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Multiplication Operators on Cesàro Function Spaces

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Abstract. In this paper, we characterize the compact, invertible, Fredholm and closed range multiplication operators on Cesàro function spaces.

1. Introduction and Preliminaries

Let (X, s, μ) be a σ-finite measure space and by $L^0(X)$ we denote the set of all equivalence classes of complex valued measurable functions defined on X where $X = [0, 1]$ or $X = [0, \infty)$. Then for $1 \leq p \leq \infty$ the Cesàro function space is denoted by $Ces_p(X)$ and is defined as

$$Ces_p(X) = \Big\{ f \in L^0(X) : \int_X \Big(\frac{1}{x} \int_0^x |f(t)| d\mu(t) \Big)^p d\mu(x) < \infty \Big\}.$$

The Cesàro function space $Ces_p(X)$ is a Banach space under the norm

$$||f|| = \Big(\int_X \Big(\frac{1}{x} \int_0^x |f(t)| d\mu(t) \Big)^p d\mu(x) \Big)^{\frac{1}{p}} \ \text{if} \ \ 1 \leq p < \infty$$

and

$$||f||_\infty = \sup_{x \in I, x > 0} \frac{1}{x} \int_0^x |f(t)| d\mu(t) < \infty \ \text{ if } \ p = \infty. \quad \text{See [3]}$$

The Cesàro function space $Ces_p[0, \infty)$ for $1 \leq p \leq \infty$ was considered by Shiue [21], Hassard and Hussein [9] and Sy, Zhang and Lee [25]. The space $Ces_\infty[0, 1]$ appeared in 1948 and it is known as the Korenblyum-Krein-Levin space K (see [13], [20]). Recently in [4] it is proved that in contrast to Cesàro sequence spaces, the Cesàro function spaces $Ces_p(X)$ on both $X = [0, 1]$ and $X = [0, \infty)$ for $1 < p < \infty$ are not reflexive and they do not have the fixed point property. In [5], Astashkin and Maligranda investigated Rademacher sums in $Ces_p[0, 1]$ for $1 \leq p \leq \infty$. The description is different for $1 \leq p < \infty$ and $p = \infty$.

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Let $u : X \to \mathbb{C}$ be a function such that $u.f \in Ces_p(X)$ for every $f \in Ces_p(X)$, then we can define a multiplication transformation $M_u : Ces_p(X) \to Ces_p(X)$ by

$$M_u f = u.f \ , \quad \forall\ f \in Ces_p(X).$$

If M_u is continuous, we call it a multiplication operator induced by u. These operators received considerable attention over the past several decades especially on L^p-spaces, Bergman spaces. From the recent literature available in operator theory we find that multiplication operators are very much intimately connected with the composition operators as most of the properties of composition operators on L^p-spaces can be stated in terms of properties of multiplication operators. For example Singh and Manhas [23] proved that a composition operator on $L^p(X, \mathbb{C})$ is compact if and only if the multiplication operator M_u is compact, where $u = \frac{d\mu T^{-1}}{d\mu}$, the Radon-Nikodym derivative of the measure μT^{-1} with respect to the measure μ. Infact the multiplication operators play an important role in the theory of Hilbert space operators. One of the main application is that every normal operator on a separable Hilbert space is unitarily equivalent to a multiplication operator. Moreover multiplication operators has its roots in the spectral theory and is being pursued today in such guises as the theory of subnormal operators and the theory of Toeplitz operators. For more details on multiplication operators we refer to ([1], [6], [2], [7], [8], [10], [14], [11], [12], [18], [19], [22], [24], [26], [27]) and refrences therein. Moreover, Compact operators on sequence spaces have recently been studied by Malkowsky [16] and Mursaleen and Noman in [17].

Definition 1.1. *A bounded linear operator $A : E \to E$ (where E is a Banach space) is called compact if $A(B_1)$ has compact closure, where B_1 denotes the closed unit ball of E.*

Definition 1.2. *A bounded linear operator $A : E \to E$ is called Fredholm if A has closed range, dim(kerA) and co-dim(ranA) are finite.*

In this paper we initiate the study of multiplication operators on Cesàro function spaces. We first prove that the set of all multiplication operators on $Ces_p(X)$ is a maximal abelian subalgebra of $B(Ces_p(X))$, the Banach algebra of all bounded linear operators on $Ces_p(X)$ into itself and after that we use this result to characterize the invertibility of multiplication operators on $Ces_p(X)$. By the symbol $A_\epsilon(u)$ we denote the set $\{x \in X : |u(x)| \geq \epsilon\}$.
The main purpose of this paper is to characterize the boundedness, compactness, closed range and Fredholmness of multiplication operators on Cesàro function spaces.

2. Invertible Multiplication Operators

The main purpose of this section is to characterize invertible multiplication operators on Cesàro function spaces. The following result (Theorem 2.1) for more general spaces (ideal Banach function spaces) has been proved by Maligranda and Persson [15]. For the sake of completeness, we give here a special case which will be further used in Theorem 2.2.

Theorem 2.1. *Let $u : X \to \mathbb{C}$ be a measurable function. Then $M_u : Ces_p(X) \to Ces_p(X)$ is a bounded operator if and only if u is an essentially bounded function. Moreover,*

$$\|M_u\| = \|u\|_\infty.$$

Proof. Suppose first that $u : X \to \mathbb{C}$ is an essentially bounded measurable function. Then for every

$f \in Ces_p(X)$, we have

$$\begin{aligned}
\|M_u f\|^p &= \int_X \Big(\frac{1}{x}\int_0^x |(u.f)(t)|d\mu(t)\Big)^p d\mu(x) \\
&= \int_X \Big(\frac{1}{x}\int_0^x |u(t)f(t)|d\mu(t)\Big)^p d\mu(x) \\
&\leq \|u\|_\infty^p \int_X \Big(\frac{1}{x}\int_0^x |f(t)|d\mu(t)\Big)^p d\mu(x) \\
&= \|u\|_\infty^p \|f\|^p.
\end{aligned}$$

Thus,

$$\|M_u f\| \leq \|u\|_\infty \|f\|, \tag{1}$$

which implies that M_u is a bounded operator.
Conversely, Suppose that M_u is a bounded operator. We show that u is essentially bounded function. Suppose u is not essentially bounded, then for every $n \in \mathbb{N}$, the set $E_n = \{x \in X : |u(x)| > n\}$ has a positive measure.
Let F_n be a measurable subset of E_n such that $\chi_{F_n} \in Ces_p(X)$, then

$$\begin{aligned}
\|M_u \chi_{F_n}\|^p &= \int_X \Big(\frac{1}{x}\int_0^x |u(t)\chi_{F_n}(t)|d\mu(t)\Big)^p d\mu(x) \\
&\geq \int_X \Big(\frac{1}{x}\int_0^x |n\chi_{F_n}(t)|d\mu(t)\Big)^p d\mu(x) \\
&= n^p \|\chi_{F_n}\|^p.
\end{aligned}$$

Hence, $\|M_u \chi_{F_n}\| > n\|\chi_{F_n}\|$. This is true for every $n \in \mathbb{N}$ which contradicts the boundedness of M_u. Thus u must be essentially bounded.
We now show that $\|M_u\| = \|u\|_\infty$. For any $\epsilon > 0$, let $E = \{x \in X : |u(x)| > (\|u\|_\infty - \epsilon)\}$. Then E has the positive measure. Now

$$\begin{aligned}
\|M_u \chi_E\|^p &= \|u.\chi_E\|^p \\
&= \int_X \Big(\frac{1}{x}\int_0^x |u(t)\chi_E(t)|d\mu(t)\Big)^p d\mu(x) \\
&\geq \int_X \Big(\frac{1}{x}\int_0^x \Big|\Big(\|u\|_\infty - \epsilon\Big)\chi_E(t)\Big|d\mu(t)\Big)^p d\mu(x) \\
&= \Big(\|u\|_\infty - \epsilon\Big)^p \int_X \Big(\frac{1}{x}\int_0^x |\chi_E(t)|d\mu(t)\Big)^p d\mu(x) \\
&= \Big(\|u\|_\infty - \epsilon\Big)^p \|\chi_E\|^p
\end{aligned}$$

Therefore $\|M_u\| \geq \|u\|_\infty - \epsilon$, but ϵ is arbitrary. Hence

$$\|M_u\| \geq \|u\|_\infty. \tag{2}$$

Finally from (1) and (2)

$$\|M_u\| = \|u\|_\infty.$$

□

Theorem 2.2. *The set of all multiplication operators on $Ces_p(X)$ is a maximal abelian subalgebra of the set $B(Ces_p(X))$.*

Proof. Let $\mathcal{H} = \Big\{M_u : u \text{ is an essentially bounded measurable function}\Big\}$ and consider the operator product

$$M_u.M_v = M_{uv},$$

where $M_u,\ M_v \in \mathcal{H}$, let us check $\mathcal{H}$ is a Banach algebra. Let $u,\ v$ are essentially bounded measurable function then $|u| \leq \|u\|_\infty$ and $|v| \leq \|v\|_\infty$,
therefore

$$\|uv\|_\infty \leq \|v\|_\infty \|u\|_\infty.$$

This implies that product is an inner operation, moreover the usual function product is associative, commutative and distributive with respect to the sum and scalar product, thus we conclude that $\mathcal{H}$ is a subalgebra of $B(Ces_p(X))$.
Now, we want to check that $\mathcal{H}$ is a maximal subalgebra, that is, given $N \in B(Ces_p(X))$, if N commute with $\mathcal{H}$ we have to prove $N \in \mathcal{H}$.
Consider the unit function $e : X \to \mathbb{C}$ defined by $e(x) = 1$ for all $x \in X$. Let $N \in B(Ces_p(X))$ be an operator which commute with $\mathcal{H}$ and let χ_E be the characteristic function of a measurable set E. Then

$$\begin{aligned} N(\chi_E) &= N\Big(M_{\chi_E}(e)\Big) \\ &= M_{\chi_E}\Big(N(e)\Big) \\ &= \chi_E.N(e) \\ &= N(e).\chi_E \\ &= M_w \chi_E, \text{ where } M_w = N(e). \end{aligned}$$

Similarly

$$N(S) = M_w(S) \tag{3}$$

for any simple function S. Now, let us check that w is essentially bounded. By way of contradiction assume that w is not essentially bounded, then the set

$$E_n = \{x \in X : |w(x)| > n\}$$

has positive measure for each $n \in \mathbb{N}$. Note that

$$M_w(\chi_{E_n}(x)) = (w\chi_{E_n}(x)) \geq n\chi_{E_n}(x)$$

for all $x \in X$. Thus

$$\begin{aligned} \|M_w \chi_{E_n}\|^p &= \|w\chi_{E_n}\|^p \\ &= \int_X \Big(\frac{1}{x}\int_0^x |w(t)\chi_{E_n}(t)| d\mu(t)\Big)^p d\mu(x) \\ &\geq n^p \int_X \Big(\frac{1}{x}\int_0^x |\chi_{E_n}(t)| d\mu(t)\Big)^p d\mu(x) \\ &= n^p \|\chi_{E_n}\|^p, \end{aligned}$$

since χ_{E_n} is a simple function then by (3) we have

$$M_w(\chi_{E_n}) = N(\chi_{E_n}).$$

Hence,

$$\|N(\chi_{E_n})\| \geq n\|\chi_{E_n}\|.$$

Therefore, N is an unbounded operator. This is a contradiction to the fact that N is bounded. Thus w is an essentially bounded measurable function and by Theorem 2.1 M_w is bounded.
Next, given $f \in Ces_p(X)$ there exists a nondecreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of measurable simple functions such that $\lim_{n\to\infty} s_n = f$, then by (3) we have

$$N(f) = N(\lim_{n\to\infty} s_n) = \lim_{n\to\infty} N(s_n) = \lim_{n\to\infty} M_w(s_n) = M_w \lim_{n\to\infty}(s_n) = M_w(f).$$

Therefore, $N(f) = M_w(f)$ for all $f \in Ces_p(X)$ and thus we conclude that $N \in \mathcal{H}$. □

Theorem 2.3. *The multiplication operator M_u is invertible if and only if u is invertible on $Ces_\infty(X)$.*

Proof. Let M_u be invertible, there exists $N \in B(Ces_p(X))$ such that

$$M_u.N = N.M_u = I, \tag{4}$$

where I represents the identity operator. Let us check that N commute with $\mathcal{H}$. Let $M_w \in \mathcal{H}$, then

$$M_w.M_u = M_u.M_w \tag{5}$$

applying N to (5) and by (4) we obtain

$$N.M_w.M_u.N = N.M_u.M_w.N$$

$$N.M_w.I = I.M_w.N$$

$$N.M_w = M_w.N$$

and thus we conclude that N commute with $\mathcal{H}$, by Theorem 2.2 $N \in \mathcal{H}$, there exists $g \in Ces_\infty(X)$ such that $N = M_g$. Hence

$$M_u.M_g = M_g.M_u = I,$$

this implies that $ug = gu = 1$, $\mu-$almost everywhere this means that u is invertible on $Ces_\infty(X)$.
On the other hand, assume u is invertible on $Ces_\infty(X)$ that is $\frac{1}{u} \in Ces_\infty(X)$, then

$$M_u.M_{\frac{1}{u}} = M_{\frac{1}{u}}.M_u = M_{\frac{1}{u}u} = M_1 = I,$$

which means that M_u is invertible on $B(Ces_p(X))$. □

Corollary 2.4. *Let $M_u \in B(Ces_p(X))$. Then M_u is invertible if and only if there exists $\epsilon > 0$ such that $|u(x)| \geq \epsilon$ for $\mu-$ almost all $x \in X$.*

Theorem 2.5. *Let $M_u \in B(Ces_p(X))$. Then M_u is an isometry if and only if $|u(x)| = 1$ a.e..*

Proof. Suppose first that $|u(x)| = 1$ a.e., then

$$\begin{aligned} \|M_u f\|^p &= \int_X \Big(\frac{1}{x}\int_0^x |u(t)f(t)|d\mu(t)\Big)^p d\mu(x) \\ &= \int_X \Big(\frac{1}{x}\int_0^x |f(t)|d\mu(t)\Big)^p d\mu(x) \\ &= \|f\|^p. \end{aligned}$$

Therefore, $\|M_u f\| = \|f\|$ and hence M_u is an isometry.
Conversely, suppose M_u is an isometry. If this is not true, then there is a measurable subset E of positive measure such that $|u(x)| < 1$ a.e. on E or there exists a measurable subset F of positive measure such that

$|u(x)| > 1$ a.e.. If $|u(x)| < 1$ a.e. we can assume that $E_\epsilon = \{x \in X : |u(x)| < 1 - \epsilon\}$ is of positive measure for some $\epsilon > 0$. We can choose a subset A of E_ϵ such that $\chi_A \in Ces_p(X)$. Now

$$\begin{aligned}
\|M_u\chi_A\|^p &= \int_X \Big(\frac{1}{x}\int_0^x |u(t)\chi_A(t)|d\mu(t)\Big)^p d\mu(x)\\
&\leq (1-\epsilon)^p \int_X \Big(\frac{1}{x}\int_0^x |\chi_A(t)|d\mu(t)\Big)^p d\mu(x)\\
&= (1-\epsilon)^p\|\chi_A\|^p\\
&< \|\chi_A\|^p, \text{ as } \epsilon \text{ is arbitrary.}
\end{aligned}$$

Therefore, $\|M_u\chi_A\| < \|\chi_A\|$, which contradicts that M_u is an isometry.
Again, if $|u(x)| > 1$ a.e. on F, then the set $F_\epsilon = \{x \in X : |u(x)| > 1 + \epsilon\}$ is of positive measure for some $\epsilon > 0$. Suppose B is a subset of F_ϵ so $\chi_B \in Ces_p(X)$. Then, obviously

$$\begin{aligned}
\|M_u\chi_B\| &\geq (1+\epsilon)\|\chi_B\|\\
&> \|\chi_B\|,
\end{aligned}$$

which again contradicts the fact that M_u is an isometry. Hence, $|u(x)| = 1$ a.e. □

3. Compact Multiplication Operators

In this section we investigate a necessary and sufficient condition for a multiplication operator to be compact.

Lemma 3.1. *Let M_u be a compact operator, for each $\epsilon > 0$, define $A_\epsilon(u) = \{x \in X : |u(x)| \geq \epsilon\}$, and $Ces_p(A_\epsilon(u)) = \{f\chi_{A_\epsilon(u)} : f \in Ces_p(X)\}$. Then $Ces_p(A_\epsilon(u))$ is a closed invariant subspace of $Ces_p(X)$ under M_u. Moreover, $M_u\Big|Ces_p(A_\epsilon(u))$ is a compact operator.*

Proof. Let $h, s \in Ces_p(A_\epsilon(u))$ and $\alpha, \beta \in \mathbb{R}$. Then $h = f\chi_{A_\epsilon(u)}$ and $s = g\chi_{A_\epsilon(u)}$, where $f, g \in Ces_p(X)$. Thus,

$$\begin{aligned}
\alpha h + \beta s &= \alpha(f\chi_{A_\epsilon(u)}) + \beta(g\chi_{A_\epsilon(u)})\\
&= (\alpha f + \beta g)\chi_{A_\epsilon(u)} \in Ces_p(A_\epsilon(u)).
\end{aligned}$$

which means that $Ces_p(A_\epsilon(u))$ is a subspace of $Ces_p(X)$. Next, for all $h \in Ces_p(A_\epsilon(u))$, we have

$$M_u h = u.h = uf\chi_{A_\epsilon(u)} = (uf)\chi_{A_\epsilon(u)},$$

where $uf \in Ces_p(X)$.
Therefore, $M_u h \in Ces_p(A_\epsilon(u))$ which means that $Ces_p(A_\epsilon(u))$ is an invariant subspace of $Ces_p(X)$ under M_u.
Now, let us show that $Ces_p(A_\epsilon(u))$ is a closed set. Indeed g be a function belonging to the closure of $Ces_p(A_\epsilon(u))$, then there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $Ces_p(A_\epsilon(u))$ such that $g_n \to g$ in $Ces_p(X)$. Just remains to exhibit that g belongs to $Ces_p(A_\epsilon(u))$. Note that

$$g = g\chi_{A_\epsilon(u)} + g\chi_{A^c_\epsilon(u)}.$$

Next, we want to show that $g\chi_{A^c_\epsilon(u)} = 0$. In fact given $\epsilon_1 > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\|g\chi_{A^c_\epsilon(u)}\| &= \|(g - g_{n_0} + g_{n_0})\chi_{A^c_\epsilon(u)}\|\\
&= \|(g - g_{n_0})\chi_{A^c_\epsilon(u)}\|\\
&\leq \|g - g_{n_0}\|\\
&< \epsilon_1.
\end{aligned}$$

Thus, $g\chi_{A^c_\epsilon(u)} = 0$, which means that $g = g\chi_{A_\epsilon(u)}$ that is $g \in Ces_p(A_\epsilon(u))$. This completes the proof of the lemma. □

Theorem 3.2. *Let $M_u \in B(Ces_p(X))$. Then M_u is compact if and only if $Ces_p(A_\epsilon(u))$ is finite dimensional for each $\epsilon > 0$.*

Proof. If $|u(x)| \geq \epsilon$, we should note that

$$|uf\chi_{A_\epsilon(u)}(x)| \geq \epsilon f\chi_{A_\epsilon(u)}(x)$$

and thus

$$\begin{aligned} \|M_u f\chi_{A_\epsilon(u)}\|^p &= \|uf\chi_{A_\epsilon(u)}\|^p \\ &= \int_X \Big(\frac{1}{x}\int_0^x |u(t)(f\chi_{A_\epsilon(u)})(t)|^p d\mu(t)\Big) d\mu(x) \\ &\geq \epsilon^p \int_X \Big(\frac{1}{x}\int_0^x |(f\chi_{A_\epsilon(u)})(t)|^p d\mu(t)\Big) d\mu(x) \\ &= \epsilon^p \|f\chi_{A_\epsilon(u)}\|^p. \end{aligned}$$

Therefore,

$$\|M_u f\chi_{A_\epsilon(u)}\| \geq \epsilon \|f\chi_{A_\epsilon(u)}\|. \tag{6}$$

Now if M_u is a compact operator, then by Lemma 3.1 $Ces_p(A_\epsilon(u))$ is closed invariant subspace of M_u and $M_u\big|Ces_p(A_\epsilon(u))$ is a compact operator. Then by (6) $M_u\big|Ces_p(A_\epsilon(u))$ has a closed range in $Ces_p(A_\epsilon(u))$ and it is invertible, being compact. Thus $Ces_p(A_\epsilon(u))$ is finite dimensional.
Conversely, Suppose that $Ces_p(A_\epsilon(u))$ is finite dimensional, for each $\epsilon > 0$. In particular for $n \in \mathbb{N}$, $Ces_p(A_{\frac{1}{n}}(u))$ is finite dimensional, then for each n, define

$$u_n : X \to \mathbb{C}$$

as

$$u_n(x) = \begin{cases} u(x), & \text{if } u(x) \geq \frac{1}{n} \\ 0 \quad , & \text{if } u(x) < \frac{1}{n}. \end{cases}$$

Then we find that

$$M_{u_n} f - M_u f = (u_n - u).f \leq \|u_n - u\|_\infty |f|$$

and thus

$$\begin{aligned} \|M_{u_n} f - M_u f\|^p &= \int_X \Big(\frac{1}{x}\int_0^x |(u_n - u)f(t)|^p d\mu(t)\Big) d\mu(x) \\ &\leq \|u_n - u\|_\infty^p \int_X \Big(\frac{1}{x}\int_0^x |f(t)|^p d\mu(t)\Big) d\mu(x) \\ &= \|u_n - u\|_\infty^p \|f\|^p. \end{aligned}$$

Therefore

$$\|M_{u_n} f - M_u f\| \leq \|u_n - u\|_\infty \|f\|.$$

Consequently,

$$\|M_{u_n} f - M_u f\| \leq \frac{1}{n}\|f\|$$

which implies that M_{u_n} converges to M_u uniformly. As $Ces_p(A_\epsilon(u))$ is finite dimensional so M_{u_n} is a finite rank operator. Therefore M_{u_n} is a compact operator and hence M_u is a compact operator. □

Proposition 3.3. *M_u is injective on $Y = Ces_p(supp(u))$, where $supp(u) = \{x \in X : u(x) \neq 0\}$.*

Proof. Let $Y = Ces_p(supp(u)) = \{f\chi_{supp(u)} : f \in Ces_p(X)\}$.
Indeed, if $M_u(\tilde{f}) = 0$ with $\tilde{f} = f\chi_{supp(u)} \in Y$, then $f(x)\chi_{supp(u)} = 0$ for all $x \in X$ and so

$$f(x)u(x) = 0, \forall x \in supp(u)$$

$$f(x) = 0, \forall x \in supp(u)$$

$$f(x)\chi_{supp(u)} = 0, \forall x \in X.$$

Then $\tilde{f} = 0$ and the proof is complete. □

4. Fredholm Multiplication Operators

In this section we first establish a condition for multiplication operator to have closed range and then we make use of it to characterize Fredholm multiplication operators.

Theorem 4.1. *Let $M_u \in B(Ces_p(X))$. Then M_u has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ μ-almost everywhere on $S = \{x \in X : u(x) \neq 0\}$ the support of u.*

Proof. If there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ μ-almost everywhere on S, then for $f \in Ces_p(X)$ we have

$$\begin{aligned} \|M_u f\chi_S\|^p &= \int_X \Big(\frac{1}{x}\int_0^x |u(t)(f.\chi_S(t)|d\mu(t)\Big)^p d\mu(x) \\ &\geq \delta^p \int_X \Big(\frac{1}{x}\int_0^x |f\chi_S(t)|d\mu(t)\Big)^p d\mu(x) \\ &= \delta^p \|f\chi_S\|^p \end{aligned}$$

Therefore, $\|M_u f\chi_S\| \geq \delta\|f\chi_S\|$. Thus M_u has closed range.
Conversely, if M_u has closed range on $Ces_p(S)$, since M_u is one-one on $Ces_p(S)$ then M_u is bounded below and thus there exists an $\delta > 0$ such that

$$\|M_u f\| \geq \delta\|f\|$$

for all $f \in Ces_p(S)$, where

$$Ces_p(S) = \{f\chi_S : f \in Ces_p(X)\}.$$

Let $E = \{x \in S : |u(x)| < \frac{\epsilon}{2}\}$. If $\mu(E) > 0$, then we can find a measurable set $F \subseteq E$ such that $\chi_F \in Ces_p(S)$. Now

$$\begin{aligned} \|M_u\chi_F\|^p &= \|u\chi_F\|^p \\ &= \int_X \Big(\frac{1}{x}\int_0^x |u(t)\chi_F(t)|d\mu(t)\Big)^p d\mu(x) \\ &\leq \Big(\frac{\epsilon}{2}\Big)^p \int_X \Big(\frac{1}{x}\int_0^x |\chi_F(t)|d\mu(t)\Big)^p d\mu(x) \\ &= \Big(\frac{\epsilon}{2}\Big)^p \|\chi_F\|^p. \end{aligned}$$

Hence

$$\|M_u\chi_F\| \leq \frac{\epsilon}{2}\|\chi_F\|,$$

which is a contradiction. Therefore $\mu(E) = 0$. This completes the proof. □

Theorem 4.2. *Suppose $M_u \in B(Ces_p(X))$. Then the following are equivalent;*
(i) $|u(x)| \geq \delta$ a.e. for some $\delta > 0$,
(ii) M_u is invertible,
(iii) M_u is Fredholm,
(iv) $ran(M_u)$ is closed and co-dim $ran(M_u) < \infty$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are obvious. We only show that $(iv) \Rightarrow (i)$. Suppose that M_u has closed range and co-dim $\mathrm{ran}(M_u) < \infty$. We claim that M_u is onto. Suppose this is not true, then for $g \in Ces_p(X)$ and $g \notin \mathrm{ran}(M_u)$ there exists a bounded linear functional $g^* \in Ces_p^*(X)$ such that

$$g^*(g) = 1 \text{ and } g^*(M_u f) = 0 \text{ for all } f \in Ces_p(X). \tag{7}$$

For $g^* \in Ces_p^*(X)$, we have by Representation theorem for continuous functionals on $Ces_p(X)$ there exists $g' \in Ces_q(X)$ where $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$g^*(g) = \int_X g(t)g'(t)d\mu(t),$$

for all $g \in Ces_p(X)$. From (7) we have

$$\int_X Re(g(t)g'(t))d\mu(t) = 1$$

and

$$g^*(M_u g) = \int_X (M_u g)(t)g'(t)d\mu(t) = 0.$$

Hence the set $\{x \in X : Re(gg')(x) \geq \delta\}(= E_\delta \text{ say })$ must have finite measure for $\delta > 0$. So we can find a sequence $\{E_n\}$ of disjoints measurable subsets of E_δ such that $0 < \mu(E_n) < \mu(E_\delta)$. Take $g_n^* = g^* \chi_{E_n}$. Then $g_n^* \in Ces_p^*(X)$ and $g_n^* \in \ker M_u^*$ because for $f \in Ces_p(X)$,

$$(M_u^* g_n^*)(f) = g_n^*(M_u f) = \int_{E_n} (M_u f)(t)g'(t)d\mu(t) = 0.$$

Then $g_n^* \in \ker M_u^*$ which proves that $\ker M_u^*$ is infinite dimensional which contradicts the fact that $\ker M_u^* =$ co-dim $\mathrm{ran}(M_u) < \infty$. Hence, M_u is onto. Therefore from the Corollary 2.4 there exists $\delta > 0$ such that $|u(x)| \geq \delta$ for $\mu-$ almost all $x \in X$. □

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