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On Statistically Convergent Sequences of Closed Sets

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Abstract. In this paper, we give the definitions of statistical inner and outer limits for sequences of closed sets in metric spaces. We investigate some properties of statistical inner and outer limits. For sequences of closed sets if its statistical outer and statistical inner limits coincide, we say that the sequence is Kuratowski statistically convergent. We prove some proporties for Kuratowski statistically convergent sequences. Also, we examine the relationship between Kuratowski statistical convergence and Hausdorff statistical convergence.

1. Introduction

Let K be a subset of positive integers $\mathbb{N}$ and $K(n) = |\{k \leq n : k \in K\}|$, where $|A|$ denotes the number of elements in A. The natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} K(n)$$

if this limit exists.

Statistical convergence of a sequence of scalars was introduced by Fast [6]. A sequence $x = (x_k)$ is said to be statistically convergent to the number L if the set

$$\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

has natural density zero for every $\varepsilon > 0$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \Big| \{k \leq n : |x_k - L| \geq \varepsilon\} \Big| = 0.$$

In this case we write

$$\text{st-}\lim_{k\to\infty} x_k = L. \tag{1}$$

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Lemma 1.1. *[15, Lemma 1.1] Statement (1) holds if and only if there exists a set*

$$K = \{k_1 < k_2 < k_3 < \cdots\} \subseteq \mathbb{N}$$

such that $\delta(K) = 1$ *and* $\lim_{n\to\infty} x_{k_n} = L$.

The concepts of statistical limit superior and statistical limit inferior were introduced by Fridy and Orhan [8]. For a sequence of real numbers $x = (x_k)$, the notions of statistical limit inferior and limit superior are defined as follows:

$$st - \liminf x := \begin{cases} \inf A_x, & A_x \neq \emptyset, \\ \infty, & \text{otherwise}, \end{cases}$$

$$st - \limsup x := \begin{cases} \sup B_x, & B_x \neq \emptyset, \\ -\infty, & \text{otherwise}, \end{cases}$$

where

$$\begin{aligned} A_x &:= \{a \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k < a\}) \neq 0\}, \\ B_x &:= \{b \in \mathbb{R} : \delta(\{k \in \mathbb{N} : x_k > b\}) \neq 0\}. \end{aligned}$$

Lemma 1.2. *[8] If* $\beta = st - \limsup x$ *is finite, then for every* $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : x_k > \beta - \varepsilon\}) \neq 0 \quad \text{and} \quad \delta(\{k \in \mathbb{N} : x_k > \beta + \varepsilon\}) = 0. \tag{2}$$

Conversely, if (2) holds for every $\varepsilon > 0$ *then* $\beta = st - \limsup x$.

The dual statement for $st - \liminf x$ is as follows:

Lemma 1.3. *[8] If* $\alpha = st - \liminf x$ *is finite, then for every* $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\}) \neq 0 \quad \text{and} \quad \delta(\{k \in \mathbb{N} : x_k < \alpha - \varepsilon\}) = 0. \tag{3}$$

Conversely, if (3) holds for every $\varepsilon > 0$ *then* $\alpha = st - \liminf x$.

The statement $\delta(K) \neq 0$ means that either $\delta(K) > 0$ or $\delta(K)$ is not defined (i.e. K does not have natural density).

The idea of statistical convergence can be extended to a sequence of points of a metric space (see [5]). We say that a sequence $x = (x_k)$ of points of a metric space (X, d) statistically converges to a point $\xi \in X$ if for each $\varepsilon > 0$ we have

$$\delta(\{k \in \mathbb{N} : d(x_k, \xi) \geq \varepsilon\}) = 0.$$

A point $\xi \in X$ is called a statistical limit point of a sequence $x = (x_k)$ if there is a set $K = \{k_1 < k_2 < k_3 < \cdots\}$ with $\delta(K) \neq 0$ such that $x_{k_n} \to \xi$ as $n \to \infty$. The set of all statistical limit points of a sequence x will be denoted by Λ_x.

A point $\xi \in X$ is called a statistical cluster point of $x = (x_k)$ if for any $\varepsilon > 0$,

$$\delta\Big(\Big\{k \in \mathbb{N} : d(x_k, \xi) < \varepsilon\Big\}\Big) \neq 0.$$

The set of all statistical cluster points of x will be denoted by Γ_x.

Let L_x denote the set of all limit points ξ (accumulation points) of the sequence x; i.e., $\xi \in L_x$ if there exists an infinite set $K = \{k_1 < k_2 < k_3 < \cdots\}$ such that $x_{k_n} \to \xi$ as $n \to \infty$.

Obviously we have $\Lambda_x \subseteq \Gamma_x \subseteq L_x$.

Lemma 1.4. *[5, Lemma 3.1] Let* (X, d) *be a metric space and K be a compact subset of X. Then, we have* $K \cap \Gamma_x \neq \emptyset$ *for every* $x = (x_n)$ *with* $\delta(\{n \in \mathbb{N} : x_n \in K\}) \neq 0$.

The concept of statistical convergence has been studied by many authors, see for instance [7, 9, 13, 15, 19].

Let (X, d) be a metric space. The distance between a subset A of X and $x \in X$ is given by

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$

where it is understood that the infimum of $d(x, .)$ is ∞ if $A = \emptyset$. For each closed subset A of X, the function $x \to d(., A)$ is Lipschitz continuous, i.e. for each $x, y \in X$

$$\left|d(x, A) - d(y, A)\right| \leq d(x, y). \tag{4}$$

The open ball with center x and radius $\varepsilon > 0$ in X is denoted by $B(x, \varepsilon)$, i.e.

$$B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Also, for any set A and $\varepsilon > 0$, we write

$$B(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}.$$

By $\Omega(x)$, we denote the set of neighborhoods of x.

Let us recall some basic properties of Kuratowski convergence. Alternatively, in the literature, convergence in this sense may be called Painlevé-Kuratowski convergence. We use the following notation:

$$\begin{aligned} \mathcal{N} &:= \{N \subseteq \mathbb{N} : \mathbb{N}\backslash N \text{ finite}\} \\ &:= \{\text{subsequences of } \mathbb{N} \text{ contain all positive integers beyond some positive integer } n_0\} \\ \mathcal{N}^{\#} &:= \{N \subseteq \mathbb{N} : N \text{ infinite}\} = \{\text{all subsequences of } \mathbb{N}\}. \end{aligned}$$

We write $\lim_{n\to\infty}$ when $n \to \infty$ as usual in $\mathbb{N}$, but $\lim_{n\in N}$ in the case of convergence of a subsequence designated by an index set N in $\mathcal{N}$ or $\mathcal{N}^{\#}$.

Definition 1.5. *(Inner and outer limits) Let (X, d) be a metric space. For a sequence (A_n) of closed subsets of X; the outer limit is the set*

$$\begin{aligned} \limsup_{n\to\infty} A_n &:= \left\{x \mid \forall\, V \in \Omega(x),\ \exists N \in \mathcal{N}^{\#},\ \forall n \in N : A_n \cap V \neq \emptyset\right\} \\ &:= \left\{x \mid \exists N \in \mathcal{N}^{\#},\ \forall n \in N,\ \exists x_n \in A_n : \lim_{n\in N} x_n = x\right\}, \end{aligned}$$

while the inner limit is the set

$$\begin{aligned} \liminf_{n\to\infty} A_n &:= \left\{x \mid \forall\, V \in \Omega(x),\ \exists N \in \mathcal{N},\ \forall n \in N : A_n \cap V \neq \emptyset\right\} \\ &:= \left\{x \mid \exists N \in \mathcal{N},\ \forall n \in N,\ \exists x_n \in A_n : \lim_{n\in N} x_n = x\right\}. \end{aligned}$$

The limit of a sequence (A_n) of closed subsets of X exists if the outer and inner limit sets are equal, that is,

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n.$$

Inner and outer limits can also be expressed in terms of distance functions or operations of intersection and union.

Proposition 1.6. *(characterizations of set limits)*

$$\begin{aligned} \limsup_{n\to\infty} A_n &= \left\{x \mid \liminf_{n\to\infty} d(x, A_n) = 0\right\}, \\ \liminf_{n\to\infty} A_n &= \left\{x \mid \limsup_{n\to\infty} d(x, A_n) = 0\right\}, \\ \limsup_{n\to\infty} A_n &= \bigcap_{N\in\mathcal{N}} cl \bigcup_{n\in N} A_n, \\ \liminf_{n\to\infty} A_n &= \bigcap_{N\in\mathcal{N}^{\#}} cl \bigcup_{n\in N} A_n. \end{aligned}$$

For more result on inner and outer limits of sequences of sets we refer to [11, 14, 16]. Concerning other types of convergence the reader could consult the book of G. Beer [3] and the survey paper of Baronti and Papini [2]. See also [4, 17, 18, 20, 21].

2. Kuratowski Statistical Convergence

In this section, we introduce Kuratowski statistical convergence of sequences of closed sets. For operational reasons in handling statements about sequences, it will be convenient to work with the following collections of subsets of $\mathbb{N}$

$$\mathcal{S} := \{N \subseteq \mathbb{N} : \delta(N) = 1\} \quad \text{and} \quad \mathcal{S}^{\#} := \{N \subseteq \mathbb{N} : \delta(N) \neq 0\}.$$

Firstly, we define the statistical analogues for inner and outer limits of a sequence of closed sets as follows.

Definition 2.1. *Let (X, d) be a metric space. The statistical outer limit and statistical inner limit of a sequence (A_n) of closed subsets of X are defined as follows:*

$$st - \limsup_{n\to\infty} A_n \quad := \quad \left\{x \mid \forall\ V \in \Omega(x),\ \exists N \in \mathcal{S}^{\#},\ \forall n \in N : A_n \cap V \neq \emptyset\right\}$$

and

$$st - \liminf_{n\to\infty} A_n \quad := \quad \left\{x \mid \forall\ V \in \Omega(x),\ \exists N \in \mathcal{S},\ \forall n \in N : A_n \cap V \neq \emptyset\right\}.$$

The statistical limit of a sequence (A_n) exists if its statistical outer and statistical inner limits coincide. In this situation we say that the sequence is Kuratowski statistically convergent and we write

$$st - \liminf_{n\to\infty} A_n = st - \limsup_{n\to\infty} A_n = st - \lim_{n\to\infty} A_n.$$

Moreover, we always have that

$$st - \liminf_{n\to\infty} A_n \subseteq st - \limsup_{n\to\infty} A_n$$

so that in fact, $st - \lim_{n\to\infty} A_n = A$ if and only if the inclusion

$$st - \limsup_{n\to\infty} A_n \subseteq A \subseteq st - \liminf_{n\to\infty} A_n$$

holds. Since $\mathcal{N} \subseteq \mathcal{S}$ and $\mathcal{S}^{\#} \subseteq \mathcal{N}^{\#}$, it is clear that

$$\liminf_{n\to\infty} A_n \subseteq st - \liminf_{n\to\infty} A_n \subseteq st - \limsup_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n. \tag{5}$$

Example 2.2. *Let us define the sequence $(A_n) \subseteq \mathbb{R}$ by*

$$A_n = \begin{cases} [-4, 1] & , \text{ if } n \text{ is an even square,} \\ [-1, 4] & , \text{ if } n \text{ is an odd square,} \\ [-3, 2] & , \text{ if } n \text{ is an even nonsquare,} \\ [-2, 3] & , \text{ if } n \text{ is an odd nonsquare.} \end{cases}$$

Then $st - \liminf_{n\to\infty} A_n = [-2, 2]$, $st - \limsup_{n\to\infty} A_n = [-3, 3]$, $\liminf_{n\to\infty} A_n = [-1, 1]$ and $\limsup_{n\to\infty} A_n = [-4, 4]$. So (A_n) is not Kuratowski statistically convergent.

From the inclusion (5), Kuratowski convergence implies Kuratowski statistical convergence, i.e.

$$\lim_{n\to\infty} A_n = A \text{ implies } st - \lim_{n\to\infty} A_n = A.$$

But, the converse of this claim does not hold in general, as seen in the following example.

Example 2.3. *Let A and B be two different nonempty closed sets in X. Define*

$$A_n := \begin{cases} A & , \quad n = k^2 \text{ for } k \in \mathbb{N}, \\ B & , \quad \text{otherwise}. \end{cases}$$

Then, $st - \lim_{n\to\infty} A_n = B$. However, $\liminf_{n\to\infty} A_n = A \cap B$ and $\limsup_{n\to\infty} A_n = A \cup B$. So, (A_n) is not Kuratowski convergent.

Without loss of generality, the neighborhoods V in Definition 2.1 can be taken to be of the form $B(x, \varepsilon)$. So, the formulas can be written just as well as

$$\begin{aligned} st - \limsup_{n\to\infty} A_n &:= \Big\{x \mid \forall \varepsilon > 0,\ \exists N \in \mathcal{S}^{\#},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\}, \\ st - \liminf_{n\to\infty} A_n &:= \Big\{x \mid \forall \varepsilon > 0,\ \exists N \in \mathcal{S},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\}. \end{aligned}$$

Proposition 2.4. *Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X. Then*

$$st - \liminf_{n\to\infty} A_n = \bigcap_{N\in\mathcal{S}^{\#}} cl \bigcup_{n\in N} A_n \quad \textit{and} \quad st - \limsup_{n\to\infty} A_n = \bigcap_{N\in\mathcal{S}} cl \bigcup_{n\in N} A_n$$

Proof. We prove only the first equality. Let $x \in st - \liminf_{n\to\infty} A_n$ be arbitrary and $N \in \mathcal{S}^{\#}$ be arbitrary. For every $\varepsilon > 0$ there exists $N_1 \in \mathcal{S}$ such that for every $n \in N_1$

$$A_n \cap B(x,\varepsilon) \neq \emptyset.$$

Since $N \cap N_1 \neq \emptyset$, there exists $n_0 \in N \cap N_1$ and $A_{n_0} \cap B(x,\varepsilon) \neq \emptyset$. Therefore,

$$\left(\bigcup_{n\in N} A_n\right) \cap B(x,\varepsilon) \neq \emptyset.$$

This means that $x \in \text{cl} \bigcup_{n\in N} A_n$. This holds for any $N \in \mathcal{S}^{\#}$. Consequently,

$$x \in \bigcap_{N\in\mathcal{S}^{\#}} \text{cl} \bigcup_{n\in N} A_n.$$

For the reverse inclusion, suppose that $x \notin st - \liminf_{n\to\infty} A_n$. Then, there exists $\varepsilon > 0$ such that

$$\delta\Big(\Big\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\}\Big) \neq 1$$

and so, the set

$$N = \Big\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) = \emptyset\Big\}$$

does not have density zero, i.e. $N \in \mathcal{S}^{\#}$. Thus

$$\left(\bigcup_{n\in N} A_n\right) \cap B(x,\varepsilon) = \emptyset.$$

This means that $x \notin \text{cl} \bigcup_{n\in N} A_n$. This completes the proof. □

As a result of Proposition 2.4, given any sequence (A_n), $st - \liminf_{n\to\infty} A_n$ and $st - \limsup_{n\to\infty} A_n$ are closed.

Proposition 2.5. *Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X. Then*

$$\begin{aligned} st-\limsup_{n\to\infty} A_n &= \Big\{x \mid st-\liminf_{n\to\infty} d(x, A_n) = 0\Big\}, \\ st-\liminf_{n\to\infty} A_n &= \Big\{x \mid st-\lim_{n\to\infty} d(x, A_n) = 0\Big\}. \end{aligned}$$

Proof. For any closed set A we have

$$d(x, A) < \varepsilon \quad \Leftrightarrow \quad A \cap B(x, \varepsilon) \neq \emptyset. \tag{6}$$

Suppose that $st - \liminf_{n\to\infty} d(x, A_n) = 0$. Then for every $\varepsilon > 0$

$$\delta\Big(\Big\{n \in \mathbb{N} : d(x, A_n) < \varepsilon\Big\}\Big) \neq 0.$$

By (6), we have $x \in st - \limsup_{n\to\infty} A_n$.

Now, we show the reverse inclusion. Let $x \in st - \limsup_{n\to\infty} A_n$. Then for every $\varepsilon > 0$ there exists $N \in \mathcal{S}^{\#}$ such that $A_n \cap B(x, \varepsilon) \neq \emptyset$ for every $n \in N$. Since

$$N \subseteq \Big\{n \in \mathbb{N} : A_n \cap B(x, \varepsilon) \neq \emptyset\Big\},$$

we obtain

$$\delta\Big(\Big\{n \in \mathbb{N} : A_n \cap B(x, \varepsilon) \neq \emptyset\Big\}\Big) \neq 0.$$

By (6) and Lemma 1.3, we have $st - \liminf_{n\to\infty} d(x, A_n) = 0$.

Similarly, for any closed set A

$$d(x, A) \geq \varepsilon \Leftrightarrow A \cap B(x, \varepsilon) = \emptyset. \tag{7}$$

Now, the second equality can be obtained by using (7). □

Proposition 2.6. *Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X. Then*

$$st-\liminf_{n\to\infty} A_n = \Big\{x \mid \exists N \in \mathcal{S},\ \forall n \in N,\ \exists y_n \in A_n : \lim_{n\to\infty} y_n = x\Big\}. \tag{8}$$

Proof. Suppose that $x \in st - \liminf_{n\to\infty} A_n$ and define K_j by

$$\begin{aligned} K_j &= \left\{n \in \mathbb{N} : A_n \cap B\left(x, \frac{1}{j}\right) \neq \emptyset\right\} \\ &= \left\{n \in \mathbb{N} : \exists y_n \in A_n \text{ and } d(x, y_n) < \frac{1}{j}\right\} \end{aligned}$$

for all $j \in \mathbb{N}$. Then, we have by Definition 2.1 that $\delta(K_j) = 1$. It is evident from the definition of K_j with $j \in \mathbb{N}$ that

$$K_1 \supset K_2 \supset K_3 \supset \cdots \supset K_j \supset K_{j+1} \cdots$$

By the proof of Lemma 1.1 in [15] we can construct the strictly increasing sequence (v_j) of positive integers that $v_j \in K_j$ for all $j \in \mathbb{N}$ and

$$\frac{K_j(n)}{n} > \frac{j-1}{j}$$

for each $n \geq v_j$. Again by the proof of Lemma 1.1 in [15], we construct the set K as follows:

$$K = \Big([1, v_1) \cap \mathbb{N}\Big) \cup \Big([v_1, v_2) \cap K_1\Big) \cup \Big([v_2, v_3) \cap K_2\Big) \cup \cdots$$

It is clear that $\delta(K) = 1$. Thus for each $n \in K$ there exists $y_n \in A_n$ such that

$$\lim_{n\in K} y_n = x.$$

Therefore x belongs to the set in the right-hand side of equality (8).

For the reverse inclusion assume that x belongs to the right-hand side set of the equality (8). Then, there exist $N \in \mathcal{S}$ and the sequence $\{y_n \mid y_n \in A_n,\ n \in N\}$ such that $\lim_{n\to\infty} y_n = x$. If ε is an arbitrary given positive number, then we can choose such a number $n_0 \in \mathbb{N}$ that for each $n > n_0,\ n \in N$, we have $y_n \in B(x, \varepsilon)$. Define the set M by $M = N \setminus \{1, 2, 3, \ldots, n_0\}$. Then, $M \in \mathcal{S}$ and $y_n \in A_n \cap B(x, \varepsilon)$ for each $n \in M$. This means that the sets A_n and $B(x, \varepsilon)$ are not disjoint. Hence, $x \in st - \liminf_{n\to\infty} A_n$. □

Corollary 2.7. *Let X be a normed linear space and (A_n) be a sequence of closed subsets of X. If there is a set $K \in \mathcal{S}$ such that A_n is convex for each $n \in K$, then $st - \liminf_{n\to\infty} A_n$ is convex and so too, when it exists, is $st - \lim_{n\to\infty} A_n$.*

Proof. Let $st - \liminf_{n\to\infty} A_n = A$. If x_1 and x_2 belong to A, by Proposition 2.6, we can find for all $n \in N$ in some set $N \in \mathcal{S}$ points y_n^1 and y_n^2 in A_n such that $\lim_{n\to\infty} y_n^1 = x_1$ and $\lim_{n\to\infty} y_n^2 = x_2$. Since $K \in \mathcal{S}$, we have $M \in \mathcal{S}$ with $M = N \cap K$. Then for arbitrary $\lambda \in [0, 1]$ and $n \in M$ let us define

$$y_n^\lambda := (1 - \lambda) y_n^1 + \lambda y_n^2 \quad \text{and} \quad x_\lambda := (1 - \lambda) x_1 + \lambda x_2.$$

Then

$$\lim_{n\in M} y_n^\lambda = x_\lambda.$$

By Proposition 2.6, we obtain $x_\lambda \in A$. This means that A is convex. □

Proposition 2.8. *Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X. Then*

$$st - \limsup_{n\to\infty} A_n = \Big\{x \mid \exists N \in \mathcal{S}^{\#},\ \forall n \in N,\ \exists y_n \in A_n : x \in \Gamma_y\Big\}. \tag{9}$$

Proof. Let $x \in st - \limsup_{n\to\infty} A_n$ be arbitrary. By Proposition 2.5,

$$st - \liminf_{n\to\infty} d(x, A_n) = 0.$$

By Lemma 1.3, for every $\varepsilon > 0$ the set

$$\Big\{n \in \mathbb{N} : d(x, A_n) < \frac{\varepsilon}{2}\Big\}$$

does not have density zero. Since A_n is closed, for $n \in \mathbb{N}$, there exists $y_n \in A_n$ such that $d(x, y_n) \le 2d(x, A_n)$. Now, we define the sequence $\{y_n \mid y_n \in A_n,\ n \in \mathbb{N}\}$. Then, clearly, x is a statistical cluster point of (y_n). That is, $x \in \Gamma_y$.

On the contrary, assume that x belongs to the right-hand side set of the equality (9). Then, there exist $N \in \mathcal{S}^{\#}$ and the sequence $\{y_n \mid y_n \in A_n, n \in N\}$ such that $x \in \Gamma_y$. That is, the set $\Big\{n \in \mathbb{N} : d(x, y_n) < \varepsilon\Big\}$ does not have density zero for every $\varepsilon > 0$. The inequality $d(x, y_n) \ge d(x, A_n)$ yields the inclusion

$$\{n \in \mathbb{N} : d(x, y_n) < \varepsilon\Big\} \subseteq \{n \in \mathbb{N} : d(x, A_n) < \varepsilon\Big\}.$$

So, the set

$$N' = \{n \in \mathbb{N} : d(x, A_n) < \varepsilon\Big\}$$

does not have density zero. That is, $N' \in \mathcal{S}^{\#}$. By (6), for every $n \in N'$ we obtain $A_n \cap B(x, \varepsilon) \neq \emptyset$. This means that $x \in st - \limsup_{n\to\infty} A_n$. □

By Proposition 2.6 and Proposition 2.8, note that $st-\liminf_{n\to\infty} A_n$ is the set of statistical limits of the sequences $(y_n)_{n\in\mathbb{N}}$ with $y_n \in A_n$ and $st-\limsup_{n\to\infty} A_n$ is the set of statistical cluster points of the sequences $(y_n)_{n\in\mathbb{N}}$ with $y_n \in A_n$.

Remark 2.9. *In Proposition 2.8 the set of statistical cluster points can not be replaced by the set of statistical limit points. Following Example 4 of [9] we let $y=(y_k)$ be the uniformly distributed sequence*

$$(y_k) = (0,1,0,1/2,1,0,1/3,2/3,1,0,1/4,2/4,3/4,1\ldots)$$

and define

$$(A_k) = (\{y_k\}) = (\{0\},\{1\},\{0\},\{1/2\},\{1\},\{0\},\{1/3\},\{2/3\},\ldots).$$

In this case $st-\limsup_{n\to\infty} A_n = [0,1]$. Because for any $x\in[0,1]$ we have

$$\left\{k\in\mathbb{N} : A_k\cap B(x,\varepsilon)\neq\emptyset\right\} = \{k\in\mathbb{N} : y_k\in(x-\varepsilon,x+\varepsilon)\}.$$

So,

$$\delta\left(\left\{k\in\mathbb{N} : A_k\cap B(x,\varepsilon)\neq\emptyset\right\}\right)\geq\varepsilon>0.$$

On the other hand, if $x\in[0,1]$ and $\{y_k \mid y_k\in A_k,\ k\in N\subseteq\mathbb{N}\}$ is a subsequence that converges to x, then we obtain $\delta(N)=0$. That is $N\notin\mathcal{S}^{\#}$. Therefore

$$\left\{x \mid \exists N\in\mathcal{S}^{\#},\ \forall n\in N,\ \exists y_n\in A_n : \lim_{n\in N} y_n = x\right\}=\emptyset.$$

Consequently

$$st-\limsup_{n\to\infty} A_n \neq \left\{x \mid \exists N\in\mathcal{S}^{\#},\ \forall n\in N,\ \exists y_n\in A_n : \lim_{n\in N} y_n = x\right\}.$$

Lemma 2.10. *Let (A_n) and (B_n) be two sequences of closed subsets of a metric space X. If there is a set $K\in\mathcal{S}$ such that $A_n\subseteq B_n$ for each $n\in K$, then the inclusions*

$$st-\liminf_{n\to\infty} A_n\subseteq st-\liminf_{n\to\infty} B_n \quad \text{and} \quad st-\limsup_{n\to\infty} A_n\subseteq st-\limsup_{n\to\infty} B_n$$

hold.

Proof. Since the second inclusion can be proved in the similar way, we prove only the first inclusion. Suppose that there exists $K\in\mathcal{S}$ such that for each $n\in K$ the inclusion $A_n\subseteq B_n$ holds. In this case for arbitrary $x\in st-\liminf_{n\to\infty} A_n$, we obtain

$$d(x,B_n)\leq d(x,A_n). \tag{10}$$

By Proposition 2.5, we have

$$st-\lim_{n\to\infty} d(x,A_n)=0. \tag{11}$$

Consequently, combining (10) and (11), we have $st-\lim_{n\to\infty} d(x,B_n)=0$. Namely $x\in st-\liminf_{n\to\infty} B_n$. This completes the proof. □

Corollary 2.11. *Let (A_n) and (B_n) be two sequences of closed subsets of a metric space X. Then, the following statements hold:*

(i) $st-\limsup_{n\to\infty}(A_n\cap B_n)\subseteq st-\limsup_{n\to\infty} A_n\cap st-\limsup_{n\to\infty} B_n$.

(ii) $st-\liminf_{n\to\infty}(A_n\cap B_n)\subseteq st-\liminf_{n\to\infty} A_n\cap st-\liminf_{n\to\infty} B_n$.

(iii) $st-\limsup_{n\to\infty}(A_n\cup B_n) = st-\limsup_{n\to\infty} A_n\cup st-\limsup_{n\to\infty} B_n$.

(iv) $st - \liminf_{n\to\infty}(A_n \cup B_n) \supseteq st - \liminf_{n\to\infty} A_n \cup st - \liminf_{n\to\infty} B_n$.

Proof. For each $n \in \mathbb{N}$, the inclusions $A_n \cap B_n \subseteq A_n$, $A_n \cap B_n \subseteq B_n$, $A_n \subseteq A_n \cup B_n$ and $B_n \subseteq A_n \cup B_n$ hold. Now, the proof is immediate by Lemma 2.10. □

We prove two theorems related to Kuratowski statistical convergence. In the finite dimensional case, Salinetti and Wets gave the corresponding results for Kuratowski convergence in [16].

Definition 2.12. *A sequence (A_k) is said to be statistically increasing if there exists a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $A_{k_n} \subseteq A_{k_{n+1}}$ for all $n \in \mathbb{N}$. Similarly, a sequence (A_k) is said to be statistically decreasing if there exists a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $A_{k_n} \supseteq A_{k_{n+1}}$ for all $n \in \mathbb{N}$.*

Theorem 2.13. *Suppose that (A_k) is a statistically increasing sequence of closed subsets of X. Then $st - \lim_{k\to\infty} A_k$ exists and*

$$st - \lim_{k\to\infty} A_k = cl \bigcup_{n\in\mathbb{N}} A_{k_n}.$$

Proof. Let (A_k) be a statistically increasing sequence of closed subsets of X and $A = cl \bigcup_{n\in\mathbb{N}} A_{k_n}$. Then, $A_{k_n} \subseteq A$ for every $n \in \mathbb{N}$. If $A = \emptyset$, then $A_{k_n} = \emptyset$ for every $n \in \mathbb{N}$. So, $st - \lim_{k\to\infty} A_k = \emptyset$. Let $A \neq \emptyset$ and $x \in cl \bigcup_{n\in\mathbb{N}} A_{k_n}$. In this case, for every $\varepsilon > 0$

$$B(x,\varepsilon) \cap \bigcup_{n\in\mathbb{N}} A_{k_n} \neq \emptyset.$$

Then there exists $n_0 \in \mathbb{N}$ such that $B(x,\varepsilon) \cap A_{k_{n_0}} \neq \emptyset$. Since (A_{k_n}) is an increasing sequence, $A_{k_{n_0}} \subseteq A_{k_n}$ for all $n \geq n_0$. Define the set M by

$$M = \{m \mid m = k_n, n \geq n_0, n \in \mathbb{N}\}.$$

Then, $\delta(M) = 1$ and $B(x,\varepsilon) \cap A_m \neq \emptyset$ for all $m \in M$. Consequently, we obtain $x \in st - \liminf_{n\to\infty} A_n$.

Now we show that $st - \limsup_{k\to\infty} A_k \subseteq A$. Let $x \in st - \limsup_{k\to\infty} A_k$ be arbitrary. Then, for every $\varepsilon > 0$ there exists $N \in \mathcal{S}^{\#}$ such that $A_k \cap B(x,\varepsilon) \neq \emptyset$ for every $k \in N$. Since $\delta(K) = 1$ and $\delta(N) \neq 0$, the set $K \cap N$ is nonempty. So, there exists $k_{n_0} \in K \cap N$ such that $B(x,\varepsilon) \cap A_{k_{n_0}} \neq \emptyset$. Therefore we obtain

$$B(x,\varepsilon) \cap \bigcup_{n\in\mathbb{N}} A_{k_n} \neq \emptyset.$$

This means that $x \in cl \bigcup_{n\in\mathbb{N}} A_{k_n}$. This completes the proof. □

Example 2.14. *Define*

$$A_k = \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq \frac{1}{k}\right\} & , \text{ if } k \text{ is a prime number,} \\ \left\{(x,y) \in \mathbb{R}^2 : \ |x| + |y| \leq \frac{k}{k+1}\right\} & , \text{ otherwise.} \end{cases}$$

Let $K = \{1, 4, 6, 8, 9, 10, 12, \ldots\}$. Then, $\delta(K) = 1$ (see [1], p.2) and $A_{k_n} \subseteq A_{k_{n+1}}$ for every $n \in \mathbb{N}$. By Definition 2.12 , we say that (A_k) is statistically increasing sequence but is not increasing. Moreover, by Theorem 2.13, (A_k) is Kuratowski statistically convergent to A, where

$$A = \left\{(x,y) \in \mathbb{R}^2 : \ |x| + |y| \leq 1\right\}.$$

Theorem 2.15. *Suppose that (A_k) is a statistically decreasing sequence of closed subsets of X. Then $st - \lim_{k\to\infty} A_k$ exists and*

$$st - \lim_{k\to\infty} A_k = \bigcap_{n\in\mathbb{N}} A_{k_n}.$$

Proof. Let $A = \bigcap_{n\in\mathbb{N}} A_{k_n}$. Clearly if $x \in A$, then $x \in A_{k_n}$ for every $n \in \mathbb{N}$. Let $M = \{m \mid m = k_n, n \in \mathbb{N}\}$. Then, $\delta(M) = 1$ and $B(x,\varepsilon) \cap A_m \neq \emptyset$ for all $\varepsilon > 0$ and $m \in M$. This means that $x \in st - \liminf_{k\to\infty} A_k$.

Now we show that $st - \limsup_{k\to\infty} A_k \subseteq A$. Let $x \in st - \limsup_{k\to\infty} A_k$ be arbitrary. Then, for every $\varepsilon > 0$ there exists $N \in \mathcal{S}^{\#}$ such that $A_k \cap B(x,\varepsilon) \neq \emptyset$ for every $k \in N$. Since $\delta(N) \neq 0$, there exists $m \in N$ such that $k_n \leq m$ for every $n \in \mathbb{N}$. Since the sequence (A_k) is decreasing, the inclusion $A_{k_n} \supseteq A_m$ holds and consequently $B(x,\varepsilon) \cap A_{k_n} \neq \emptyset$. This means that $x \in clA_{k_n}$. Since A_{k_n} is closed, $x \in A_{k_n}$. Therefore $x \in \bigcap_{n\in\mathbb{N}} A_{k_n}$. This completes the proof. □

Proposition 2.16. *[2, Proposition 10] Let X be a finite-dimensional normed linear space and (A_n) be a sequence of closed convex subsets of X. If $\lim_{n\to\infty} A_n = A \neq \emptyset$ with A compact. Then, $\bigcup_{n=1}^{\infty} A_n$ is bounded.*

Now, we give an example which shows that Proposition 2.16 is not valid for Kuratowski statistical convergence.

Example 2.17. *Define (A_n) by*

$$A_n := \begin{cases} [-n,n] & , \quad n = k^2 \text{ for } k \in \mathbb{N}, \\ [1,2] & , \quad \text{otherwise.} \end{cases}$$

Then, (A_n) is a sequence of closed convex subsets of $\mathbb{R}$ and $st - \lim_{n\to\infty} A_n = [1,2]$. However, $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}$ is not bounded.

In the next section we compare Kuratowski statistical convergence with Hausdorff statistical convergence, introduced by Nuray and Rhoades [12].

3. Hausdorff Statistical Convergence

We mention some references related to Hausdorff convergence: [2, 3, 10, 16, 17]. The Hausdorff distance $h(E,F)$ between the subsets E and F of X is defined as follows:

$$h(E,F) = \max\{D(E,F), D(F,E)\},$$

where

$$D(E,F) = \sup_{x\in E} d(x,F) = \inf\{\varepsilon > 0 : E \subseteq B(F,\varepsilon)\}$$

unless both E and F are empty in which case $h(E,F) = 0$. Note that if only one of the two sets is empty then $h(E,F) = \infty$.

It is known, for a long time (see [2, 10]), that

$$h(E,F) = \sup_{x\in X} |d(x,E) - d(x,F)|. \tag{12}$$

Definition 3.1. *[12] Let (X,d) be a metric space and (A_n) be a sequence of closed subsets of X. We say that the sequence (A_n) is Hausdorff statistically convergent to a closed subset A of X if*

$$st - \lim_{n\to\infty} h(A_n, A) = 0. \tag{13}$$

In this case, we write $A = st_H - \lim_{n\to\infty} A_n$.

Lemma 3.2. *Suppose that $\{A; A_n, n \in \mathbb{N}\}$ is a family of closed subsets of X. Then $A = st_H - \lim_{n\to\infty} A_n$ if and only if either there exists $M \in \mathcal{S}$ such that A and A_n are empty for all $n \in M$ or for any $\varepsilon > 0$ the sets*

$$\left\{n \in \mathbb{N} : A \not\subseteq B(A_n,\varepsilon)\right\} \quad \text{and} \quad \left\{n \in \mathbb{N} : A_n \not\subseteq B(A,\varepsilon)\right\} \tag{14}$$

have density zero.

Proof. If $A = \emptyset$, then

$$\left\{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon\right\} = \left\{n \in \mathbb{N} : A_n \neq \emptyset\right\}.$$

Thus,

$$\delta\left(\left\{n \in \mathbb{N} : A_n \neq \emptyset\right\}\right) = 0.$$

Namely

$$\delta\left(\left\{n \in \mathbb{N} : A_n = \emptyset\right\}\right) = 1.$$

Conversely, there exists $M \in \mathcal{S}$ such that A_n is empty for all $n \in M$. Then, it is clear that $A = \emptyset$.

On the other hand if $A \neq \emptyset$, then (13) holds if and only if

$$\delta\left(\left\{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon\right\}\right) = 0$$

or equivalently,

$$\delta\left(\left\{n \in \mathbb{N} : h(A_n, A) < \varepsilon\right\}\right) = 1$$

for any $\varepsilon > 0$. By the definition of Hausdorff metric,

$$\delta\left(\left\{n \in \mathbb{N} : A \subseteq B(A_n, \varepsilon) \text{ and } A_n \subseteq B(A, \varepsilon)\right\}\right) = 1.$$

Consequently, we obtain that the sets in (14) have density zero. □

Theorem 3.3. *Suppose that $\{A; A_n, n \in \mathbb{N}\}$ is a family of closed subsets of X with A nonempty. Then Hausdorff statistical convergence implies Kuratowski statistical convergence, i.e. $st_H - \lim_{n\to\infty} A_n = A$ implies $st - \lim_{n\to\infty} A_n = A$.*

Proof. If $A = \emptyset$, then $\delta\left(\left\{n \in \mathbb{N} : A_n = \emptyset\right\}\right) = 1$. Hence, $st - \lim_{n\to\infty} A_n = \emptyset$. Let us suppose that A and A_n are nonempty for every $n \in \mathbb{N}$. By the equality (12),

$$st - \lim_{n\to\infty} d(x, A_n) = d(x, A) \text{ for each } x \in X. \tag{15}$$

Take $x \in A$. Then, we have

$$st - \lim_{n\to\infty} d(x, A_n) = d(x, A) = 0.$$

By Proposition 2.5, this implies $x \in st - \liminf_{n\to\infty} A_n$. Consequently, we obtain

$$A \subseteq st - \liminf_{n\to\infty} A_n.$$

Conversely, take $x \in st - \limsup_{n\to\infty} A_n$. Again, one can derive from Proposition 2.5 that

$$st - \liminf_{n\to\infty} d(x, A_n) = 0.$$

By (15), we obtain

$$d(x, A) = st - \lim_{n\to\infty} d(x, A_n) = 0.$$

So, $x \in A$. Therefore, we conclude from the inclusion relation

$$st - \limsup_{n\to\infty} A_n \subseteq A \subseteq st - \liminf_{n\to\infty} A_n$$

that $A = st - \lim_{n\to\infty} A_n$. □

The converse of Theorem 3.3 does not hold, in general. To see this, we give the following example.

Example 3.4. *Define the sequence* (A_n) *by*

$$A_n := \begin{cases} [2,3] & , \quad n = k^3 \text{ for } k \in \mathbb{N}, \\ [0,1] \cup \{n\} & , \quad \text{otherwise}. \end{cases}$$

(A_n) *is Kuratowski statistically convergent to* $[0,1]$*. However,* (A_n) *is not Hausdorff statistically convergent.*

Definition 3.5. *The sequence* (A_n) *is said to be statistically bounded if there exists a compact set* K *such that* $\delta\Big(\Big\{n \in \mathbb{N} : A_n \nsubseteq K\Big\}\Big) = 0$.

It is natural to ask under which conditions Kuratowski and Hausdorff statistical convergence are identical? The answer is given by the following theorem.

Theorem 3.6. *Let* (A_n) *be a statistically bounded sequence of closed subsets of* X*. If* $st - \lim_{n\to\infty} A_n = A$ *with* $A \neq \emptyset$*, then* $st_H - \lim_{n\to\infty} A_n = A$.

Proof. Let (A_n) be a statistically bounded sequence of closed subsets of X. Then, there exist a compact set K and $M \in \mathcal{S}$ such that $A_n \subseteq K$ for all $n \in M$. By Lemma 2.10, $st - \lim_{n\to\infty} A_n = A \subseteq K$. So, the closed set A is compact. Then given $\varepsilon > 0$, A has a finite cover with open balls of radius ε; i.e. there exists $\{x_1, x_2, x_3 \ldots, x_n\}$ with $x_i \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{\varepsilon}{2}\right).$$

Since $st - \lim_{n\to\infty} A_n = A$ and $x_i \in A$ for $i \in \{1, 2, \ldots, n\}$, we obtain $st - \lim_{n\to\infty} d(x_i, A_n) = 0$. Therefore, there exists $N_i = \{n \in \mathbb{N} : d(x_i, A_n) < \varepsilon/2\}$ such that $\delta(N_i) = 1$ for each i. Let us define $N = \bigcap_{i=1}^{n} N_i$. Then, $\delta(N) = 1$. Thus, we obtain $d(y, A_n) \leq d(y, x_i) + d(x_i, A_n) < \varepsilon$ for any $y \in A$ and $n \in N$. So, $A \subseteq B(A_n, \varepsilon)$ for every $n \in N$. This means that $\delta\left(\left\{n \in \mathbb{N} : A \nsubseteq B(A_n, \varepsilon)\right\}\right) = 0$. Now, suppose that $C = \left\{n \in \mathbb{N} : A_n \nsubseteq B(A, \varepsilon)\right\}$ for some $\varepsilon > 0$ does not have density zero. Since $\delta(M) = 1$, we have $\delta(M \cap C) \neq 0$. Hence, there exists a sequence $\{y_k \mid y_k \in A_k \backslash B(A, \varepsilon),\ k \in M \cap C\} \subseteq K$. By Lemma 1.4, the sequence (y_n) has at least statistical cluster point that belongs to $st - \limsup_{n\to\infty} A_n = A$ but does not belong to $B(A, \varepsilon) \supseteq A$, a contradiction. This gives that $\delta\left(\left\{n \in \mathbb{N} : A_n \nsubseteq B(A, \varepsilon)\right\}\right) = 0$, which completes the proof. □

Conclusion

In literature, there are different definitions for convergence of set-valued sequences. The best known of them are Kuratowski convergence, Hausdorff convergence, Wijsman convergence, Mosco convergence and scalar convergence. Statistical convergence for set-valued sequences was first defined by Nuray and Rhoades [12]. They studied Hausdorff and Wijsman statistical convergence. In the present paper, based on the definitions of inner and outer limits given by Rockafellar and Wets [14], we introduce the statistical inner and outer limits, and investigate some of their properties. Later, we define the Kuratowski statistical convergence and give the results related to this concept corresponding to the results on Kuratowski convergence due to Salinetti and Wets [16]. Furthermore, we compare the Hausdorff statistical convergence with the Kuratowski statistical convergence.

It is natural to study statistical convergence for other types of convergence of set-valued sequences. In the light of the main results of our paper, one can provide ways of statistically approximating set-valued mappings through convergence of graphs and extended real-valued functions through convergence of epigraphs.

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