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Characterization of Some Classes of Compact Operators between Certain Matrix Domains of Triangles

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Abstract. In this paper, we characterize the classes $((\ell_1)_T, (\ell_1)_{\tilde{T}})$ and $(c_T, c_{\tilde{T}})$ where $T = (t_{nk})_{n,k=0}^{\infty}$ and $\tilde{T} = (\tilde{t}_{nk})_{n,k=0}^{\infty}$ are arbitrary triangles. We establish identities or estimates for the Hausdorff measure of noncompactness of operators given by matrices in the classes $((\ell_1)_T, (\ell_1)_{\tilde{T}})$ and $(c_T, c_{\tilde{T}})$. Furthermore we give sufficient conditions for such matrix operators to be Fredholm operators on $(\ell_1)_T$ and c_T. As an application of our results, we consider the class (bv, bv) and the corresponding classes of matrix operators. Our results are complementary to those in [2] and some of them are generalization for those in [3].

1. Introduction and Notation

As usual, let ω, ϕ, c and c_0 denote the sets of all complex, finite, convergent and null sequences $x = (x_k)_{k=0}^{\infty}$, respectively, and $\ell_1 = \{x \in \omega : \sum_{k=0}^{\infty} |x_k| < \infty\}$ be the set of all absolutely convergent series. We write e and $e^{(n)}$ $(n = 0, 1, \dots)$ for the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$, respectively.

The β–dual of a subset X of ω is the set $X^\beta = \{a \in \omega : \sum_{k=0}^{\infty} a_k x_k \text{ converges for all } x \in X\}$.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers, X and Y be subsets of ω and $x \in \omega$. We write $A_n = (a_{nk})_{k=0}^{\infty}$ for the sequence in the n–th row of A, $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ and $Ax = (A_n x)_{n=0}^{\infty}$ (provided all the series $A_n x$ converge). The set $X_A = \{x \in \omega : Ax \in X\}$ is called the matrix domain of A in X. Also (X, Y) is the class of all matrices A such that $X \subset Y_A$; so $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all n and $Ax \in Y$ for all $x \in X$.

A Banach space $X \subset \omega$ is a BK space if each projection $x \mapsto x_n$ on the n–th coordinate is continuous. A BK space $X \supset \phi$ is said to have AK if $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \to x$ $(m \to \infty)$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$.

If $X \supset \phi$ is a BK space and $a \in \omega$ we write

$$\|a\|_X^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : \|x\| = 1\right\}$$

provided the expression on the righthand side is defined and finite which is the case whenever $a \in X^\beta$ ([12, Theorem 7.2.9]).

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If X and Y are Banach spaces, then we write, as usual, $\mathcal{B}(X, Y)$ for the set of all bounded linear operators $L : X \to Y$ with the operator norm $\|\cdot\|$ defined by $\|L\| = \sup_{\|x\|=1}\{\|L(x)\|\}$.

The following result is very important for our research.

Lemma 1.1. *Let X and Y be BK spaces.*
(a) Then we have $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ where $L_A(x) = Ax$ for all $x \in X$ ([5, Theorem 1.23] or [12, Theorem 4.2.8]).
(b) If X has AK then we have $\mathcal{B}(X, Y) \subset (X, Y)$, that is, every $L \in \mathcal{B}(X, Y)$ is given by a matrix $A \in (X, Y)$ such that $Ax = L(x)$ for all $x \in X$ ([4, Theorem 1.9.]).

In this paper we consider the classes $((\ell_1)_T, (\ell_1)_{\tilde{T}})$ and $(c_T, c_{\tilde{T}})$ where $T = (t_{nk})_{n,k=0}^{\infty}$ and $\tilde{T} = (\tilde{t}_{nk})_{n,k=0}^{\infty}$ are triangles. A matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is said to be a triangle if $t_{nk} = 0$ for all $k > n$ and $t_{nn} \neq 0$ $(n = 0, 1 \ldots)$. Throughout, let T denote a triangle, S its inverse and $R = S^t$, the transpose of S. We remark that the inverse of a triangle exists, is unique and a triangle ([12, 1.4.8, p. 9] and [9, Remark 22 (a), p. 22]).

In [2], the authors considered the space $(c_0)_T = \{x \in \omega : Tx \in c_0\}$, generalized some results on matrix transformations and compact operators on $(c_0)_T$, and finally gave a sufficient condition for a linear operator on $(c_0)_T$ defined by an infinite matrix to be a Fredholm operator.

Since many recently defined sequence spaces arise from the concept of matrix domains of triangles in classical sequence spaces, we classify the following classes: $((c_0)_T, (c_0)_{\tilde{T}})$, $(c_T, c_{\tilde{T}})$, $((\ell_1)_T, (\ell_1)_{\tilde{T}})$. The class $((c_0)_T, (c_0)_{\tilde{T}})$ was the subject of research in [2].

Hence, it remains to consider the classes $(c_T, c_{\tilde{T}})$ and $((\ell_1)_T, (\ell_1)_{\tilde{T}})$. In this way we extend existing results. Our results are complementary to those in [2] and some of them are generalization for those in [3].

As in [2] and [3], this will be achieved in three steps. First we will characterize the classes $(c_T, c_{\tilde{T}})$ and $((\ell_1)_T, (\ell_1)_{\tilde{T}})$, then establish identities or inequalities for the Hausdorff measure of noncompactness of the corresponding matrix operators, and give necessary and sufficient conditions for these operators to be compact, and finally, establish sufficient conditions for an infinite matrix to be a Fredholm operator on c_T and $(\ell_1)_T$.

2. Matrix Transformations

In this section, we characterize the classes $(c_T, c_{\tilde{T}})$ and $((\ell_1)_T, (\ell_1)_{\tilde{T}})$. These characterizations will be reduced to the well–known characterizations of matrix transformations between the classical sequence spaces ([11, 12]).

The following results play an important role in our research.

Lemma 2.1. ([5, Theorem 1.23]) *Let X be a BK space and $A \in (X, \ell_\infty)$. Then we have*

$$\|L_A\| = \sup_n \|A_n\|^*.$$

Lemma 2.2. ([5, Theorem 3.8]) *Let T be triangle.*
(a) Then, for arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.
(b) If X and Y are BK spaces and $A \in (X, Y_T)$, then $\|L_A\| = \|L_B\|$.

Now we establish a slightly improved version of [6, Theorem 3.4].

Lemma 2.3. *Let X be a BK space with AK, Y be an arbitrary subset of ω and $R = S^t$. Then $A \in (X_T, Y)$ if and only if $\hat{A} \in (X, Y)$ and $W^{(A_n)} \in (X, \ell_\infty)$ for all $n = 0, 1, \ldots$, where $\hat{A}$ is the matrix with the rows $\hat{A}_n = RA_n$ for $(n = 0, 1, \ldots)$ and the triangles $W^{(A_n)}$ $(n = 0, 1, \ldots)$ are defined by*

$$w_{mk}^{(A_n)} = \begin{cases} \sum_{j=m}^{\infty} a_{nj}s_{jk} & (0 \leq k \leq m) \\ 0 & (k > m) \end{cases} \quad (m = 0, 1, \ldots). \tag{2.1}$$

Moreover, if $A \in (X_T, Y)$ then we have

$$Az = \hat{A}(Tz) \text{ for all } z \in Z = X_T. \tag{2.2}$$

Proof. First we assume $A \in (X_T, Y)$. Then it follows from [6, Theorem 3.4] that $\hat{A} \in (X, Y)$ and $W^{(A_n)} \in (X, c_0)$ for all N, and so $W^{(A_n)} \in (X, \ell_\infty)$.
Conversely we assume that $\hat{A} \in (X, Y)$ and $W^{(A_n)} \in (X, \ell_\infty)$ for all n. Then the series $\hat{a}_{nk} = \sum_{j=k}^{\infty} a_{nj}s_{jk}$ converge for each k and n, and so

$$\lim_{m\to\infty} \sum_{j=m}^{\infty} a_{nj}s_{jk} = \lim_{m\to\infty} w_{mk}^{(A_n)} = 0 \text{ for each } n \text{ and each fixed } k,$$

and this and $W^{(A_n)} \in (X, \ell_\infty)$ together imply $W^{(A_n)} \in (X, c_0)$ by [12, 8.3.6], since X is an FK space with AK and c_0 is a closed subspace of ℓ_∞. Now it follows from [6, Theorem 3.4] that $A \in (X_T, Y)$.
The last part is now obvious. □

We also need the next result.

Lemma 2.4. ([6, Theorem 3.6]) *Let X and Y be BK spaces and X have AK. If $A \in (X_T, Y)$ then we have*

$$\|L_A\| = \|L_{\hat{A}}\| \tag{2.3}$$

where $\hat{A}$ is the matrix defined in Lemma 2.3.

Before characterizing the class $((\ell_1)_T, (\ell_1)_{\tilde{T}})$, it is useful to observe the following.

Remark 2.5. *Let X be a BK space with AK and Y be an arbitrary subset of ω. Applying Lemma 2.2 (a) first and then Lemma 2.3, we obtain $A \in (X_T, X_{\tilde{T}})$ in and only if*

$$\begin{cases} \hat{B} \in (X, Y) & \text{and} \\ W^{(B_n)} \in (X, \ell_\infty) & \text{for all } n, \end{cases} \tag{2.4}$$

where $B = \tilde{T}A$, $\hat{b}_{nk} = \sum_{j=k}^{\infty} s_{jk}b_{nj}$ for all n and k, and

$$w_{mk}^{(B_n)} = \begin{cases} \sum\limits_{j=m}^{\infty} s_{jk}b_{nj} & (0 \le k \le m) \\ 0 & (k > m) \end{cases} \qquad (m = 0, 1, \dots).$$

On the other hand, applying Lemma 2.3 first and then Lemma 2.2 (a), we obtain $A \in (X_T, Y_{\tilde{T}})$ if and only if

$$\begin{cases} \hat{C} = \tilde{T}\hat{A} \in (X, Y) & \text{and} \\ W^{(A_n)} \in (X, \ell_\infty) & \text{for all } n, \end{cases} \tag{2.5}$$

Then the conditions in (2.4) and (2.5) are equivalent.

Proof. First we assume that the conditions in (2.4) are satisfied. Then the series

$$\hat{b}_{nk} = \sum_{j=k}^{\infty} s_{jk}b_{nj} = \sum_{j=k}^{\infty} s_{jk} \sum_{i=0}^{n} \tilde{t}_{ni}a_{ij} \text{ converge for all } n \text{ and } k. \tag{2.6}$$

If we fix k then it is easy to show by mathematical induction with respect to n that the series

$$\hat{a}_{nk} = \sum_{j=k}^{\infty} s_{jk}a_{nj} \text{ converge for each } n \ge 0. \tag{2.7}$$

It follows from (2.7) that

$$\hat{b}_{nk} = \sum_{j=k}^{\infty} s_{jk} \sum_{i=0}^{n} \tilde{t}_{ni} a_{ij} = \sum_{i=0}^{n} \tilde{t}_{ni} \sum_{j=k}^{\infty} s_{jk} a_{ij} = \sum_{i=0}^{n} \tilde{t}_{ni} \hat{a}_{ik} = \hat{c}_{nk} \text{ for all } n \text{ and } k,$$

and consequently the condition $\hat{B} \in (X, Y)$ implies $\hat{C} \in (X, Y)$.
Similarly it can be shown that the convergence of

$$w_{mk}^{(B_n)} = \sum_{j=m}^{\infty} s_{jk} b_{nj} \text{ for all } n,\ 0 \le k \le m \text{ and all } m$$

implies that of

$$w_{mk}^{(A_n)} = \sum_{j=m}^{\infty} s_{jk} a_{nj} \text{ for all } n,\ 0 \le k \le m \text{ and all } m$$

and that

$$w_{mk}^{(B_n)} = \sum_{j=m}^{\infty} s_{jk} b_{nj} = \sum_{j=m}^{\infty} s_{jk} \sum_{i=0}^{n} \tilde{t}_{ni} a_{ij} = \sum_{i=0}^{n} \tilde{t}_{ni} \sum_{j=m}^{\infty} s_{jk} a_{ij} = \sum_{i=0}^{n} \tilde{t}_{ni} w_{mk}^{(A_i)}. \tag{2.8}$$

Writing $\tilde{S}$ for the inverse of the triangle $\tilde{T}$ we obtain for all n, m and k with $0 \le k \le m$

$$\sum_{l=0}^{n} \tilde{s}_{nl} w_{mk}^{(B_l)} = \sum_{l=0}^{n} \tilde{s}_{nl} \sum_{i=0}^{l} \tilde{t}_{li} w_{mk}^{(A_i)} = \sum_{i=0}^{n} w_{mk}^{(A_i)} \sum_{l=i}^{n} \tilde{s}_{nl} \tilde{t}_{li} = \sum_{i=0}^{n} w_{mk}^{(A_i)} \delta_{ni} = w_{mk}^{(A_n)},$$

and so $W^{(B_n)} \in (X, \ell_\infty)$ for all n implies $W^{(A_n)} \in (X, \ell_\infty)$ for all n.
Thus we have established that the conditions in (2.4) imply those in (2.5).
Conversely we assume that the conditions in (2.5) are satisfied. Then the series in (2.7) converge for all n and k, hence

$$\hat{b}_{nk} = \sum_{j=k}^{\infty} s_{jk} b_{nj} = \sum_{j=k}^{\infty} s_{jk} \sum_{i=0}^{n} \tilde{t}_{ni} a_{ij} = \sum_{i=0}^{n} \tilde{t}_{ni} \sum_{j=k}^{\infty} s_{jk} a_{ij} = \sum_{i=0}^{n} \tilde{t}_{ni} \hat{a}_{ik} = \hat{c}_{nk}. \tag{2.9}$$

Thus $\hat{C} \in (X, Y)$ implies $\hat{B} \in (X, Y)$. Similarly it can be shown that (2.8) holds for all n, m and $0 \le k \le m$, and consequently $A_n \in (X, \ell_\infty)$ for all n implies $B^{(n)} \in (X, \ell_\infty)$ for all n. □

Now we prove our first main result.

Theorem 2.6. *Let T and $\tilde{T}$ be triangles. Then we have $A \in ((\ell_1)_T, (\ell_1)_{\tilde{T}})$ if and only if*

$$\sup_k \sum_{n=0}^{\infty} \left| \sum_{j=k}^{\infty} s_{jk} \sum_{i=0}^{n} \tilde{t}_{ni} a_{ij} \right| < \infty, \tag{2.10}$$

and

$$\sup_{m,k} \left| \sum_{j=m}^{\infty} s_{jk} a_{nj} \right| < \infty \text{ for all } n = 0, 1, \ldots \tag{2.11}$$

Proof. Since ℓ_1 is a BK space with AK, it follows from Lemma 2.3 that $A \in ((\ell_1)_T, (\ell_1)_{\tilde{T}})$ if and only if $\hat{A} \in (\ell_1, (\ell_1)_{\tilde{T}})$ and $W^{(A_n)} \in (\ell_1, \ell_\infty)$ for all n. First, we have by Lemma 2.2 that $\hat{A} \in (\ell_1, (\ell_1)_{\tilde{T}})$ if and only if $B = \tilde{T}\hat{A} \in (\ell_1, \ell_1)$ which is the case if and only if ([12, 8.4.1D])

$$\sup_k \sum_{n=0}^{\infty} |b_{nk}| < \infty. \tag{2.12}$$

It follows from the definition of the matrices B and $\hat{A}$ that

$$b_{nk} = \sum_{i=0}^{n} \tilde{t}_{ni}\hat{a}_{ik} = \sum_{i=0}^{n} \tilde{t}_{ni} \sum_{j=k}^{\infty} s_{jk}a_{ij} = \sum_{j=k}^{\infty} s_{jk} \sum_{i=0}^{n} \tilde{t}_{ni}a_{ij},$$

and so the conditions in (2.12) and (2.10) are the same. Furthermore, we have $W^{(A_n)} \in (\ell_1, \ell_\infty)$ by [12, 8.4.1A] if and only if

$$\sup_{m,k} \left|w_{mk}^{(A_n)}\right| = \sup_{m,k} \left|\sum_{j=m}^{\infty} a_{nj}s_{jk}\right| < \infty,$$

which is is (2.11). □

The space c of convergent sequences is not an AK space, so we cannot apply Lemma 2.3 in the characterization of the class $(c_T, c_{\tilde{T}})$. This is why we need the next result.

Lemma 2.7. ([6, Remark 3.5 (b)]) *Let Y be a linear subspace of ω. Then we have $A \in (c_T, Y)$ if and only if*

$$\hat{A} \in (c_0, Y),\ \ W^{(A_n)} \in (c, c) \text{ for all } n \tag{2.13}$$

and

$$\hat{A}e - (\alpha_n)_{n=0}^{\infty} \in Y \text{ where } \alpha_n = \omega_n(A) = \lim_{m\to\infty} \sum_{k=0}^{m} w_{mk}^{(A_n)} \text{ for } n = 0, 1, \dots. \tag{2.14}$$

Remark 2.8. *Let Y be an arbitrary subspace of ω. Applying Lemma 2.2 (a) first and then Lemma 2.7, we obtain $A \in (c_T, c_{\tilde{T}})$ if and only if*

$$\begin{cases} \hat{B} \in (c_0, c) & \\ W^{(B_n)} \in (c, c) & \text{for all } n \\ \hat{B}e - (\beta_n)_{n=0}^{\infty} \in c & \text{where } \beta_n = \omega_n(B) = \lim_{m\to\infty} \sum_{k=0}^{m} w_{mk}^{(B_n)} \text{ for } n = 0, 1, \dots. \end{cases} \tag{2.15}$$

On the other hand, applying Lemma 2.7 first and then Lemma 2.2 (a), we obtain $A \in (c_T, c_{\tilde{T}})$ if and only if

$$\begin{cases} \hat{C} = \tilde{T}\hat{A} \in (c_0, c) & \\ W^{(A_n)} \in (c, c) & \text{for all } n \\ \tilde{T}\left(\hat{A}e - (\alpha_n)_{n=0}^{\infty}\right) \in c & \text{with } \alpha_n \text{ from (2.14) for } n = 0, 1, \dots. \end{cases} \tag{2.16}$$

Then conditions in (2.15) and (2.16) are equivalent.

Proof. First we assume that the conditions in (2.15) are satisfied. Then it follows as in Remark 2.5 that the first two conditions in (2.16) are satisfied and $\hat{c}_{nk} = \hat{b}_{nk}$ for all n and k. Furthermore $\hat{C} \in (c_0, c) \subset (c_0, \ell_\infty) = (\ell_\infty, \ell_\infty)$, that is, $\hat{A} \in (\ell_\infty, (\ell_\infty)_{\tilde{T}})$, implies $e \in \omega_{\tilde{A}}$, and so by [12, Theorem 1.4.1 (i)]

$$\hat{C}e = (\tilde{T}A)e = \tilde{T}(Ae).$$

Since $W^{(A_n)} \in (c,c)$ for all n, the limits α_n exist for all n, and so we obtain by (2.8) for all n

$$\begin{aligned}\tilde{T}_n\left((\alpha_j)_{j=0}^\infty\right) &= \sum_{i=0}^n \tilde{t}_{ni}\alpha_i = \sum_{i=0}^n \tilde{t}_{ni}\lim_{m\to\infty}\sum_{k=0}^m w_{mk}^{(A_i)} = \lim_{m\to\infty}\sum_{k=0}^m\sum_{i=0}^n \tilde{t}_{ni}w_{mk}^{(A_i)} \\ &= \lim_{m\to\infty}\sum_{k=0}^m w_{mk}^{(B_n)} = \beta_n.\end{aligned}$$

Consequently it follows from the third condition in (2.15) that

$$\tilde{T}(\hat{A}e - (\alpha_n)) = \tilde{T}(\hat{A}e) - \tilde{T}(\alpha_n) = \hat{C}e - (\beta_n) = \hat{B}e - (\beta_n) \in c.$$

Thus we have shown that the conditions in (2.15) imply those in (2.16).
Conversely we assume that the conditions in (2.16) are satisfied. Then it follows as above, that the first two conditions in (2.15) are satisfied, and $\hat{b}_{nk} = \hat{c}_k$ for all n and k. Also $\hat{B} \in (c_0, c)$ implies $\hat{B} \in (\ell_\infty, \ell_\infty)$ and so $e \in \omega_{\hat{B}}$, and $W^{(B_n)} \in (c,c)$ for all n implies that the limits β_n exist for all n. Again we have $\beta_n = \tilde{T}_n((\alpha_j)_{j=0}^\infty)$ for all n and $\hat{B}e - (\beta_n) = \tilde{T}(\hat{A}e - (\alpha_n))$, and the third condition in (2.16) implies the third condition in (2.15). □

Now, we can prove our results of the next theorem.

Theorem 2.9. *Let T and $\tilde{T}$ be triangles. Then we have $A \in (c_T, c_{\tilde{T}})$ if and only if the following conditions hold:*

$$\sup_n \sum_{k=0}^\infty \left|\hat{b}_{nk}\right| < \infty, \tag{2.17}$$

$$\lim_{n\to\infty} \hat{b}_{nk} = \hat{\beta}_k \text{ exists for all } k, \tag{2.18}$$

$$\sup_m \sum_{k=0}^m \left|w_{mk}^{(B_n)}\right| < \infty \text{ for each } n, \tag{2.19}$$

$$\lim_{m\to\infty} \sum_{k=0}^m w_{mk}^{(B_n)} = \beta_n \text{ exists for each } n, \tag{2.20}$$

$$\lim_{n\to\infty}\left(\sum_{k=0}^\infty \hat{b}_{nk} - \beta_n\right) = \eta \text{ exists.} \tag{2.21}$$

where the matrices B, $\hat{B}$ and $W^{(B_n)}$ are defined as in Remark 2.5.

Proof. Applying Lemma 2.2 (a) we have $A \in (c_T, c_{\tilde{T}})$ if and only if $B = \tilde{T}A \in (c_T, c)$, which by Lemma 2.7 is equivalent to $\hat{B} \in (c_0, c)$, $W^{(B_n)} \in (c,c)$ for all n and $\hat{B}e - (\beta_n)_{n=0}^\infty \in c$. Furthermore, we have $\hat{B} \in (c_0, c)$ if and only if ([12, 8.4.5A])

$$\sup_n \sum_{k=0}^\infty |\hat{b}_{nk}| < \infty \text{ and } \lim_{n\to\infty} \hat{b}_{nk} = \hat{\beta}_k \text{ for all } k = 0, 1, \dots, \tag{2.22}$$

that is, (2.17) and (2.18). Also $\hat{B}e - (\beta_n)_{n=0}^\infty \in c$ is the condition in (2.21). Furthermore, we have by [12, 8.4.5A] that $\hat{B} \in (c_0, c)$ if and only if the conditions in (2.17) and (2.18) hold. Finally we have $W^{(B_n)} \in (c,c)$ by [12, 8.4.5A] if and only if the conditions in (2.19) and (2.20) hold and $\lim_{m\to\infty} w_{mk}^{(B_n)}$ exists for each k, the last condition obviously being redundant. □

3. Compact Operators

We recall some definitions and results which are important for this section.

If X and Y are Banach spaces then a linear operator $L : X \to Y$ is said to be compact if its domain is all of X and for every bounded sequence $(x_n)_{n=0}^\infty$ in X, the sequence $(L(x_n))_{n=0}^\infty$ has a convergent subsequence in Y. We denote the class of such operators by $\mathcal{C}(X,Y)$. If $X = Y$, we write $\mathcal{C}(X)$, for short. The most effective way to find conditions for a linear operator L to be compact is by applying the Hausdorff measure of noncompactness.

Let (X,d) be a metric space, $\mathcal{M}_X$ denote the class of bounded subsets of X and $B(x,r) = \{y \in X : d(x,y) < r\}$ be the open ball of radius $r > 0$ with its centre in x. Then the *Hausdorff measure of noncompactness of the set* $Q \in \mathcal{M}_X$, denoted by $\chi(Q)$, is given by

$$\chi(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X,\ r_i < \epsilon\ (i = 1, \dots, n),\ n \in \mathbb{N}\};$$

the function χ is called the *Hausdorff measure of noncompactness*.

Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures of noncompactness on X and Y. Then the operator $L : X \to Y$ is called (χ_1,χ_2)*-bounded* if $L(Q) \in \mathcal{M}_Y$ for every $Q \in \mathcal{M}_X$ and there exists a positive constant C such that $\chi_2(L(Q)) \leq C\chi_1(Q)$ for every $Q \in \mathcal{M}_X$. If an operator L is (χ_1,χ_2)–bounded then the number $\|L\|_{(\chi_1,\chi_2)} = \inf\{C > 0 : \chi_2(L(Q)) \leq C\chi_1(Q)$ for all $Q \in \mathcal{M}_X\}$ is called the (χ_1,χ_2)*- measure of noncompactness of* L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

We need the following results.

Lemma 3.1. *Let X and Y be Banach spaces and* $L \in \mathcal{B}(X,Y)$, $S_X = \{x \in X : \|x\| = 1\}$ *and* $\bar{B}_X = \{x \in X : \|x\| \leq 1\}$. *Then we have*

$$\|L\|_\chi = \chi(L(\bar{B}_X)) = \chi(L(S_X)) \quad ([5, \text{Theorem } 2.25]); \tag{3.1}$$

$$L \in \mathcal{C}(X,Y) \quad \textit{if anf only if} \quad \|L\|_\chi = 0 \quad ([5, \text{Corollary } 2.26\ (2.58)]); \tag{3.2}$$

Lemma 3.2 (Goldenštein, Gohberg, Markus). ([5, Theorem 2.23]) *Let X be a Banach space with a Schauder basis* $(b_n)_{n=0}^\infty$, $Q \in \mathcal{M}_X$ *and* $P_n : X \to X$ *be the projector onto the linear span of* $\{b_1, b_2, ..., b_n\}$. *Then we have*

$$\frac{1}{a}\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right) \leq \chi(Q) \leq \limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|\right), \tag{3.3}$$

where $a = \limsup_{n\to\infty}\|I - P_n\|$. *(Let us mention that if* $X = c$ *then* $a = 2$*).*

Lemma 3.3. ([8, Theorem 2.8.]) *Let* $Q \in \mathcal{M}_X$ *where X is* ℓ_p *for* $1 \leq p < \infty$ *or* c_0. *If* $P_n : X \to X$ *is the operator defined by* $P_n(x) = x^{[m]}$ *for all* $x = (x_k)_{k=0}^\infty \in X$, *then*

$$\chi(Q) = \lim_{n\to\infty}(\sup_{x\in Q}\|(I-P_n)(x)\|).$$

Lemma 3.4. ([6, Theorem 4.2]) Let X be a normed sequence space and χ_T and χ denote the Hausdorff measures of noncompactness on $\mathcal{M}_{X_T}$ and $\mathcal{M}_X$. Then we have $\chi_{X_T}(Q) = \chi(T(Q))$ for all $Q \in \mathcal{M}_{X_T}$.

Lemma 3.5. *Let X and Y be Banach sequence spaces,* $\tilde{T}$ *be a triangle and* $L \in \mathcal{B}(X, Y_{\tilde{T}})$. *Then we have*

$$\|L\|_{(\chi,\chi_{\tilde{T}})} = \|L_{\tilde{T}} \circ L\|_\chi.$$

Proof. We have by (3.1) and Lemma 3.4

$$\|L\|_{(\chi,\chi_{\tilde{T}})} = \chi_{\tilde{T}}(L(S_X)) = \chi\left(\tilde{T}(L(S_X))\right) = \chi\left((L_{\tilde{T}} \circ L)(S_X)\right) = \|L_{\tilde{T}} \circ L\|_\chi.$$

□

Now we establish an identity for the Hausdorff measure of noncompactness of matrix operators in $\mathcal{B}((\ell_1)_T, (\ell_1)_{\tilde{T}})$ and necessary and sufficient conditions for such operators to be compact.

Theorem 3.6. *Let T and $\tilde{T}$ be triangles and the operator $L_A \in \mathcal{B}((\ell_1)_T, (\ell_1)_{\tilde{T}})$ be given by a matrix $A \in ((\ell_1)_T, (\ell_1)_{\tilde{T}})$. Then we have*

$$\|L_A\|_{(\chi_T, \chi_{\tilde{T}})} = \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}^{\infty}|\hat{b}_{nk}|\right)\right) \quad \text{where } \hat{b}_{nk} = \sum_{j=k}^{\infty} s_{jk}\sum_{i=0}^{n}\tilde{t}_{ni}a_{ij} \text{ for all } n \text{ and } k.$$

Furthermore, L_A is compact if and only if

$$\lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}^{\infty}\left|\sum_{j=k}^{\infty}s_{jk}\sum_{i=0}^{n}\tilde{t}_{ni}a_{ij}\right|\right)\right) = 0.$$

Proof. Applying Lemma 3.5, [1, 1, Corollary 3.6 (b), (3.14)] and (2.9), we obtain with $B = \tilde{T}A$

$$\|L_A\|_{(\chi_T, \chi_{\tilde{T}})} = \|L_{\tilde{T}} \circ L_A\|_{(\chi_T, \chi)} = \|L_B\|_{(\chi_T, \chi)} = \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}^{\infty}|\hat{b}_{nk}|\right)\right),$$

that is, the first identity of the theorem. The characterization of compact matrix operators now follows by (3.2). □

Now we establish an inequality for the Hausdorff measure of noncompactness of matrix operators in $\mathcal{B}(c_T, c_T)$ and necessary and sufficient conditions for such operators to be compact.

Theorem 3.7. *Let T and $\tilde{T}$ be triangles and the operator $L_A \in \mathcal{B}(c_T, c_{\tilde{T}})$ be given by a matrix $A \in (c_T, c_{\tilde{T}})$. Then we have*

$$\frac{1}{2}\cdot M \le \|L_A\|_{\chi} \le M$$

where

$$M = \lim_{r\to\infty}\sup_{n>r}\left(\sum_{k=0}^{\infty}\left|\hat{b}_{nk} - \hat{\beta}_k\right| + \left|\sum_{k=0}^{\infty}\hat{\beta}_k - \eta - \beta_n\right|\right)$$

and $\hat{b}_{nk}$, β_k, η and $\hat{\beta}_k$ are defined in Theorem 2.9.
L_A is compact if and only if

$$\lim_{r\to\infty}\sup_{n>r}\left(\sum_{k=0}^{\infty}\left|\hat{b}_{nk} - \hat{\beta}_k\right| + \left|\sum_{k=0}^{\infty}\hat{\beta}_k - \eta - \beta_n\right|\right) = 0.$$

Proof. We put for arbitrary matrices D

$$\phi_n(D) = \sum_{k=0}^{\infty}|\hat{d}_{nk} - \hat{\delta}_k| + \left|\sum_{k=0}^{\infty}\hat{\delta}_k - \eta(D) - \delta_n\right| \text{ for } n = 0, 1, \ldots,$$

where

$$\hat{\delta}_k = \lim_{n\to\infty}\hat{d}_{nk} \text{ for } k = 0, 1, \ldots,$$

$$\delta_n = \omega_n(D) = \lim_{m\to\infty}\sum_{k=0}^{m} w_{mk}^{(D_n)} \text{ for } n = 0, 1, \ldots$$

and

$$\eta(D) = \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\hat{d}_{nk} - \delta_n\right).$$

We obtain from Lemma 3.5 and [1, 1, Corollary 3.6 (b), (3.16)–(3.19)] that if $A \in (c_T, c_{\tilde{T}})$ then

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{r\geq n}\phi_n(C)\right) \leq \|L\|_\chi \leq \lim_{r\to\infty}\left(\sup_{r\geq n}\phi_n(C)\right), \text{ where } C = \tilde{T}A.$$

We have $\hat{c}_{nk} = \hat{b}_{nk}$ for all n and k by (2.9) and so $\hat{\gamma}_k = \lim_{n\to\infty}\hat{c}_{nk} = \lim_{n\to\infty}\hat{b}_{nk} = \hat{\beta}_k$ for $k = 0, 1, \ldots$ ((2.18) in Theorem 2.9). Furthermore, by definition,

$$w_{mk}^{(B_n)} = \sum_{j=m}^{\infty} s_{jk}b_{nj} = \sum_{j=m}^{\infty} s_{jk}\sum_{i=0}^{n}\tilde{t}_{ni}a_{ij} = \sum_{j=m}^{\infty} s_{jk}c_{nj} = w_{mk}^{(C_n)} \text{ for all } n, m \text{ and } k;$$

this implies

$$\gamma_n = \lim_{m\to\infty}\sum_{k=0}^{m} w_{mk}^{(C_n)} = \lim_{m\to\infty}\sum_{k=0}^{m} w_{mk}^{(B_n)} = \beta_n \text{ ((2.20)) for all } n,$$

and finally

$$\eta(C) = \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\hat{c}_{nk} - \gamma_n\right) = \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\hat{b}_{nk} - \beta_n\right) = \eta \text{ ((2.21)).}$$

Thus we have

$$M = \lim_{r\to\infty}\left(\sup_{r\geq n}\phi_n(C)\right),$$

and the statements of the theorem are an immediate consequence. □

4. Fredholm Operators on c_T and $(\ell_1)_T$

Now we establish sufficient conditions for an operator in $\mathcal{B}(X) = \mathcal{B}(X, X)$ to be a Fredholm operator when $X = \ell_1$ or $X = c$.

We recall the definition of a Fredholm operator.

Definition 4.1. *Let X and Y be Banach spaces and $L \in \mathcal{B}(X, Y)$. We denote the null and the range spaces of L by $N(L)$ and $R(L)$. Then T is said to be a Fredholm operator if the following conditions hold:*

(1) *$N(L)$ is finite dimensional;*
(2) *$R(L)$ is closed;*
(3) *$Y/R(L)$ is finite dimensional.*

The set of Fredholm operators from X to Y is denoted by $\Phi(X, Y)$ and we write $\Phi(X) = \Phi(X, X)$, for short.

The next result ([10], p.106) is of greater importance for our studies than the definition itself: if $X = Y$ and $L \in \mathcal{C}(X)$, then $I - L$ is a Fredholm operator where I is the identity operator on X.

We establish sufficient conditions for an operator L_A given by a matrix $A \in (X_T, X_T)$ to be a Fredholm operator when $X = \ell_1$ or $X = c$. Again we use the Hausdorff measure of noncompactness.

We consider infinite matrix $C = (c_{nk})_{n,k=0}^{\infty}$ associated with infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ and defined as follows:

$$c_{nk} = \begin{cases} -a_{nk} & (n \neq k) \\ 1 - a_{nn} & (n = k). \end{cases} \tag{4.1}$$

Then we have that if the operator L_C given by the infinite matrix C is compact then the operator L_A given by the infinite matrix A is a Fredholm operator. Taking this into account, we obtain the following new results:

Theorem 4.2. *(a) Let $L_A \in \mathcal{B}((\ell_1)_T)$ be given by a matrix A. We write $D = TC$ and $\hat{d}_{nk} = \sum_{j=k}^{\infty} s_{jk} d_{nj}$ for all n and k. If*

$$\lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty} |\hat{d}_{nk}|\right) = \lim_{r\to\infty}\left(\sup_k \left(\sum_{n=r}^{\infty}\left|\sum_{j=k}^{\infty} s_{jk}\left(t_{nj} - \sum_{i=0}^{n} t_{ni}a_{ij}\right)\right|\right)\right) = 0,$$

then we have $L_A \in \Phi((\ell_1)_T)$.
(b) Let $L_A \in \mathcal{B}(c_T)$ be given by a matrix A and $D = TC$. If

$$\lim_{r\to\infty} \sup_{n\geq r} \phi_n(D) = \lim_{r\to\infty} \sup_{n>r}\left(\sum_{k=0}^{\infty}\left|\hat{d}_{nk} - \hat{\delta}_k\right| + \left|\sum_{k=0}^{\infty}\hat{d}_{nk} - \eta(D) - \delta_n\right|\right) = 0,$$

then $L_A \in \Phi(c_T)$, where $\hat{\delta}_k$ $(k = 0, 1, \dots)$, δ_n $(n = 0, 1, \dots)$ and $\eta(D)$ are defined as in the proof of Theorem 3.7.

Proof. Defining the matrix C as in (4.1), we obtain

$$\sum_{i=0}^{n} t_{ni}c_{ij} = t_{nj}(1 - a_{jj}) - \sum_{i=0, i\neq j}^{n} t_{ni}a_{ij} = t_{nj} - \sum_{i=0}^{n} t_{ni}a_{ij}.$$

Now, if we apply Theorem 3.6 in (a) and Theorem 3.7 in (b), the proof is obvious since the operator L_A given by the infinite matrix A is Fredholm if the operator L_C given by the infinite matrix C is compact. □

We close with an application of our results.

Example 4.3. *We write $bv = \{x \in \omega : \sum_{k=0}^{\infty}|x_k - x_{k-1}| < \infty\}$ for the set of all sequences of bounded variation. Let $T = \tilde{T} = \Delta$ be the matrix of the operator of the first difference, that is, $\Delta_{nn} = 1$, $\Delta_{n,n-1} = -1$ and $\Delta_{nk} = 0$ for $k \neq n, n-1$ $(n = 0, 1, \dots)$. Then we have $S = \Sigma$ where $\Sigma_{nk} = 1$ for $0 \leq k \leq n$ and $\Sigma_{nk} = 0$ for $k > n$ $(n = 0, 1, \dots)$, and $bv = (\ell_1)_\Delta$. Now, applying Theorem 2.6 we obtain*

$$A \in (bv, bv) \text{ if and only if } \sup_k \sum_{n=0}^{\infty}\left|\sum_{j=k}^{\infty}(a_{nj} - a_{n-1,j})\right| < \infty \text{ ([11, \textbf{99.}(99.2)])},$$

since the condition in (2.11) becomes redundant in this case. Also applying Theorem 3.6, we obtain that the matrix operator $L_A \in \mathcal{B}((\ell_1)_T)$ is compact if and only if

$$\lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty}\left|\sum_{j=k}^{\infty}(a_{nj} - a_{n-1,j})\right|\right) = 0.$$

Finally, applying and Theorem 4.2 (b), we obtain that $L_A \in \Phi((\ell_1)_T)$ if

$$\lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty}\left|\sum_{j=k}^{\infty}\left(\Delta_{nj} - (a_{nj} - a_{n-1,j})\right)\right|\right) = 0.$$

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