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On Tauberian Theorems for Statistical Weighted Mean Method of Summability

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Abstract. In this paper we establish some new Tauberian theorems for the statistical weighted mean method of summability via the weighted general control modulo of the oscillatory behavior of nonnegative integer order of a real sequence. The main results improve the well-known classical Tauberian theorems which are given for weighted mean method of summability and statistical convergence.

1. Introduction

Let $p = (p_n)$ be a sequence of nonnegative numbers with $p_0 > 0$ and

$$P_n := \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty.$$

Let $u = (u_n)$ be a sequence of real numbers. The weighted means of (u_n) are defined by

$$\sigma_{n,p}^{(1)}(u) := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k$$

for all nonnegative integers n. The difference between u_n and $\sigma_{n,p}^{(1)}(u)$ which is called the weighted Kronecker identity is given by the identity

$$u_n - \sigma_{n,p}^{(1)}(u) = V_{n,p}^{(0)}(\Delta u) \tag{1}$$

where $V_{n,p}^{(0)}(\Delta u) := \dfrac{1}{P_n} \displaystyle\sum_{k=1}^{n} P_{k-1} \Delta u_k$, and $\Delta u_n = u_n - u_{n-1}$ with $u_{-1} = 0$ (see [2]). The weighted Kronecker identity is crucial and will be repeatedly used in proofs.

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For each integer $m \geq 0$, we define $\sigma_{n,p}^{(m)}(u)$ and $V_{n,p}^{(m)}(\Delta u)$ by

$$\sigma_{n,p}^{(m)}(u) = \begin{cases} \dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n} p_k \sigma_{k,p}^{(m-1)}(u) & \text{if } m \geq 1 \\ u_n & \text{if } m = 0 \end{cases}$$

and

$$V_{n,p}^{(m)}(\Delta u) = \begin{cases} \dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n} p_k V_{k,p}^{(m-1)}(\Delta u) & \text{if } m \geq 1 \\ V_{n,p}^{(0)}(\Delta u) & \text{if } m = 0 \end{cases}$$

respectively. The weighted classical control modulo of a sequence (u_n) is denoted by $\omega_{n,p}^{(0)}(u) = \dfrac{P_{n-1}}{p_n}\Delta u_n$. Totur and Çanak [16, Lemma 1] proved the identity that for a sequence (u_n) and any integer $m \geq 1$,

$$\frac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(m)}(u) = V_{n,p}^{(m-1)}(\Delta u).$$

The weighted general control modulo of the oscillatory behavior of integer order $m \geq 1$ of a sequence (u_n) is defined by $\omega_{n,p}^{(m)}(u) = \omega_{n,p}^{(m-1)}(u) - \sigma_{n,p}^{(1)}(\omega^{(m-1)}(u))$. The weighted general control modulo of order m of (u_n) has been used recently as Tauberian conditions for various summability methods in [1–3, 16].

For a sequence (u_n) and any positive integer m, we define

$$\left(\frac{P_{n-1}}{p_n}\Delta\right)_m u_n = \left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1}\left(\frac{P_{n-1}}{p_n}\Delta u_n\right) = \frac{P_{n-1}}{p_n}\Delta\left(\left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1} u_n\right)$$

where $\left(\frac{P_{n-1}}{p_n}\Delta\right)_0 u_n = u_n$, and $\left(\frac{P_{n-1}}{p_n}\Delta\right)_1 u_n = \frac{P_{n-1}}{p_n}\Delta u_n$.

The different writing of the weighted general control modulo of integer order $m \geq 1$ of (u_n) is obtained in [16, Lemma 4] by

$$\omega_{n,p}^{(m)}(u) = \left(\frac{P_{n-1}}{p_n}\Delta\right)_m V_{n,p}^{(m-1)}(\Delta u).$$

2. Development of Tauberian Theory for Weighted Mean Method of Summability

A sequence (u_n) is said to be summable to a finite number s by the weighted mean method determined by the sequence p, in short, $(\overline{N}, p)$ summable to s, if

$$\lim_{n\to\infty} \sigma_{n,p}^{(1)}(u) = s. \tag{2}$$

Notice that since $p_n = 1$ for all nonnegative integers n, then $(\overline{N}, 1)$ summability method equivalent to $(C, 1)$ summability method.

If the limit

$$\lim_{n\to\infty} u_n = s \tag{3}$$

exists, then (2) also exists. The converse is false, in general. However, the conditional converse of this statement holds if we add some certain conditions on the sequence (u_n). Such conditions are called Tauberian conditions and the resulting theorem is called a Tauberian theorem.

Hardy [8] showed that if (u_n) is $(\overline{N}, p)$ summable to s and

$$\omega_{n,p}^{(0)}(u) = O(1) \tag{4}$$

then (u_n) converges to s.

Çanak and Totur [1] replaced the Hardy's Tauberian condition by

$$\omega_{n,p}^{(1)}(u) \geq -H \quad (n = 0, 1, 2, ...) \tag{5}$$

for some $H > 0$, and showed that if (u_n) is $(\overline{N}, p)$ summable to s and the condition (5) is satisfied, then (u_n) converges to s with some certain conditions on (p_n). Çanak and Totur [16, Theorem 2] used the condition

$$\omega_{n,p}^{(m)}(u) = O(1) \quad (n = 0, 1, 2, ...) \tag{6}$$

for some nonnegative integer m as a general Tauberian condition for $(\overline{N}, p)$ summability method.

The condition (6) generalizes the Hardy's condition (4) for any integer $m \geq 0$.

Throughout this work, the symbol $[\lambda n]$ denotes the integer part of the product λn.

A sequence (u_n) is said to be slowly decreasing [9] if

$$\lim_{\lambda \to 1+} \liminf_{n \to \infty} \min_{n+1 \leq k \leq [\lambda n]} (u_k - u_n) \geq 0. \tag{7}$$

The condition (7) is satisfied if there exists a constant $C > 0$ such that

$$\frac{P_{n-1}}{p_n} \Delta u_n \geq -C \tag{8}$$

with $\frac{P_{n-1}}{p_n} = O(n)$. Indeed, we can estimate as follows. For any $k > n$, we have

$$u_k - u_n = \sum_{j=n+1}^{k} \Delta u_j \geq -C \sum_{j=n+1}^{k} \frac{p_j}{P_{j-1}} \geq -C \sum_{j=n+1}^{k} \frac{1}{j} \geq -C \log\left(\frac{k}{n}\right)$$

whence we conclude that

$$\liminf_{n \to \infty} \min_{n+1 \leq k \leq [\lambda n]} (u_k - u_n) \geq -C \log \lambda, \quad \lambda > 1.$$

Letting $\lambda \to 1+$, the inequality (7) follows immediately. Note that we used C to denote a constant, possibly different at each occurrence.

The following lemma gives a relation between the sequences (u_n) and $(\sigma_{n,p}^{(1)}(u))$.

Lemma 2.1. *Let (p_n) satisfy the condition*

$$\frac{P_{n-1}}{p_n} = O(n). \tag{9}$$

If (u_n) is slowly decreasing, then $(\sigma_{n,p}^{(1)}(u))$ is slowly decreasing.

Proof. In [12, page 568], it is shown that if (u_n) is slowly decreasing and the condition (9) holds, then there exist numbers $a > 0$, $b > 0$ for which

$$\sigma_{m,p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u) \geq \frac{P_m - P_n}{P_m} \left[-a\left(\ln\left(\frac{P_m}{P_n}\right) - \ln C\right) - b\right] \tag{10}$$

for all $m \geq n \geq 0$. In (10), if m and n are replaced by n and $n-1$, respectively, we get

$$\Delta\sigma_{n,p}^{(1)}(u) \geq \frac{p_n}{P_n} \left[-a\left(\ln\left(\frac{P_n}{P_{n-1}}\right) - \ln C\right) - b\right]. \tag{11}$$

Multiplying the both sides of (11) by $\frac{P_{n-1}}{p_n}$ and taking (9) into consideration, we have

$$V_{n,p}^{(0)}(\Delta u) \geq -C$$

for some $C > 0$.

It easily follows from the identity $V_{n,p}^{(0)}(\Delta u) = \frac{P_{n-1}}{p_n} \Delta\sigma_{n,p}^{(1)}(u)$ that $(\sigma_{n,p}^{(1)}(u))$ is slowly decreasing. □

Móricz and Rhoades [13] improved the Tauberian condition (4) replacing it by slow decrease of (u_n).

Theorem 2.2. *Let*

$$1 < \liminf_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} \leq \limsup_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} < \infty, \quad for \quad \lambda > 1 \tag{12}$$

and

$$1 < \liminf_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} \leq \limsup_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} < \infty, \quad for \quad 0 < \lambda < 1. \tag{13}$$

If (u_n) is $(\overline{N}, p)$ summable to s and (u_n) is slowly decreasing, then (u_n) converges to s.

3. Weighted Statistical Convergence and Tauberian Theorems

The notion of statistical convergence was introduced by Fast [5].

A real sequence (u_n) is said to be statistical convergent to s provided that for arbitrary $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \leq n : |u_k - s| \geq \epsilon\}| = 0$$

where the notation $|\mathcal{A}|$ indicates the number of the elements of the set $\mathcal{A}$. In this case, we write

$$st - \lim u_n = s. \tag{14}$$

Let (v_n) be a real sequence and α be a constant. If $st-\lim u_n = s$ and $st-\lim v_n = t$, then $st-\lim(\alpha u_n+v_n) = \alpha s+t$.

If the limit (3) exists, then (u_n) is statistically convergent to s. The converse is not necessarily true. For example, the sequence

$$u_n = \begin{cases} \sqrt{k} & \text{if } k = n^2 \\ 1 & \text{if } k \neq n^2 \end{cases}$$

is statistically convergent to 1, but not convergent in ordinary sense. Also, (14) may imply (3) by adding some suitable Tauberian conditions on the sequence (u_n).

A sequence (u_n) is said to be $(\overline{N}, p, k)$-statistically convergent to s for each nonnegative integer k if

$$st - \lim \sigma_{n,p}^{(k)}(u) = s.$$

For $k = 1$, $(\overline{N}, p, k)$-statistical convergence reduces to $(\overline{N}, p)$ statistical convergence. In addition, if $p_n = 1$ for all n, we have $(C, 1)$-statistical convergence.

Kolk proved the following lemma.

Lemma 3.1. *([10]) Let (u_n) be bounded. If $st - \lim u_n = s$, then $st - \lim \sigma_{n,p}^{(1)}(u) = s$.*

Remark 3.2. *By Lemma 3.1, clear that if (u_n) is $(\overline{N}, p, k-1)$-statistically convergent to s, then (u_n) is $(\overline{N}, p, k)$-statistically convergent to s for any integer $k \geq 1$. However, $(\overline{N}, p, k)$-statistical convergence of (u_n) does not imply $(\overline{N}, p, k-1)$-statistical convergence of (u_n). For example, if we take*

$$\sigma_{n,p}^{(k-1)}(u) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

it is clear that (u_n) is $(\overline{N}, p, k)$-statistically convergent to $\frac{1}{2}$ by $p_n = 1$ for all n, but it is not $(\overline{N}, p, k-1)$-statistically convergent.

Fridy [6] demonstrated that the condition (4) is a Tauberian condition for statistical convergence. That is, if (u_n) is statistically convergent to s and the condition (4) is satisfied, then (u_n) is convergent to s.

Móricz and Orhan [15] proved a Tauberian theorem that if $st-\lim \sigma_{n,p}^{(1)}(u)=s$ and either the condition (8) or the condition (7) is satisfied, then (u_n) is statistically convergent to s.

Later, Chen and Chang [4] generalized Móricz and Orhan's Tauberian theorem in the following theorem.

Theorem 3.3. *Let (p_n) satisfy the conditions (12) and (13). If (u_n) is statistically convergent to s and (u_n) is slowly decreasing, then (u_n) converges to s.*

Also, Chen and Chang [4] proved the following Tauberian theorem.

Theorem 3.4. *Let (p_n) satisfy the conditions (12) and (13). If (u_n) is $(\overline{N},p)$-statistically convergent to s and (u_n) is slowly decreasing, then (u_n) converges to s.*

There are also some studies associated with Tauberian theorems in which statistical convergence is used (see [7, 11, 14]).

In this paper, we extend and generalize Theorem 3.4. We establish converge of (u_n) in ordinary sense from $(\overline{N},p,k)$-statistical convergence of (u_n) for some nonnegative integer k by adding some conditions upon which of the general control modulo of the oscillatory behavior of integer order $m\geq 0$ of (u_n). Our results generalize classical type Tauberian theorems for $(\overline{N},p)$-statistical convergence of (u_n).

4. Main Results

The following theorem generalizes Theorem 3.4 given by Chen and Chang[4].

Theorem 4.1. *Let (p_n) satisfy the conditions (9), (12) and (13). If (u_n) is $(\overline{N},p,k)$-statistically convergent to s for some integer $k\geq 0$ and (u_n) is slowly decreasing, then (u_n) converges to s.*

Proof. Since (u_n) is slowly decreasing, then by Lemma 2.1

$$(\sigma_{n,p}^{(r)}(u)) \text{ is slowly decreasing,} \tag{15}$$

for each integer $r\geq 1$. By assumption, $st-\lim_n \sigma_{n,p}^{(k)}(u)=s$. Taking $r=k$ in (15), by Theorem 3.3, we obtain that $(\sigma_{n,p}^{(k)}(u))$ converges to s. This means that $(\sigma_{n,p}^{(k-1)}(u))$ is $(\overline{N},p)$ summable to s. After taking $r=k-1$ in (15), by Theorem 2.2, $(\sigma_{n,p}^{(k-1)}(u))$ converges to s. Continuing in this vein, it follows that $(\sigma_{n,p}^{(1)}(u))$ converges to s. Finally, (u_n) is $(\overline{N},p)$ summable to s. By hypothesis, the proof is completed by Theorem 2.2. □

Theorem 4.2 extends Theorem 3.4.

Theorem 4.2. *Let (p_n) satisfy the conditions (9), (12) and (13). If (u_n) is bounded and statistically convergent to s, and $(\sigma_{n,p}^{(1)}(\omega^{(r)}(u)))$ is slowly decreasing for some nonnegative integer r, then (u_n) converges to s.*

Proof. Since $st-\lim_n u_n=s$ and (u_n) is bounded, then using the weighted Kronecker identity (1), we have, by Lemma 3.1,

$$st-\lim_n \sigma_{n,p}^{(1)}(\omega^{(r)}(u))=0 \tag{16}$$

for each nonnegative integer r. Suppose that $(\sigma_{n,p}^{(1)}(\omega^{(r)}(u)))$ is slowly decreasing. We see by applying Theorem 3.3 to the sequence $(\sigma_{n,p}^{(1)}(\omega^{(r)}(u)))$ that $\sigma_{n,p}^{(1)}(\omega^{(r)}(u))=o(1)$. From the identity

$$\sigma_{n,p}^{(1)}(\omega^{(r)}(u))=n\Delta\sigma_{n,p}^{(2)}(\omega^{(r-1)}(u))$$

we have

$$n\Delta\sigma_{n,p}^{(2)}(\omega^{(r-1)}(u))\geq -H \tag{17}$$

for some $H \geq 0$. Hence, we get the sequence $(\sigma^{(2)}_{n,p}(\omega^{(r-1)}(u)))$ is slowly decreasing. It follows by applying the Kronecker identity (1) to the sequence $(\sigma^{(1)}_{n,p}(\omega^{(r-1)}(u)))$ that we have $\sigma^{(1)}_{n,p}(\omega^{(r)}(u)) = \sigma^{(1)}_{n,p}(\omega^{(r-1)}(u)) - \sigma^{(2)}_{n,p}(\omega^{(r-1)}(u))$. Therefore, we obtain that $(\sigma^{(1)}_{n,p}(\omega^{(r-1)}(u)))$ is slowly decreasing.

Continuing in this vein, we obtain $(\sigma^{(1)}_{n,p}(\omega^{(0)}(u)))$ is slowly decreasing. Taking $r = 0$ in (16), we have $st - \lim_n \sigma^{(1)}_{n,p}(\omega^{(0)}(u)) = 0$. By Theorem 3.3, we get

$$\sigma^{(1)}_{n,p}(\omega^{(0)}(u)) = V^{(0)}_{n,p}(\Delta u) = o(1).$$

From the identity $V^{(0)}_{n,p}(\Delta u) = \frac{P_{n-1}}{p_n}\Delta\sigma^{(1)}_{n,p}(u)$, we have that $(\sigma^{(1)}_{n,p}(u))$ is slowly decreasing. On the other hand, $st - \lim_n u_n = s$ implies $st - \lim_n \sigma^{(1)}_{n,p}(u) = s$ from Lemma 2.1. Finally, we obtain $\lim_n \sigma^{(1)}_{n,p}(u) = s$ from Theorem 3.3, and the proof is completed by weighted Kronecker identity.□

Corollary 4.3. *Let (p_n) satisfy the conditions (9), (12) and (13). If (u_n) be bounded and statistically convergent to s, and*

$$\omega^{(r+1)}_{n,p}(u) \geq -H$$

for some $H > 0$ and nonnegative integer r, then (u_n) converges to s.

Proof. From the identity

$$\omega^{(r+1)}_{n,p}(u) = \frac{P_{n-1}}{p_n}\Delta\sigma^{(1)}_{n,p}(\omega^{(r)}(u))$$

we obtain that $(\sigma^{(1)}_{n,p}(\omega^{(r)}(u)))$ is slowly decreasing. □

Convergence of a sequence can be obtained by using Corollary 4.3.

Example 4.4. *The sequence $u = (u_n)$ defined by*

$$u_n = \begin{cases} \displaystyle\sum_{j=1}^{k} \frac{j}{2^j} & \text{if } k = n^2 \\ 2 & \text{if } k \neq n^2 \end{cases}$$

is statistically convergent to 2 and bounded. On the other hand, if we choose $p_n = 1$ for each nonnegative integer n in Corollary 4.3, then the conditions (9), (12) and (13) are satisfied. Since the sequence $(n\Delta u_n)$ is bounded, then $\omega^{(r+1)}_{n,p}(u) \geq -H$ for some $H > 0$ and nonnegative integer r by the Kronecker identity (1). Hence, (u_n) converges to 2 by Corollary 4.3.

The following theorem recovers converge of (u_n) from $(\overline{N}, p, k)$-statistical convergence of (u_n) for some integer $k \geq 0$ under which the condition slow decrease of $(\sigma^{(1)}_n(\omega^{(m)}(u)))$. The result generalizes and extends Theorem 3.4.

Theorem 4.5. *Let (p_n) satisfy the conditions (9), (12) and (13). If (u_n) be bounded and $(\overline{N}, p, k)$-statistically convergent to s for some integer $k \geq 0$, and $(\sigma^{(1)}_{n,p}(\omega^{(m)}(u)))$ is slowly decreasing for some nonnegative integer m, then (u_n) converges to s.*

Proof. If taking the weighted means of the sequence $(\sigma^{(1)}_{n,p}(\omega^{(m)}(u)))$ is repeated $j-$times, we obtain from the identity

$$\sigma^{(j+1)}_{n,p}(\omega^{(m)}(u)) = \sigma^{(1)}_{n,p}(\omega^{(m)}(\sigma^{(j)}(u))) \tag{18}$$

that $(\sigma^{(1)}_{n,p}(\omega^{(m)}(\sigma^{(j)}(u))))$ is slowly decreasing.

We note that the statement (u_n) is $(\overline{N}, p, k)$-statistical convergent to s is equivalent to the statement $(\sigma_{n,p}^{(k)}(u))$ is statistical convergent to s for some nonnegative integer k. After taking $j = k$ and $r = m$ in Theorem 4.2, we obtain

$$\lim_n \sigma_{n,p}^{(k)}(u) = s.$$

This means that $(\sigma_{n,p}^{(k-1)}(u))$ is $(\overline{N}, p)$ summable to s, therefore $(\sigma_{n,p}^{(k-1)}(u))$ is $(\overline{N}, p)$-statistical convergent to s. After taking $j = k - 1$ in (18) and $r = m$ in Theorem 4.2, we have from the identity

$$\omega_{n,p}^{(m)}(\sigma^{(k)}(u)) = \sigma_{n,p}^{(1)}(\omega^{(m)}(\sigma^{(k-1)}(u))) \tag{19}$$

that

$$\lim_n \sigma_{n,p}^{(k-1)}(u) = s.$$

Continuing in this vein, we obtain $\lim_n \sigma_{n,p}^{(1)}(u) = s$. Therefore, since (u_n) is $(\overline{N}, p)$ summable to s, hence (u_n) is $(\overline{N}, p)$-statistical convergent to s. By hypothesis, since $(\sigma_{n,p}^{(1)}(\omega^{(m)}(u)))$ is slowly decreasing, then (u_n) converges to s. The proof is completed.□

Corollary 4.6. *Let (p_n) satisfy the conditions (9), (12) and (13). If (u_n) is bounded and $(\overline{N}, p, k)$-statistically convergent to s for some integer $k \geq 0$, and*

$$\omega_{n,p}^{(m+1)}(u) \geq -H \tag{20}$$

for some $H > 0$ and nonnegative integer m, then (u_n) converges to s.

The following example is an application of Corollary 4.6.

Example 4.7. *The sequence $u = (u_n)$ defined by*

$$u_n = \begin{cases} \sum_{j=1}^{k} \frac{j}{(j+1)!} & \text{if } k = n^2 \\ 1 & \text{if } k \neq n^2 \end{cases}$$

is $(\overline{N}, p, 1)$-statistically convergent to 1 and bounded. On the other hand, if we choose $p_n = 1$ for each nonnegative integer n in Corollary 4.6, then the conditions (9), (12) and (13) are satisfied. Since the sequence $(n\Delta u_n)$ is bounded, then $\omega_{n,p}^{(m+1)}(u) \geq -H$ for some $H > 0$ and nonnegative integer m by Kronecker identity (1). Hence, (u_n) converges to 1 by Corollary 4.6.

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