i2}))\sharp e_i^* f(e_{i7})\\
&= \sum_i ((a \cdot \underbrace{e_{i4} S^{-1}(e_{i1})})\sharp_\sigma h \underbrace{e_{i3} S^{-1}(e_{i2})})\sharp e_i^* f(e_{i5})\\
&= \sum_i (a\sharp_\sigma h)\sharp e_i^* f(e_i) = (a\sharp_\sigma h)\sharp f.
\end{aligned}
$$

So we get $\psi \circ \varphi = \iota$. As to $\varphi \circ \psi = \iota$, we proceed the proof as follows:

$$
\begin{aligned}
&\varphi \circ \psi(T)(a\sharp_\sigma h)\\
&= \sum_i \varphi((T(\sigma^{-1}(e_{i4}, S^{-1}(e_{i3}) \cdot e_{i2}\sharp_\sigma e_{i5}))(1_A \sharp_\sigma S^{-1}(e_{i1})))\sharp e_i^*)(a\sharp_\sigma h)\\
&= \sum_i T(\sigma^{-1}(e_{i4}, S^{-1}(e_{i3})) \cdot e_{i2}\sharp_\sigma e_{i5})(1_A\sharp_\sigma S^{-1}(e_{i1}))(a\sharp_\sigma h_1)e_i^*(h_2)\\
&= T(\sigma^{-1}(h_8, S^{-1}(h_7)) \cdot h_6\sharp_\sigma h_9)((S^{-1}(h_5) \cdot a)\sigma(S^{-1}(h_4), h_1)\sharp_\sigma \underbrace{S^{-1}(h_3)h_2)}\\
&= T(\sigma^{-1}(h_6, S^{-1}(h_5)) \cdot h_4\sharp_\sigma h_7)((S^{-1}(h_3) \cdot a)\sigma(S^{-1}(h_2), h_1)\sharp_\sigma 1_H)\\
&= T((\sigma^{-1}(h_6, S^{-1}(h_5)) \cdot h_4\sharp_\sigma h_7)((S^{-1}(h_3) \cdot a)\sigma(S^{-1}(h_2), h_1)\sharp_\sigma 1_H))\\
&= T((\sigma^{-1} \underbrace{(h_6, S^{-1}(h_5)) \cdot h_4})(h_7 \cdot (\underbrace{S^{-1}(h_3)} \cdot a))(h_8 \cdot \sigma(S^{-1}(h_2), h_1))\sharp_\sigma h_8))\\
&= T((\sigma^{-1}(h_6, S^{-1}(h_5)) \cdot h_3)(h_7 \cdot (S^{-1}(h_4) \cdot a))(h_8 \cdot \sigma(S^{-1}(h_2), h_1))\sharp_\sigma h_8))\\
&\overset{(4.1)}{=} T((\underbrace{h_6 S^{-1}(h_5)} \cdot a)(\sigma^{-1}(h_7 \cdot S^{-1}(h_4)) \cdot h_3))(h_8 \cdot \sigma(S^{-1}(h_2), h_1))\sharp_\sigma h_9))\\
&= T(a(\sigma^{-1}(h_5 \cdot S^{-1}(h_4)) \cdot h_3))(h_6 \cdot \sigma(S^{-1}(h_2), h_1))\sharp_\sigma h_7))
\end{aligned}
$$

$$= T(a(\sigma^{-1}(h_5 \cdot S^{-1}(h_4)) \cdot h_2))(h_6 \cdot \sigma(S^{-1}(h_3), h_1))\sharp_\sigma h_7))$$

$$\overset{(4.2)}{=} T(a\sigma(\underbrace{h_5 S^{-1}(h_4)}, h_1)\sigma^{-1}(\underbrace{h_6 S^{-1}(h_3)}, h_2)\sharp_\sigma \underbrace{\ h_7\ })$$

$$= T(a\sharp_\sigma h).$$

This proof is completed. □

From the lemmas above, we can get the following main result in this section.

Theorem 4.5. *Let H be a finite dimensional Hopf algebra and $A\sharp_\sigma H$ be the L-R crossed product with convolution inverse σ such that H is σ-cocommutative. Then there is a canonical isomorphism between the algebras $(A\sharp_\sigma H)\sharp H^*$ and $End(A\sharp_\sigma H)_A$.*

5. Applications

In this section, we shall give some applications of Theorem 4.5, some classical results in several materials can be obtained.

5.1. Crossed Products

If the right H-module action of A is trivial, that is, $a \cdot h = a\varepsilon_H(t)$ for any $a \in A$ and $h \in H$, then A is an H-bimodule and (3.2) holds, and $A\sharp_\sigma H$ recovers to the usual crossed product in sense of [4]. From Theorem 4.5, we have

Corollary 5.1. *([5]) Let H be a finite dimensional Hopf algebra and $A\sharp_\sigma H$ be the usual crossed product with convolution inverse σ. Then there is a canonical isomorphism between the algebras $(A\sharp_\sigma H)\sharp H^*$ and $End(A\sharp_\sigma H)_A$.*

5.2. L-R Smash Products

If σ is trivial, that is, $\sigma(h, g) = \varepsilon_H(h)\varepsilon_H(g)1_A$, then $A\sharp_\sigma H$ reduces to the usual L-R smash product. From Theorem 4.5, we have

Corollary 5.2. *([12]) Let H be a finite dimensional Hopf algebra and $A\sharp H$ be the usual L-R smash product. Then there is a canonical isomorphism between the algebras $(A\sharp H)\sharp H^*$ and $End(A\sharp H)_A$.*

Furthermore, if the right H-module action of A is trivial, then L-R smash product $A\sharp H$ is exactly the usual smash product. From Corollary 5.2, we have

Corollary 5.3. *([3]) Let H be a finite dimensional Hopf algebra and $A\sharp H$ be the usual smash product. Then there is a canonical isomorphism between the algebras $(A\sharp H)\sharp H^*$ and $End(A\sharp H)$.*

References

[1] Y. Nakagami, M. Takesaki: Duality for crossed products of Von Neumann algebras, Lecture Notes in Math. 731, Springer Verlag (1979).
[2] M. Cohen, S. Montgomery: Group-graded rings, smash products, and group actions, Trans. Amer. Math. Soc., 282(1984), 237-258.
[3] R. J. Blattner, S. Montgomery: A duality theorem for Hopf module algebras, J. Algebra, 95(1985), 153-172
[4] R. Blattner, M. Cohen, S. Montgomery: Crossed products and inner actions of Hopf algebras, Trans. Amer. Math. Soc., 298(1986), 671-711.
[5] M. Koppinen: A duality theorem for crossed products of Hopf algebras, J. Algebra, 146(1992), 153-174.
[6] S. H. Wang: A duality theorem for Hopf comodule coalgebra, Chinese Science Bulletin, 39(1994),1239-1239
[7] S. H. Wang: A duality theorem for crossed coproduct for Hopf algebras, Science in China, 38(1995), 1-7.
[8] C. Lomp: Duality for Partial Group Actions, Int. Electron. J. Algebra, 4(2008), 53-62.
[9] B. L. Shen: Maschke-type theorem, Duality theorem, and the global dimension for weak crossed products, Comm. Algebra, 40(2012), 1802-1820.
[10] B. L. Shen, S. H. Wang: On group crossed coproduct, Int. Electron. J. Algebra, 4(2008), 177-188.
[11] B. L. Shen, S. H. Wang: Blattner-Cohen-Montgomery's Duality Theorem for (Weak) Group Smash Products, Comm. Algebra, Comm. Algebra, 36(2008), 2387-2409.

[12] X. Y. Zhou, Q. Li, L. Y. Zhang: Duality theorem for weak L-R smash products, Appl. Math. J. Chinese Univ., 25(2010), 481-48.
[13] P. Bonneau, M. Gerstenhaber, A. Giaquinto, D. Sternheimer, Quantum groups and deformation quantization: explicit approaches and implicit aspects, J. Math. Phys., 45(2004), 3703-3741.
[14] P. Bonneau, D. Sternheimer, Topological Hopf algebras, quantum groups and deformation quantization. In: Hopf Algebras in Noncommutative Geometry and Physics. In: Lecture Notes in Pure and Appl. Math. , Vol. 239, New York: Marcel Dekker, (2005) pp.55-70.
[15] L. Y. Zhang: L-R smash products for bimodule algebras, Prog. Nat. Sci., 16(2006), 580-587.
[16] F. Panaite, F. Van Oystaeyen: L-R-smash product for (quasi-)Hopf algebras. J. Algebra, 309(2007), 168-191.
[17] S. Montgomery: Hopf algebras and their actions on rings. CBMS, Lect. Notes, 1993.
[18] A. L. Agore: Coquasitriangular structures for extensions of Hopf algebras. Applications. Glasgow Math. J., 55(2013), 201-215.