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L-Subquantales and L-Filters of Quantales Based on Quasi-Coincidence of Fuzzy Points with Parameters

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Abstract. The paper investigates fuzziness of quantales by means of quasi-coincidence of fuzzy points with two parameters based on L-sets and developes two more generalized fuzzy structures, called $(\in_g, \in_g \vee q_h)$-L-subquantale and $(\in_g, \in_g \vee q_h)$-L-filter. Some intrinsic connections between $(\in_g, \in_g \vee q_h)$-L-subquantales and crisp subquantales are established, and relationships between $(\in_g, \in_g \vee q_h)$-L-filters of quantales and their extensions (especially the essential connections between $(\in_g, \in_g \vee q_h)$-L-subquantales and $(\in_g, \in_g \vee q_h)$-L-filters of quantales) are studied by employing the new characterizations of $(\in_g, \in_g \vee q_h)$-L-filters of quantales. Also, sufficient conditions for the extension of an $(\in_g, \in_g \vee q_h)$-L-filter to be an $(\in_g, \in_g \vee q_h)$-L-filter of a quantale are also offered. In particular, it is proved that the category **GLFquant** (resp., **GFFQant**) of $(\in_g, \in_g \vee q_h)$ L-subquantales (resp., L-filters) is of a topological construct on **Quant** and posses equalizers and pullbacks.

1. Introduction

Fuzzy set theory, originally proposed by Zadeh [53], has provided a useful mathematical tool for the description of the behaviors of those systems which are too complex or uncertain to be precisely analyzed by classical mathematical methods and tools. Furthermore, to describe those situations involving uncertainties or ambiguities more concretely, Goguen [12] replaced the unit interval [0,1] by a lattice and proposed L-fuzzy sets (or L-sets for short). Since than, fuzzy set theory has opened up keen insights and applications in a wide range of scientific fields such as information systems, control engineering, expert systems, management science, operations research, pattern recognition and others. How to apply fuzzy sets to the lattice-ordered environment, as an important branch of this field, has attracted widespread attention of researchers and has become a rapidly progressing research field (see [8, 10, 40, 47, 50]) in recent years since fuzzy lattices have been widely used in engineering, computer science, topology, logic and so on ([17, 31, 32]). On the other hand, fuzzy algebra has also become a promising topic (see [6, 7, 43–46, 49, 52]) since fuzzy algebraic structures have been successfully applied to many other fields such as information science, coding theory, topology logic, measure theories, etc.

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As all know, quantales with both lattice-ordered structures and algebraic structures provide a lattice setting of the study of non-commutative C^*-algebras and constructive foundations of the study of quantum mechanics (see [28]). In 1990, Yetter applied quantale theory to linear logic and provided a sound and complete class of models for linear intuitionstic logic [42]. Henceforth, quantales have invoked many interesting research topics in theoretical computer science [33], algebraic theory [15, 18, 51], rough set theory [24, 41], groupoid theory [33], linear logic (see [11, 34]), topological theory [14], etc. Based on the aforementioned analysis, the study combining fuzzy sets and quantales may become a promising topic that deserves further investigation. Recently, Ma et al. studied quantales based on fuzzy sets (see [27]). Before long, Liang introduced L-fuzzy quantales based on L-sets in [19]. On the other hand, many researchers have been generalizing some different types of fuzzy mathematics by the quasi-coincidence of a fuzzy point with a fuzzy set mentioned in [30] (see [4, 25, 26, 29, 52]). Inspired by this, Xiao further generalized L-fuzzy quantales to $(\in, \in \vee q)$-L-quantales by quasi-coincidence of a fuzzy point based on an L-set and discussed related properties (see [38]). Some results in [38] are obviously important and interesting, but not complete. So, there are at least the following two problems need to be considered:

Question 1.1 Can we find a new kind of L-quantale that posses more abundant sheaf structures than those of [38]?

Question 1.2 If the answer for Question 1.1 is "yes", then what is the characterization of it?

One of our main purposes is to answer the above questions. We define a new concept of quasi-coincidence of a fuzzy point with two parameters on an L-set, which has broken the limitation that quasi-coincidence of a fuzzy point with two parameters must depend on fuzzy sets. We then apply it to quantale theory and present a more generalized structure, called $(\in_g, \in_g \vee q_h)$-L-subquantales, which provides a solid background for the subsequent researches. This approach will be helpful for us to make a more accurate understanding for the quantale operations occurring in fuzzy points of L-sets.

It is no doubt that filters are very important tools in many areas of classical mathematics such as topology theory and measure theory. From a logical point of view, different filters correspond to different sets of provable formulae (see [48]). Moreover, filters are closely related to congruence relations (see [9]). Therefore, more and more researchers have been focusing on this topic (see [16, 20–22, 39, 45, 49]). Furthermore Wang et al. introduced fuzzy filters on quantales in [37]. Motivated by the idea of the generalized L-fuzzy subquantales, our another main aim in this paper is to study a new kind of fuzzy filter named $(\in_g, \in_g \vee q_h) - L$-filter which is of course a reasonable generalization of fuzzy filter in [37]. We also hope that the fuzziness of filters can induce some new applications in the fields of logic, computer science, topology, etc.

Now, a natural question arises:

Question 1.3 Are there any connections between $(\in_g, \in_g \vee q_h)$-L-subquantales and $(\in_g, \in_g \vee q_h)$-L-filters of quantales?

We will in this paper discuss some intrinsic connections between $(\in_g, \in_g \vee q_h)$-L-subquantales and $(\in_g, \in_g \vee q_h)$-L-filters of quantales (see Theorem 4.17 and Remark 4.18).

It is well known that category theory is not only a tool commonly used by many pure mathematicians, but also a tie which can easily connect mathematics and theoretical computer science (see [1, 3, 36]). In the past few years, some researchers have endeavored to establish connections between the quantale theory and category theory. Many interesting results have been obtained (see [2, 5, 13, 23]). Inspired by this, we will also further investigate the characterizations of $(\in_g, \in_g \vee q_h)$-L-subquantales and $(\in_g, \in_g \vee q_h)$-L-filters of quantales based on the category theory.

The rest contents of this paper are arranged as follows: In section 2, we introduce the basic notions and properties which will be used in the paper. Section 3 is devoted to presenting the concept of $(\in_g, \in_g \vee q_h)$-L-subquantales of quantales and discussing related properties of them. $(\in_g, \in_g \vee q_h)$-L-filters of quantales are studied in section 4. Finally, in section 5, we further investigate the properties of the category **GLFquant** (resp., **GFFQant**) of $(\in_g, \in_g \vee q_h)$-L-subquantales (resp., L-filters) of quantales.

2. Preliminaries

We in this section mostly recall some elementary notions and facts related to quantales, L-sets and L-quantales (see [12, 34, 38]) which will be often used in this paper.

Definition 2.1. ([33]) A quantale $Q = (Q, \&)$ is a complete lattice Q with an associative binary operation & satisfying

$$a\&\left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a\&b_i) \quad \text{and} \quad \left(\bigvee_{i\in I} a_i\right)\&b = \bigvee_{i\in I}(a_i\&b)$$

for all $a, b, a_i, b_i \in Q$ $(i \in I)$, where I is an index set.

It is easy to see from Definition 2.1 that in a quantale Q, for all $a, b, c \in Q$, we have the following results.

$$b \leq c \Longrightarrow a\&b \leq a\&c \quad \text{and} \quad b\&a \leq c\&a.$$

Definition 2.2. ([33]) Let Q be a quantale. A subset S of Q is said to be a subquantale of Q if S is closed under sups and &.

Definition 2.3. ([33]) Let Q be a quantale. A nonempty subset B of Q is said to be a filter of Q if B is an upper set and closed under &.

Definition 2.4. ([33]) Let P and Q be quantales. A function $f : P \to Q$ is said to be a homomorphism of quantales if f preserves arbitrary sups and the operation &.

Remark 2.5. The category of quantales and homomorphisms is denote by **Quant**.

Definition 2.6. ([12]) Let $\mu \in L^X$ and $a \in L$. Then the L-set $a\mu$, defined by $a\mu(x) = a \wedge \mu(x)$, is called a-layer of A (briefly, a layer of A).

Definition 2.7. ([38]) Let μ, ν be L-sets of a quantale Q. Then $\mu\&\nu$, $\vee\mu$ and $\mu \vee \nu$ are defined as $(\mu\&\nu)(x) = \sup\limits_{x=a\&b}(\mu(a) \wedge \nu(b))$, $\vee\mu(x) = \bigvee\limits_{x=\vee x_i}(\wedge\mu(x_i))$ and $(\mu \vee \nu)(x) = \mu(x) \vee \nu(x)$, respectively.

Throughout this paper, Q and L denote a quantale and a complete lattice, respectively. We consider that $g, h, t, m \in L$ and $g < h$.

The standard terminology of category theories see [1].

3. $(\in_g, \in_g \vee q_h)$-L-Subquantales of Quantales

In this section, $(\in_g, \in_g \vee q_h)$-L-subquantales of quantales will be discussed. Meanwhile, the relationships between $(\in_g, \in_g \vee q_h)$-L-subquantales of quantales and crisp subquantales are going to be established.

Definition 3.1. [38] An L-set of a set X with the form

$$\mu(y) = \begin{cases} t(\neq 0) & \text{if} \quad y = x, \\ 0 & \text{if} \quad y \neq x, \end{cases}$$

is called a fuzzy point with support x and value t, denoted by x_t. Let 1 and 0 be the top and bottom elements of L respectively. When $\mu(x) \geq t$ (resp., $\mu(x) \vee t = 1$), a fuzzy point x_t is called "belong to "(resp., "quasi-coincident with") an L-set μ, written as $x_t \in \mu$ (resp., $x_t \in q\mu$). We say $x_t \in \vee q\mu$ if $x_t \in \mu$ or $x_t \in q\mu$.

Definition 3.2. ([38]) Let Q be a quantale and L be a complete lattice. An L-set μ of Q is called an $(\in, \in \vee q)$-L-quantale if it satisfies the following conditions:

(i) For each $x, y \in Q$ and every $t, s \in L$, $(x\&y)_{t\wedge s} \in \vee q\mu$ whenever $x_t \in \mu$ and $y_s \in \mu$;

(ii) For each $\{x_i\}_{i\in I} \subseteq Q$ and every $\{t_i\}_{i\in I} \subseteq L$, $\left(\bigvee\limits_{i\in I} x_i\right)_{\bigwedge\limits_{i\in I} t_i} \in \vee q\mu$ whenever $\{(x_i)_{t_i}\}_{i\in I} \subseteq \mu$.

Taking full advantages of parameters, we generalize the relations, called " belong to" and "quasi-coincident with", on L-sets as follows .

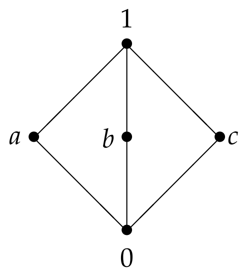
Definition 3.3. For a fuzzy point x_r and an L-set μ of a nonempty set X, we denote that
(i) $x_r \in_g \mu$ if $\mu(x) \geq r > g$;
(ii) $x_r q_h \mu$ if $\mu(x) \vee r \geq h$;
(iii) $x_r \in_g \vee q_h \mu$ if $x_r \in_g \mu$ or $x_r q_h \mu$;
(iv) $x_r \overline{\in_g \vee q_h} \mu$ if $x_r \overline{\in_g} \mu$ and $x_r \overline{q_h} \mu$.

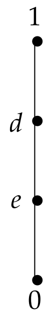
Definition 3.4. If from the condition $x_t \in_g \mu$ we have $x_t \in_g \vee q_h \nu$ for all $x \in X, r \in (g, 1], \mu, \nu \in L^X$. Then we say that $\mu \Subset_{(g,h)} \nu$.

Definition 3.5. An L-set μ of Q is called an $(\in_g, \in_g \vee q_h)$-L-subquantale if it satisfies that
$(1A)$ For every $x, y \in Q$ and each $m, n \in L$, if from conditions $x_m \in_g \mu$ and $y_n \in_g \mu$ we have $(x \& y)_{m \wedge n} \in_g \vee q_h \mu$;
$(2A)$ For every $\{x_i\}_{i \in I} \subseteq Q$ and each $\{m_i\}_{i \in I} \subseteq L$, if from the condition $\{(x_i)_{m_i}\}_{i \in I} \in_g \mu$ we can obtain that $(\bigvee\limits_{i \in I} x_i)_{\bigwedge\limits_{i \in I} m_i} \in_g \vee q_h \mu$.

Example 3.6. Let $Q = \{0, a, b, c, 1\}$ and $L = \{0, d, e, 1\}$. The partial-order of Q and L, as well as the binary operation & of Q, are given as

&	0	a	b	c	1
0	0	0	0	0	0
a	0	b	c	a	1
b	0	c	a	b	1
c	0	a	b	c	1
1	1	1	1	1	1





It is then easy to verify that $(Q, \&)$ is a quantale and L is a complete lattice. Define an L-set μ of Q as $\mu(1) = \mu(0) = \mu(c) = 1, \mu(a) = \mu(b) = d$. Then it is not difficult to check that μ is an $(\in_e, \in_e \vee q_d)$-L-subquantale.

Lemma 3.7. *Suppose that μ is an L-set of Q. Then $(1A)$ holds if and only if one of the following conditions holds:*
$(1B)$ For each $x, y \in Q$, we can obtain that $\mu(x \& y) \vee g \geq \mu(x) \wedge \mu(y) \wedge h$;
$(1C)$ $(\mu \& \mu) \Subset_{(g,h)} \mu$.

Proof. $(1A) \Rightarrow (1B)$ For each $x, y \in Q$, suppose that $\mu(x \& y) \vee g < m \leq \mu(x) \wedge \mu(y) \wedge h$. We can then acquire that $\mu(x) \geq m > g, \mu(y) \geq m > g$ and $\mu(x \& y) < m < h$. Therefore, $x_m \in_g \mu$, $y_m \in_g \mu$, but $(x \& y)_m \overline{\in_g \vee q_h} \mu$, which contradicts $(1A)$. Thus $(1B)$ follows.

$(1B) \Rightarrow (1C)$ Let $(\mu \& \mu) \overline{\Subset_{(g,h)}} \mu$. Then there exist $x_m \in_g (\mu \& \mu)$ such that $x_m \overline{\in_g \vee q_h} \mu$. Thus, $(\mu \& \mu)(x) \geq m > g, \mu(x) < m$ and $\mu(x) \vee m < h$. It follows that $\mu(x) < h$. If $x = a \& b$ for some $a, b \in Q$, then by $(1B)$, we infer

$$\mu(x) \vee g \geq \mu(a) \wedge \mu(b) \wedge h.$$

Moreover, since $\mu(x) < h$ and $g < h$, $\mu(x) \vee g \geq \mu(a) \wedge \mu(b)$. So we obtain that

$$\begin{aligned} m \leq (\mu \& \mu)(x) &= \sup_{x = a \& b} (\mu(a) \wedge \mu(b)) \\ &\leq \sup_{x = a \& b} (\mu(x) \vee g) \\ &= \mu(x) \vee g. \end{aligned}$$

It is obvious a contradiction. Therefore $(1C)$ follows.

$(1C) \Rightarrow (1A)$ Consider any $m, n \in L$ and $x, y \in Q$ with properties $x_m \in_g \mu$ and $y_n \in_g \mu$. Then $\mu(x) \geq m > g$ and $\mu(y) \geq n > g$. Hence for all $z = x\&y$, we can acquire that

$$(\mu\&\mu)(z) = \sup_{z=x\&y} (\mu(x) \wedge \mu(y)) \geq \mu(x) \wedge \mu(y) \geq m \wedge n > g.$$

It follows that $(x\&y)_{m\wedge n} \in_g (\mu\&\mu)$. By (1C), we know that $(x\&y)_{m\wedge n} \in_g \vee q_h \mu$. So condition $(1A)$ holds. □

Lemma 3.8. *Let μ be an L-set of Q. Then* $(2A)$ *holds if and only if one of the following conditions holds:*
$(2B)$ *For every* $\{x_i\}_{i\in I} \subseteq Q$, $\mu(\bigvee_{i\in I} x_i) \vee g \geq (\bigwedge_{i\in I} \mu(x_i)) \wedge h$;
$(2C)$ $\vee\, \mu \Subset_{(g,h)} \mu$.

Proof. The proof is similar to that of Lemma 3.7. □

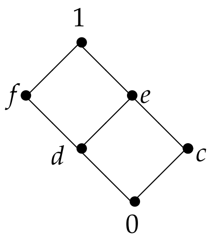
From the above discussion we can immediately obtain the following results.

Theorem 3.9. *An L-set μ of Q is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q if and only if it satisfies* $(1B)$ *and* $(2B)$.

Theorem 3.10. *An L-set μ of Q is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q if and only if it satisfies* $(1C)$ *and* $(2C)$.

Remark 3.11. When $g = 0$ and $h = 1$, the $(\in_g, \in_g \vee q_h)$-L-subquantale of Q is just an $(\in, \in \vee q)$-L-quantale of Q mentioned in [38]. However, the converse does not hold in general. This can be seen in the following example.

Example 3.12. Consider the quantale Q in Example 3.6. Let $L = \{0, c, d, e, f, 1\}$ and the partial-order of L is given as



Define an L-set of Q as $\mu(1) = \mu(c) = \mu(0) = 1, \mu(a) = f, \mu(b) = e$. Then it is easy to prove that μ is an $(\in_e, \in_e \vee q_1)$-L-subquantale but not an $(\in, \in \vee q)$-L-quantale, as $\mu(a\&a) = \mu(b) = e \not\geq f = \mu(a) \wedge \mu(a)$.

In order to investigate the properties of $(\in_g, \in_g \vee q_h)$-L-subquantale of Q, we first give the following notations.

$\mu_t^g = \{x \in Q | x_t \in_g \mu\}$, $\mu_t^h = \{x \in Q | x_t q_h \mu\}$ and $[\mu]_t^h = \{x \in Q | x_t \in_g \vee q_h \mu\}$, where $\mu \in L^Q$ and $t, g, h \in L$ with properties $g < t \leq 1$ and $g < h$.

The following theorem proclaims the relationships between $(\in_g, \in_g \vee q_h)$-L-subquantales and crisp quantales of Q.

Theorem 3.13. *Assume that L is a completely distributive lattice and μ is an L-set of Q. Then we have*
(i) μ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q if and only if μ_t^g is a subquantale of Q for all $t \in L$ and $g < t \leq h$;
(ii) μ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q if and only if μ_t^h is a subquantale of Q for all $t \in L$ and $0 \leq t < h$;
(iii) μ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q if and only if $[\mu]_t^h$ is a subquantale of Q for all $t \in L$ and $g < t < h$.

Proof. (i) Let us first assume that μ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q and $x, y \in \mu_t^g$. We can obtain that $x_t \in_g \mu$ and $y_t \in_g \mu$, that is, $\mu(x) \geq t > g$ and $\mu(y) \geq t > g$. Then by $(1B)$, we know that $\mu(x\&y) \geq \mu(x) \wedge \mu(y) \wedge h$. So we can acquire that $\mu(x\&y) \geq \mu(x) \wedge \mu(y) \geq t > g$. Thus $x\&y \in \mu_t^g$. Similarly, we can prove that μ_t^g is closed under *sups*.

Conversely, suppose that μ_t^g is a subquantale of Q and $\mu(x\&y) \vee g < t \leq \mu(x) \wedge \mu(y) \wedge h$ for all $x, y \in Q, g < t \leq h$. Then $\mu(x) \geq t > g, \mu(y) \geq t > g$ and $\mu(x\&y) < t$, that is, $x \in \mu_t^g, y \in \mu_t^g$ but $x\&y \notin \mu_t^g$, which is obvious a contradiction. Thus (1B) holds. By the same argument, we can show that (2B) holds. Therefore, μ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q.

(ii) Let μ be an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q and $x, y \in \mu_t^h, t \in L$ with the property $0 \le t < h$. Then $x_t q_h \mu$ and $y_t q_h \mu$, that is, $\mu(x) \vee t \ge h$ and $\mu(y) \vee t \ge h$. Since μ is an $(\in_g, \in_g \vee q_h)$-L-subquantale, we have $\mu(x\&y) \vee g \ge \mu(x) \wedge \mu(y) \wedge h$. So we infer

$$\begin{aligned}(\mu(x\&y) \vee t) \vee (g \vee t) &= (\mu(x\&y) \vee g) \vee t\\ &\ge (\mu(x) \wedge \mu(y) \wedge h) \vee t\\ &= (\mu(x) \vee t) \wedge (\mu(y) \vee t) \wedge (h \vee t)\\ &\ge h.\end{aligned}$$

From $t < h$ and $g < h$, we conclude that $\mu(x\&y) \vee t \ge h$, that is, $x\&y \in \mu_t^h$. Analogously, we can prove that μ_t^h is also closed under *sups*. From this fact, we can easily know that μ is a subquantale of Q.

Conversely, assume that the given conditions hold. If there exist $x, y \in Q$ such that $\mu(x\&y) \vee g < t \le \mu(x) \wedge \mu(y) \wedge h$, then $\mu(x\&y) < h, \mu(x) \ge t$ and $\mu(y) \ge t$. Taking $\mu(x) \ge h$ and $\mu(y) \ge h$, we then obtain that $\mu(x) \vee t \ge h, \mu(y) \vee t \ge h$ and $\mu(x\&y) \vee t < h$, for every $t < h$. Therefore, we have $x_t q_h \mu$, $y_t q_h \mu$ and $(x\&y)_t \overline{q_h} \mu$, that is, $x \in \mu_t^h, y \in \mu_t^h$ and $(x\&y) \notin \mu_t^h$. This is evident a contradiction. Thus μ satisfies (1B). In the same way, we can show that (2B) holds. Thus μ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q.

(iii) Let μ be an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q and $x, y \in [\mu]_t^h$ for some $g < t < h$. Then we can easily obtain that $x_t \in_g \vee q_h \mu$ and $y_t \in_g \vee q_h \mu$, that is, $\mu(x) \ge t > g$ or $\mu(x) \vee t \ge h$ and $\mu(y) \ge t > g$ or $\mu(y) \vee t \ge h$. Meanwhile, from the conditions $\mu(x) \vee t \ge h$ and $t < h$, we have $\mu(x) \ge h > g$. Likewise, if $\mu(y) \vee t \ge h$, then $\mu(y) > g$. Therefore, we have $\mu(x) \wedge \mu(y) \wedge h > g$. On the other hand, because μ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q, we can easily know that $\mu(x\&y) \vee g \ge \mu(x) \wedge \mu(y) \wedge h$. So we have $\mu(x\&y) \ge \mu(x) \wedge \mu(y) \wedge h$. We now consider the following cases.
Case 1: When $\mu(x) \ge t > g$ and $\mu(y) \ge t > g$, we can easily obtain that

$$\mu(x\&y) \ge \mu(x) \wedge \mu(y) \wedge h \ge t > g.$$

So we deduce $(x\&y)_t \in_g \mu$.
Case 2: When $\mu(x) \vee t \ge h$ and $\mu(y) \vee t \ge h$, We can acquire that

$$\mu(x\&y) \vee t \ge (\mu(x) \wedge \mu(y) \wedge h) \vee t = (\mu(x) \vee t) \wedge (\mu(y) \vee t) \wedge (h \vee t) \ge h.$$

Therefore, we have $(x\&y)_t q_h \mu$.
Case 3: When $\mu(x) \ge t > g$ and $\mu(y) \vee t \ge h$, from the condition $g < t < h$, we can easily know that $\mu(y) \ge h > t > g$. It follows that

$$\mu(x\&y) \ge \mu(x) \wedge \mu(y) \wedge h \ge t > g.$$

Hence, we conclude $(x\&y)_t \in_g \mu$.
Case 4: When $\mu(x) \vee t > h$ and $\mu(y) \ge t > g$, similar to Case 3, we have $(x\&y)_t \in_g \mu$.
So we can draw the conclusion that $(x\&y)_t \in_g \vee q_h \mu$ in any case, that is, $x\&y \in [\mu]_t^h$. In the same way, we can prove that $[\mu]_t^h$ is closed under *sups*. Thus $[\mu]_t^h$ is a subquantale of Q.

Conversely, Let $[\mu]_t^h$ be a subquantale of Q for all $g < t < h$ and $\mu(x\&y) \vee g < t \le \mu(x) \wedge \mu(y) \wedge h$ for all $x, y \in Q$. Then we have $\mu(x) \ge t > g, \mu(y) \ge t > g$ and $\mu(x\&y) < t \le h$, that is, $x_t \in_g \mu, y_t \in_g \mu$ but $\mu(x\&y)_t \overline{\in_g \vee q_h} \mu$, which means that $x, y \in [\mu]_t^h$ but $x\&y \notin [\mu]_t^h$. This is obvious a contradiction. Whence, we have $\mu(x\&y) \vee g \ge \mu(x) \wedge \mu(y) \wedge h$, that is, (1$B$) holds. Similarly we can show that (2B) is valid. Thus μ is an $(\in_g, \in_g \vee q_h)$-L-subquantale. □

We now discuss the relationships of two $(\in_g, \in_g \vee q_h)$-L-subquantales of Q on binary operations & and $\vee$.

Theorem 3.14. *Let μ and ν be two $(\in_g, \in_g \vee q_h)$-L-subquantales of Q and L be a completely distributive lattice. Then $\mu\&\nu$ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q.*

Proof. Assume that μ and ν are two $(\in_g, \in_g \vee q_h)$-L-subquantales of Q and $x, y \in Q$. Then we can obtain that

$$\begin{aligned}(\mu\&\nu)(x) \wedge (\mu\&\nu)(y) \wedge h &= \bigvee_{x=a\&b} (\mu(a) \wedge \nu(b)) \wedge \bigvee_{y=c\&d} (\mu(c) \wedge \nu(d)) \wedge h \\ &= \bigvee_{x=a\&b, y=c\&d} ((\mu(a) \wedge \mu(c) \wedge h) \wedge (\nu(b) \wedge \nu(d) \wedge h)) \\ &\leq \bigvee_{x\&y=a\&b\&c\&d} ((\mu(a\&c) \vee g) \wedge (\nu(b\&d) \vee g)) \\ &\leq \bigvee_{x\&y=e\&f} (\mu(e) \wedge \nu(f) \vee g) \\ &= (\mu\&\nu)(x\&y) \vee g.\end{aligned}$$

So $(1B)$ is valid.

On the other hand, let $\{x_i\}_{i\in I} \subseteq Q$. Then we have

$$\begin{aligned}(\bigwedge_{i\in I}(\mu\&\nu)(x_i)) \wedge h &= \left(\bigwedge_{i\in I} \bigvee_{x_i=a_i\&b_i} \mu(a_i) \wedge \nu(b_i)\right) \wedge h \\ &= \bigvee_{x_i=a_i\&b_i} \left(\left((\bigwedge_{i\in I} \mu(a_i)) \wedge h\right) \wedge \left((\bigwedge_{i\in I} \nu(b_i)) \wedge h\right)\right) \\ &\leq \bigvee_{\vee x_i=(\vee a_i)\&(\vee b_i)} ((\mu(\vee a_i) \vee g) \wedge (\nu(\vee b_i) \vee g)) \\ &= \bigvee_{\vee x_i=(\vee a_i)\&(\vee b_i)} ((\mu(\vee a_i) \wedge \nu(\vee b_i)) \vee g) \\ &= (\mu\&\mu)(\vee x_i) \vee g.\end{aligned}$$

Whence, $(2B)$ holds. Summing up the above statements, we can easily know that $\mu\&\nu$ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q. □

Theorem 3.15. *Let μ and ν be two $(\in_g, \in_g \vee q_h)$-L-subquantales of Q and L be a completely distributive lattice. Then $\mu \vee \nu$ is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q.*

Proof. The proof runs parallel to that of Theorem 3.14. □

4. $(\in_g, \in_g \vee q_h)$-L-Filters of Quantales

In this section, we investigate an $(\in_g, \in_g \vee q_h)$-L-filter of a quantale which is a generalization of an L-filter mentioned in [37]. Furthermore, we compare $(\in_g, \in_g \vee q_h)$-L-filters of quantales with their extensions and give the conditions which can guarantee the extension of an $(\in_g, \in_g \vee q_h)$-L-filter to be an $(\in_g, \in_g \vee q_h)$-L-filter of a quantale. Particularly, the intrinsic connections between $(\in_g, \in_g \vee q_h)$-L-subquantales and $(\in_g, \in_g \vee q_h)$-L-filters of quantales are established.

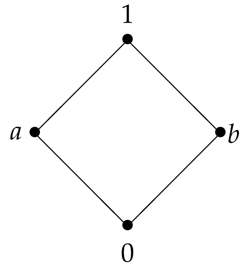
Definition 4.1. ([37]) An L-set F of Q is called an L-filter if it satisfies that
(*i*) $F(1_Q) = 1_L$;
(*ii*) For each $x, y \in Q, x \leq y$ implies $F(x) \leq F(y)$;
(*iii*) For every $x, y \in Q, F(x) \wedge F(y) \leq F(x\&y)$.
We call the pair (Q, F) an L-filtered quantale.

In view of the characterizations of $(\in_g, \in_g \vee q_h)$-L-subquantales, we continue to study a new kind of L-filters called $(\in_g, \in_g \vee q_h)$-L-filter.

Definition 4.2. An L-set F of Q is said to be an $(\in_g, \in_g \vee q_h)$-L-filter if it satisfies that
(F_1) For each $x, y \in Q$, from the condition $x \leq y$, we can obtain that $F(x) \leq F(y)$;
(F_2) For every $x, y \in Q$, we have $F(x\&y) \vee g \geq F(x) \wedge F(y) \wedge h$.
The pair (Q, F) is called an $(\in_g, \in_g \vee q_h)$-L-filtered quantale.

Example 4.3. Consider the complete lattice L mentioned in Example 3.12. Let $Q = \{0, a, b, 1\}$ with the partial-order and the binary operation as follows

&	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1



Defining an L-set F of Q as $F(1) = 1, F(a) = f, F(b) = d, F(0) = 0$, we can easily show that F is an $(\in_d, \in_d \vee q_f)$-L-filter of Q.

Remark 4.4. If we chose $F(1_Q) = 1_L, h = 1$ and $g = 0$ in Definition 4.2, then $(\in_g, \in_g \vee q_h)$-L-filter of Q is an L-filter of Q defined in Definition 4.1.

A natural question is whether an $(\in_g, \in_g \vee q_h)$-L-filter of Q is an L-filter of Q. The following example gives us a negative answer.

Example 4.5. In Example 4.3, we redefine an L-set $F : Q \to L$ by $F(1) = 1, F(a) = f, F(b) = d, F(0) = 0$. Then it is easy to check that F is an $(\in_d, \in_d \vee q_e)$-L-filter of Q. On the other hand, since $F(a\&b) = F(0) = 0 \ngeq d = F(a) \wedge F(b)$, it follows that F is not an L-filter of Q.

Now we come to discuss the properties of $(\in_g, \in_g \vee q_h)$-L-filters of quantales.

Proposition 4.6. *Let L be a distributive lattice and $F \in L^X$. For all $a \in L$, if F is an $(\in_g, \in_g \vee q_h)$-L-filter of Q, so is the layer of F.*

Proof. It follows immediately from Definition 2.6 and 4.2. □

Proposition 4.7. *Define an L-set $F : Q \to L$ by $F(1) \neq 0$ and*

$$F(x) = \begin{cases} \alpha & x \in A, \\ \beta & x \notin A, \end{cases}$$

where $\emptyset \neq A \subseteq Q, \alpha, \beta \in L$ with $h > \alpha > \beta > g$. Then A is a filter of Q if and only if F is an $(\in_g, \in_g \vee q_h)$-L-filter of Q.

Proof. Suppose that A is a filter of Q. Then by Definition 4.2, to complete the proof of the necessity, we only need to show that (F_1) and (F_2) hold. Firstly, we prove that $(F1)$ is valid. For all $x, y \in Q$ with $x \leq y$, only the following two cases need to consider: (i) If $F(x) = \beta$, it is trivial. (ii) If $F(x) = \alpha$, then $x \in A$. Since A is a filter and $(A, \leq)$ is a upper set, it follows that $y \in A$. Whence, $F(y) = \alpha$, which means that $F(x) \leq F(y)$. So $(F1)$ is valid. Next, for all $x, y \in Q$, we consider the following two cases to show that $(F2)$ holds.
Case 1: If $F(x) = \beta$ or $F(y) = \beta$, then by $h > \alpha > \beta > g$, we can obtain that $F(x\&y) \vee g \geq F(x) \wedge F(y) \wedge h$.
Case 2: If $F(x) = F(y) = \alpha$, then $x, y \in A$. Since A is a filter, we have $x\&y \in A$, that is, $F(x\&y) = \alpha$. So $F(x\&y) \vee g \geq F(x) \wedge F(y) \wedge h$.
Therefore $(F2)$ is valid. In conclusion, F is an $(\in_g, \in_g \vee q_h)$-L-filter of Q.

We now consider the sufficiency. Assume that F is an $(\in_g, \in_g \vee q_h)$-L-filter of Q. We first show that A is an upper set. For every $x \in A$ and $y \in Q$ with $x \leq y$, we have $F(x) = \alpha$. By $(F1)$, we can acquire that $F(x) \leq F(y)$. Whence, $F(y) = \alpha$, namely, $y \in A$, which means that A is an upper set. In addition, for all

$x, y \in A$, we prove that $x\&y \in A$. By the definition, we have $F(x) = F(y) = \alpha$. Then by $h > \alpha > \beta > g$, we have $F(x\&y) \vee g \geq F(x) \wedge F(y) \wedge h = \alpha$. Hence $F(x\&y) = \alpha$, that is, $x\&y \in A$. So A is a filter of Q. □

From Theorem 4.7 we can readily obtain the following results.

Corollary 4.8. *A is a filter of Q if and only if its characteristic function χ_A is an $(\in_g, \in_g \vee q_h)$-L-filter of Q.*

Definition 4.9. Let (Q, F) and (A, G) be two $(\in_g, \in_g \vee q_h)$-L-filtered quantales. Then the mapping $\varphi : Q \to A$ is an order-preserving quantale homomorphism if it satisfies that
(i) φ is a quantale homomorphism;
(ii) For each $x \in Q, F(x) \leq G(\varphi(x))$.

Proposition 4.10. *Let (Q, F) and (A, G) be two $(\in_g, \in_g \vee q_h)$-L-filtered quantales, $f : Q \to A$ be an order-preserving quantale homomorphism. Then we have*
(i) For a given mapping $f^{\to}(F) : A \to L$ by $a \to F(\bar{f}(a))$ for all $a \in A$, where $\bar{f}$ is the right adjoint of f, $f^{\to}(F)$ is an $(\in_g, \in_g \vee q_h)$-L-filter of A.
(ii) For a given mapping $f^{\leftarrow}(G) : Q \to L$ by $f^{\leftarrow}(G)(x) = G(f(x))$ for all $x \in Q$, $f^{\leftarrow}(G)$ is an $(\in_g, \in_g \vee q_h)$-L-filter of Q.

Proof. (i) Firstly, let $x, y \in A$ with $x \leq y$. Then $\bar{f}(x) \leq \bar{f}(y)$. Since F is an $(\in_g, \in_g \vee q_h)$-L-filter of Q, we have $F(\bar{f}(x)) \leq F(\bar{f}(y))$, which implies that $f^{\to}(F)(x) \leq f^{\to}(F)(y)$. This shows that (F_1) holds. Secondly, we prove that (F_2) is valid. In fact, for all $x, y \in A$, we have

$$\begin{aligned} f^{\to}(F)(x) \wedge f^{\to}(F)(y) \wedge h &= F(\bar{f}(x)) \wedge F(\bar{f}(y)) \wedge h \\ &\leq F(\bar{f}(x)\&\bar{f}(y)) \vee g \\ &= F(\bar{f}(x\&y)) \vee g. \end{aligned}$$

Therefore, $f^{\to}(F)$ is an $(\in_g, \in_g \vee q_h)$-L-filter of A.

(ii) The proof is similar to (i). □

Proposition 4.11. *Let $\{F_i\}_{j\in J}$ be a family of $(\in_g, \in_g \vee q_h)$-L-filters of Q. Define a mapping $F : Q \to L$ by $F(x) = \bigwedge_{j\in J} F_j(x)$ for all $x \in Q$. Then F is an $(\in_g, \in_g \vee q_h)$-L-filter of Q.*

Next we investigate the extension of an $(\in_g, \in_g \vee q_h)$-L-filter of Q.

Definition 4.12. Let $F \in L^Q$. Then an L-set defined by

$$\langle F, x\rangle : Q \to L, \qquad y \to F(y\&x),$$

where $x, y \in Q$, is called a left extension of F with respect to x.

Similarly we can define the right extension of F with respect to x. If $\langle F, x\rangle$ is both a left extension and a right extension of F with respect to x, we call $\langle F, x\rangle$ is an extension of F with respect to x, denoted by $< F >_x$.

To build relationships between $(\in_g, \in_g \vee q_h)$-L-filter F and the extension of F with respect to x, we need to consider the next problem. Is the extension of an $(\in_g, \in_g \vee q_h)$-L-filter still an $(\in_g, \in_g \vee q_h)$-L-filter? The following example shows us a negative answer.

Example 4.13. Consider the quantale Q in Example 3.6 and the complete lattice L in Example 3.12. If we define an L-set $F : Q \to L$ by $F(0) = 0, F(1) = F(c) = 1, F(a) = d, F(b) = d$, then F is an $(\in_c, \in_c \vee q_f)$-L-filter, but its left extension is not an $(\in_c, \in_c \vee q_f)$-L-filter for $< F, b > (a\&a) \vee c = F((a\&a)\&b) \vee c = e \not\geq f = F(a\&b) \wedge f = (< F, b > (a)) \wedge (< F, b > (a)) \wedge f$.

Next we give conditions which can guarantee the extension of an $(\in_g, \in_g \vee q_h)$-L-filter to be an $(\in_g, \in_g \vee q_h)$-L-filter.

Theorem 4.14. *Let Q be an commutative and idempotent quantale and F be an $(\in_g, \in_g \vee q_h)$-L-filter. Then for all $x \in Q$, the left (resp., right) extension of F with respect to x is an $(\in_g, \in_g \vee q_h)$-L-filter of Q.*

Proof. We only need to consider the case of left extension of F with respect to x, since the case of right extension of F is similar. Assume that F is an $(\in_g, \in_g \vee q_h)$-L-filter of Q. For all $a, b \in Q$ with $a \leq b$, we have $a\&x \leq b\&x$. Since F is an $(\in_g, \in_g \vee q_h)$-L-filter of Q, it follows from $(F1)$ that $F(a\&x) \leq F(b\&x)$, that is, $\langle F, x\rangle(a) \leq \langle F, x\rangle(b)$. On the other hand, for each $a, b \in Q$, we have

$$\begin{aligned}
(\langle F, x\rangle(a\&b)) \vee g &= F((a\&b)\&x) \vee g \\
&= F((a\&b)\&(x\&x)) \vee g \\
&= F((a\&x)\&(b\&x)) \vee g \\
&\geq F(a\&x) \wedge F(b\&x) \wedge h \\
&= (\langle F, x\rangle(a)) \wedge (\langle F, x\rangle(b)) \wedge h.
\end{aligned}$$

Therefore, $(F2)$ follows. Summing up the above statements, we can know that $\langle F, x\rangle$ is an $(\in_g, \in_g \vee q_h)$-L-filter of Q. □

Proposition 4.15. *Assume that F is an $(\in_g, \in_g \vee q_h)$-L-filter of Q and $x, y \in Q$ with $x \leq y$. Then we have $\langle F, x\rangle \leq < F, y >$.*

Proposition 4.16. *Let F and G be two $(\in_g, \in_g \vee q_h)$-L-filters of Q and $F \leq G$. Then we have $\langle F, x\rangle \leq < G, x >$ for each $x \in Q$.*

From the aforementioned discussion, we know that an $(\in_g, \in_g \vee q_h)$-L-subquantale and an $(\in_g, \in_g \vee q_h)$-L-filter are two important substructures of quantales which characterize the properties of quantales. And we naturally want to know if there exist some relationships between them. Next, we will concentrate on considering this problems.

Theorem 4.17. *An $(\in_g, \in_g \vee q_h)$-L-filter of Q is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q.*

Proof. Let F be an $(\in_g, \in_g \vee q_h)$-L-filter. By the definition, it is easy to show that F satisfies $(2B)$. For any $\{x_i\}_{i\in I} \subseteq Q$ and $i \in I$, by $(F2)$, we have $F(\vee x_i) \geq F(x_i)$. Consequently, we can obtain that $F(\vee x_i) \geq \wedge F(x_i)$. It follows that $F(\vee x_i) \vee g \geq F(\vee x_i) \geq \wedge F(x_i) \geq (\wedge F(x_i)) \wedge h$. Therefore, $(2B)$ is valid. □

Remark 4.18. In general, the converse of Theorem 4.17 may not hold. For instance, in Example 3.12, μ is an $(\in_e, \in_e \vee q_d)$-L-subquantale but not an $(\in_e, \in_e \vee q_d)$-L-filer of Q.

Then the following problem is obvious worth to consider.

Open problem 4.19. Can we give some reasonable conditions which guarantee an $(\in_g, \in_g \vee q_h)$-L-subquantale to be an $(\in_g, \in_g \vee q_h)$-L-filter of a quantale?

5. The category of $(\in_g, \in_g \vee q_h)$-L-subquantales (resp., L-filters) over quantales

We further introduce the characterizations of $(\in_g, \in_g \vee q_h)$-L-subquantales and $(\in_g, \in_g \vee q_h)$-L-filters of quantales by means of category theories, in this section.

In what follows we will use the symbol **GLFquant** (resp., **GFFQuant**) to represent the category of $(\in_g, \in_g \vee q_h)$-L-subquantales (resp., L-filters) and order-preserving quantale homomorphisms.

Theorem 5.1. *Let L be a completely distributive lattice. Then* **GLFquant** *is of a topological construct on* **Quant**.

Proof. Let $\{(Q_i,\mu_i)\}_{i\in I}\in \mathcal{O}b(\mathbf{GLFquant})$ and $\{f_i : Q\to Q_i\}_{i\in I}$ be a family of quantale homomorphisms. Define an L-set as follows

$$\mu : Q\to L, \qquad x\to \bigwedge_{i\in I}\mu_i(f_i(x)).$$

To prove **GLFquant** is topological, we just need to show that $\{f_i : (Q,\mu)\to (Q_i,\mu_i)\}_{i\in I}$ is the unique **GLFquant** initial lift of $\{f_i : Q\to Q_i\}_{i\in I}$.

Step 1 We show that $\{f_i : (Q,\mu)\to (Q_i,\mu_i)\}_{i\in I}$ is a **GLFquant** initial lift of $\{f_i : Q\to Q_i\}_{i\in I}$.

We first prove that (Q,μ) belongs to $\mathcal{O}b(\mathbf{GLFquant})$. Let $x,y\in Q$, considering $\{(Q_i,\mu_i)\}_{i\in I}\in \mathcal{O}b(\mathbf{GLFquant})$, we can obtain that $\{\mu_i\}_{i\in I}$ are $(\in_g,\in_g\vee q_h)$-L-quantales of $\{Q_i\}_{i\in I}$. Further, since L is a complete distributive lattice, it follows that

$$\begin{aligned}
\mu(x\&y)\vee g &= (\bigwedge_{i\in I}\mu_i(f_i(x\&y)))\vee g\\
&= \bigwedge_{i\in I}(\mu_i(f_i(x)\&f_i(y))\vee g)\\
&\geq \bigwedge_{i\in I}(\mu_i(f_i(x))\wedge\mu_i(f_i(y))\wedge h)\\
&= (\bigwedge_{i\in I}\mu_i(f_i(x)))\wedge(\bigwedge_{i\in I}\mu_i(f_i(y)))\wedge h\\
&= \mu(x)\wedge\mu(y)\wedge h.
\end{aligned}$$

Therefore, $(1B)$ follows. By the same argument, we can show that $(2B)$ holds. Thus μ is an $(\in_g,\in_g\vee q_h)$-L-subquantale of Q. By the definition, we can then easily know that (Q,μ) belongs to $\mathcal{O}b(\mathbf{GLFquant})$. We then show that $f_i\in \mathcal{M}or(\mathbf{GLFquant})$for every $i\in I$. Let $x\in Q$. Then $\mu(x)=\bigwedge_{i\in I}\mu_i(f_i(x))\leq \mu_i(f_i(x))$. In addition, $f_i : Q\to Q_i$ are quantale homomorphisms. Thus $f_i\in \mathcal{M}or$ (**GLFquant**) for each $i\in I$. On the other hand, assume that (Q_1,ν) is an object of **GLFquant**, $g : Q_1\to Q$ is a quantale homomorphism such that $g_i=f_i\circ g$ for all $i\in I$ and $g_i\in \mathcal{M}or$ (**GLFquant**), then we have $\nu(x)\leq\mu_i(g_i(x))$ for all $i\in I$ and $x\in Q_1$. It follows that

$$\nu(x)\leq\bigwedge_{i\in I}(\mu_i(g_i(x)))=\bigwedge_{i\in I}(\mu_i(f_i\circ g)(x))=\bigwedge_{i\in I}(\mu_i(f_i(g(x))))=\mu(g(x)).$$

Whence $g\in \mathcal{M}or$ (**GLFquant**). Based on the above conclusions, we can know that $\{f_i : (Q,\mu)\to (Q_i,\mu_i)\}_{i\in I}$ is a **GLFquant** initial lift of $\{f_i : Q\to Q_i\}_{i\in I}$.

Step 2 We show the uniqueness of the initial lift.

If $\{f_i : (Q,\bar{\mu})\to (Q_i,\mu_i)\}_{i\in I}$ is also a **GLFquant** initial lift of $\{f_i : Q\to Q_i\}_{i\in I}$ which is different from $\{f_i : (Q,\mu)\to (Q_i,\mu_i)\}_{i\in I}$, then $f_i : (Q,\bar{\mu})\to (Q_i,\mu_i)\}\in \mathcal{M}or(\mathbf{GLFquant})$. It is immediate that $\bar{\mu}(x)\leq\mu_i(f_i(x))$ for every $i\in I$ and $x\in Q$. Thus $\bar{\mu}(x)\leq\bigwedge_{i\in I}\mu_i(f_i(x))=\mu(x)$, i.e., $\bar{\mu}\leq\mu$. On the other hand, for the **GLFquant** object (Q,μ) and **Quant** morphism $id_Q : Q\to Q$, since $\{f_i : (Q,\bar{\mu})\to (Q_i,\mu_i)\}_{i\in I}$ is a initial lift of $\{f_i : Q\to Q_i\}_{i\in I}$, we have $f_i\circ id_Q=f_i\in \mathcal{M}or(\mathbf{GLFquant})$and $id_Q : (Q,\mu)\to (Q,\bar{\mu})\in \mathcal{M}or(\mathbf{GLFquant})$. Thus, for all $x\in Q$, $\mu(x)\leq\bar{\mu}(id_Q(x))=\bar{\mu}(x)$, which means $\mu\leq\bar{\mu}$. Therefore, $\mu=\bar{\mu}$.

Based on Step 1 and Step 2, we can now complete the proof of Theorem 5.1. □

Similar to the proof of Theorem 5.1, from Theorem 4.17 and the definition of the category **GFFQuant** we can also obtain the following result about **GFFQuant**.

Corollary 5.2. *Let L be a completely distributive lattice. Then* **GFFQuant** *is is of a topological construct on* **Quant**.

Theorem 5.3. **GLFquant** *has equalizers.*

$$\begin{tikzcd}
(Q'_3, \mu'_3) \arrow[d, "\bar{e}"'] \arrow[dr, "e'"] & & \\
(Q_3, \mu_3) \arrow[r, "e"'] & (Q_1, \mu_1) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Q_2, \mu_2)
\end{tikzcd}$$

Proof. Suppose that $(Q_1, \mu_1), (Q_2, \mu_2) \in \mathcal{O}b(\mathbf{GLFquant})$, f and g are **GLFquant** morphisms from (Q_1, μ_1) to (Q_2, μ_2). Define $Q_3 = \{x \in Q_1 | f(x) = g(x)\}$ with the binary operation & as the same as that of Q_1. If $e : Q_3 \to Q_1$ is an embedding and $\mu_3 = \mu_1 \circ e$. We next show that $((Q_3, \mu_3), e)$ is the equalizer of f and g.

Step 1 We first show that Q_3 is a subquantale of Q_1.

Assume that $x, y \in Q_3$. Then we have $f(x) = g(x)$ and $f(y) = g(y)$. It follows that $f(x\&y) = f(x)\&f(y) = g(x)\&g(y) = g(x\&y)$. Whence we can acquire that $x\&y \in Q_3$. Analogously, for all $x_i \in Q_3$, we have $\vee x_i \in Q_3$. Thus Q_3 is a subquantale of Q_1.

Step 2 We then show that $(Q_3, \mu_3) \in \mathcal{O}b(\mathbf{GLFquant})$.

Since $\mu_3 = \mu_1 \circ e$, μ_1 is an $(\in_g, \in_g \vee q_h)$-L-subquantale of Q_1 and e is embedding, for all $x, y \in Q_3$, we have

$$\begin{aligned}
\mu_3(x\&y) \vee g &= (\mu_1 \circ e)(x\&y) \vee g \\
&= \mu_1(x\&y) \vee g \\
&\geq \mu_1(x) \wedge \mu_1(y) \wedge h \\
&= \mu_1(e(x)) \wedge \mu_1(e(y)) \wedge h \\
&= \mu_3(x) \wedge \mu_3(y) \wedge h.
\end{aligned}$$

Similarly, $\mu_3(\bigvee_{i\in I} x_i) \vee g \geq (\bigwedge_{i\in I} \mu_3(x_i)) \wedge h$ for each $\{x_i\}_{i\in I} \subseteq Q_3$. Therefore, we can obtain that $(Q_3, \mu_3) \in \mathcal{O}b(\mathbf{GLFquant})$.

Step 3 We further prove that $e \in \mathcal{M}or(\mathbf{GLFquant})$.

From the definition of e and μ_3 with $\mu_3 = \mu_1 \circ e$, we can easily know that e is a quantale homomorphism from Q_3 to Q_1. For all $x \in Q_3$, we have $\mu_3(x) = (\mu_1 \circ e)(x) = \mu_1(e(x))$. We then acquire that $e \in \mathcal{M}or(\mathbf{GLFquant})$.

Step 4 We finally show that **GLFquant** has equalizers.

Suppose that $(Q'_3, \mu'_3) \in \mathcal{O}b(\mathbf{GLFquant})$ and e' is a **GLFquant** morphism from (Q'_3, μ'_3) to (Q_1, μ_1) satisfying $f \circ e' = g \circ e'$. Define a mapping $\bar{e} : Q'_3 \to Q_3$ and $\bar{e} = e'$. We next focus on showing that $\bar{e}$ is a **GLFquant** morphism from (Q'_3, μ'_3) to (Q_3, μ_3) and $e = e' \circ \bar{e}$. Firstly, let $x \in Q_3$. By $f \circ e' = g \circ e'$, we can obtain that $f(e'(x)) = g(e'(x))$ for each $x \in Q_3$. So we have $e'(x) \in Q_3$, which means that $\bar{e} = e'$ is well defined. Secondly, let $x, y \in Q'_3$. Since e' is a quantale homomorphism, we have

$$\bar{e}(x\&y) = e'(x\&y) = e'(x)\&e'(y) = \bar{e}(x)\&\bar{e}(y).$$

By the same argument, we can show that $\bar{e}(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} \bar{e}(x_i)$. We can then acquire that $\bar{e}$ is a quantale homomorphism. Thirdly, assume that $x \in Q'_3$. Since e' is a **GLFquant** morphism from (Q'_3, μ'_3) to (Q_1, μ_1), it follows that $\mu'_3(x) \leq \mu_1(e'(x))$. We then have

$$\mu'_3(x) \leq \mu_1(e'(x)) = \mu_1(e(e'(x))) = (\mu_1 \circ e)(e'(x)) = \mu_3(e'(x)) = \mu_3(\bar{e}(x)).$$

Namely, $\bar{e}$ is a **GLFquant** morphism from (Q'_3, μ'_3) to (Q_3, μ_3). At last, from the assumption, we can easily know that $e' = e \circ \bar{e}$ and the uniqueness of $\bar{e}$ is obvious. This completes the proof.

Analogously, we can also obtain the following result. □

Corollary 5.4. **GFFquant** *has equalizers.*

Theorem 5.5. *Let L be a completely distributive lattice. Then the category* **GLFquant** *has pullbacks.*

Proof. Assume that $(Q_1,\mu_1),(Q_2,\mu_2),(Q_3,\mu_3)\in \mathcal{O}b(\mathbf{GLFquant})$, f is a **GLFquant** morphism from (Q_1,μ_1) to (Q_3,μ_3) and g is a **GLFquant** morphism from (Q_2,μ_2) to (Q_3,μ_3). Define $H=\{(x,y)\in Q_1\times Q_2|f(x)=g(x)\}$. The binary operations & and $\vee$ of H are defined as

$$(x,y)\&(\mu,\nu)=(x\&\mu,y\&\nu),\qquad (x,y)\vee(\mu,\nu)=(x\vee\mu,y\vee\nu),$$

where $(x,y),(\mu,\nu)\in H\times H$. By the definition, it is easy to check that H is a quantale. Define four mappings as

$$\begin{array}{lllllll} \mu: H & \to & L & \qquad & p_1: H & \to & Q_3\\ (x,y) & \to & \mu_1(x)\wedge\mu_2(y), & & (x,y) & \to & x,\\ p_2: H & \to & Q_2 & & p_0: H & \to & Q_3\\ (x,y) & \to & Y, & & (x,y) & \to & (f\circ p_1)(x,y)=(g\circ p_2)(x,y). \end{array}$$

To show that the category **GLFquant** has pullbacks, we only need to show that $((H,\mu),\{p_i\}_{i=0,1,2})$ is the limit of f and g.

Step 1 We proved that $(H,\mu)\in\mathcal{O}b\mathbf{GLFquant}$.

Since $(Q_1,\mu_1),(Q_2,\mu_2)\in\mathcal{O}b(\mathbf{GLFquant})$, μ_1 and μ_2 are $(\in_g,\in_g\vee q_h)$-L-subquantale of Q_1 and Q_2, respectively. Let $(x,y),(u,v)\in H$. We have

$$\begin{aligned} \mu((x,y)\&(u,v))\vee g &= \mu(x\&u,y\&v)\vee g\\ &=(\mu_1(x\&u)\wedge\mu_2(y\&v))\vee g\\ &=(\mu_1(x\&u)\vee g)\wedge(\mu_2(y\&v)\vee g)\\ &\geq(\mu_1(x)\wedge\mu_1(u)\wedge h)\wedge(\mu_2(y)\wedge\mu_2(v)\wedge h)\\ &=(\mu_1(x)\wedge\mu_2(y))\wedge(\mu_1(u)\wedge\mu_2(v))\wedge h\\ &=\mu(x,y)\wedge\mu(u,v)\wedge h. \end{aligned}$$

Analogously, for each $\{(x_i,y_i)\}_{i\in I}\subseteq H$, we can acquire that $\mu(\bigvee\limits_{i\in I}(x_i,y_i))\vee g\geq(\bigwedge\limits_{i\in I}\mu(x_i,y_i))\wedge h$. So $(H,\mu)\in\mathcal{O}b\mathbf{GLFquant}$.

Step 2 We further show that $((H,\mu),\{p_i\}_{i=0,1,2})$ is the natural source with respect to the functor $F:\mathcal{I}\to$**GLFquant** on **GLFquant**, where $\mathcal{I}=$
$$\begin{tikzcd}[row sep=small] 1\bullet \arrow[dr] & \\ & \bullet\, 0 \\ 2\bullet \arrow[ur] & \end{tikzcd}$$
. We first show that $\{P_i\}_{(i=0,1,2)}$ are morphisms of **GLFquant**. Indeed, for all $(x,y),(u,v)\in H$, we have $p_1((x,y)\&(u,v))=p_1(x\&u,y\&v)=x\&u=p_1(x,y)\&p_1(u,v)$. Analogously, for all $(x_i,y_i)_{i\in I}\subseteq H$, it is easy to check that $p_1(\bigvee\limits_{i\in I}(x_i,y_i))=\bigvee\limits_{i\in I}p_1(x_i,y_i)$. Thus, p_1 is a quantale homomorphism. On the other hand, $\mu(x,y)=\mu_1(x)\wedge\mu_2(y)\leq\mu_1(x)=\mu_1(p_1(x,y))$. Hence, p_1 is a morphism of **GLFquant**. By the same argument, p_2 is also a morphism of **GLFquant**. For p_0, let $(x,y)\in H$. Then $(f\circ p_1)(x,y)=f(p_1(x,y))=f(x)=g(y)=g(p_2(x,y))=(g\circ p_2)(x,y)$. So p_0 is well defined. Since f is a **GLFquant** morphism from (Q_1,μ_1) to (Q_3,μ_3) and g is a **GLFquant** morphism from (Q_2,μ_2) to (Q_3,μ_3), for all $x\in Q_1,y\in Q_2$, we have $\mu_1(x)\leq\mu_3(f(x)),\mu_2(y)\leq\mu_3(g(y))$. It follows that

$$\begin{aligned} \mu(x,y)&=\mu_1(x)\wedge\mu_2(y)\\ &\leq\mu_3(f(x))\wedge\mu_3(g(y))\leq\mu_3(f(x))=\mu_3(f(p_1(x,y)))\\ &=\mu_3((f\circ p_1)(x,y)))=\mu_3(p_0(x,y)). \end{aligned}$$

On the other hand, for every $(x,y),(a,b)\in H, p_0((x,y)\&(a,b))=(f\circ p_1)((x,y)\&(a,b))=f(p_1(x\&a,y\&b))=f(x\&a)=f(x)\&f(a)=f(p_1(x,y))\&f(p_1(a,b))=p_0(x,y)\&p_0(a,b)$. Similarly, for all $(x_i,y_i)_{i\in I}\subseteq H$, we can obtain that $p_0(\bigvee\limits_{i\in I}(x_i,y_i))=\bigvee\limits_{i\in I}p_0(x_i,y_i)$. Whence p_0 is a quantale homomorphism. By the definition, p_0 is a morphism of **GLFquant** and $((H,\mu),\{p_i\})$ is the natural source with respect to the functor $F:\mathcal{I}\to$**GLFquant**, that is, the diagram

$$\begin{tikzcd}
(H,\mu) \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, "p_0"] & (Q_1,\mu_1) \arrow[d, "f"] \\
(Q_2,\mu_2) \arrow[r, "g"'] & (Q_3,\mu_3)
\end{tikzcd}$$

is commutative.

Step 3 Let $((\bar{H},\bar{\mu}),\{h_i\}_{i=0,1,2})$ be any natural source with respect to the functor $F:\mathcal{I}\rightarrow$**GLFquant**, where $\mathcal{I}= \begin{tikzcd}[row sep=tiny] 1\bullet \arrow[dr] & \\ & \bullet\, 0 \\ 2\bullet \arrow[ur] & \end{tikzcd}$. That is, the following diagram

$$\begin{tikzcd}
(\bar{H},\bar{\mu}) \arrow[r, "h_1"] \arrow[d, "h_2"'] \arrow[dr, "h_0"] & (Q_1,\mu_1) \arrow[d, "f"] \\
(Q_2,\mu_2) \arrow[r, "g"'] & (Q_3,\mu_3)
\end{tikzcd}$$

is commutative. Firstly, define a mapping by

$$\begin{array}{rcl} h:\bar{H} & \rightarrow & H \\ x & \rightarrow & (h_1(x),h_2(x)). \end{array}$$

Then it is easy to show that h is well defined. In fact, since $f\circ h_1=g\circ h_2=h_0$, we can obtain that $f(h_1(x))=g(h_2(x))$ for all $x\in\bar{H}$. Then $(h_1(x),h_2(x))\in H$. Secondly, h is a **GLFquant** morphism from $(\bar{H},\bar{\mu})$ to (H,μ). Indeed, by the definition of natural source, h_1 is a **GLFquant** morphism from $(\bar{H},\bar{\mu})$ to (Q_1,μ_1) and h_2 is a **GLFquant** morphism from $(\bar{H},\bar{\mu})$ to (Q_2,μ_2). Thus $h_1:\bar{H}\rightarrow Q_1$ and $h_2:\bar{H}\rightarrow Q_2$ are all quantale homomorphisms. For all $x\in\bar{H}$, we can obtain $\bar{\mu}(x)\le\mu_1(h_1(x))$ and $\bar{\mu}(x)\le\mu_2(h_2(x))$. Whence, for all $x,y\in\bar{H}$, we can easily acquire that

$$\begin{aligned} h(x\&y) &= (h_1(x\&y),h_2(x\&y)) \\ &= (h_1(x)\&h_1(y),h_2(x)\&h_2(y)) \\ &= (h_1(x),h_2(x))\&(h_1(y),h_2(y)) \\ &= h(x)\&h(y). \end{aligned}$$

With a similar argument, for all $\{x_i\}_{i\in I}\subseteq\bar{H}$, we have $h(\bigvee\limits_{i\in I}x_i)=\bigvee\limits_{i\in I}h(x_i)$. Thus h is a quantale homomorphism. On the other hand, for each $x\in\bar{H}$, we have

$$\bar{\mu}(x)\le\mu_1(h_1(x))\wedge\mu_2(h_2(x))=\mu(h_1(x),h_2(x))=\mu(h(x)).$$

Therefore, h is a **GLFquant** morphism from $(\bar{H},\bar{\mu})$ to (H,μ). Thirdly, for all $x\in\bar{H}$, we have $p_1(h(x))=p_1(h_1(x),h_2(x))=h_1(x)$, which means that $p_1\circ h=h_1$. Analogously, we can acquire that $p_2\circ h=h_2$. It follows that the following diagram

$$\begin{tikzcd}
(\bar{H}, \bar{\mu}) \arrow[dr, "h"] \arrow[drr, "h_1"] \arrow[ddr, "h_2"'] & & \\
& (H, \mu) \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, "p_0"] & (Q_1, \mu_1) \arrow[d, "f"] \\
& (Q_2, \mu_2) \arrow[r, "g"'] & (Q_3, \mu_3)
\end{tikzcd}$$

is commutative. At last, the uniqueness of h is obvious. In conclusion, $((H, \mu), \{p_i\}_{i=0,1,2})$ is the limit of f and g. Based on above results, we now completes the proof. □

Similarly, we can get the following result.

Corollary 5.6. *Let L be a completely distributive lattice. Then* **GFFquant** *has pullbacks.*

6. Conclusions

In the present paper, we developed two more generalized fuzzy structures, called an $(\in_g, \in_g \vee q_h)$-L-subquantale and an $(\in_g, \in_g \vee q_h)$-L-filter of a quantale. By Example 4.13 and Theorem 4.14, we showed the relationships between $(\in_g, \in_g \vee q_h)$-L-filters and their extensions. Particularly, we discovered some connections between $(\in_g, \in_g \vee q_h)$-L-subquantales and $(\in_g, \in_g \vee q_h)$-L-filters of quantales. At last, we further studied the characterizations of $(\in_g, \in_g \vee q_h)$-L-subquantales and $(\in_g, \in_g \vee q_h)$-L-filters of quantales by category theory. Related research methods in this paper may provide a useful tool for the research of fuzzy mathematics. We also hope that our works can invoke some research topics for L-set theory and provide more applications in the fields such as logic, engineering, computer science, information science, topology and so on. Our future work will focus on this field. Apart from the open problems, the following topics may be explored:(1) To consider $(\in_g, \in_g \vee q_h)$-L-ideals of quantales. (2) To study $(\in_g, \in_g \vee q_h)$-L-prime ideals of quantales. (3) To establish $(\in_g, \in_g \vee q_h)$-L-spectrum of quantales. (4) To study about applications, especially in information sciences and general systems.

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