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Half Lightlike Submanifolds of a Semi-Riemannian Manifold of Quasi-Constant Curvature

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Abstract. We study the geometry of half lightlike submanifolds $(M, g, S(TM), S(TM^{\perp}))$ of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of quasi-constant curvature subject to the following conditions; (1) the curvature vector field ζ of $\widetilde{M}$ is tangent to M, (2) the screen distribution $S(TM)$ of M is either totally geodesic or totally umbilical in M, and (3) the co-screen distribution $S(TM^{\perp})$ of M is a conformal Killing distribution.

1. Introduction

In the generalization from the theory of submanifolds in Riemannian to the theory of submanifolds in semi-Riemannian manifolds, the induced metric on submanifolds may be degenerate(lightlike) therefore there is a natural existence of lightlike submanifolds and for which the local and global geometry is completely different than non-degenerate case. In lightlike case, the standard text book definitions do not make sense and one fails to use the theory of non-degenerate geometry in the usual way. The primary difference between the lightlike submanifolds and non-degenerate submanifolds is that in the first case, the normal vector bundle intersects with the tangent bundle. Thus, the study of lightlike submanifolds becomes more difficult and different from the study of non-degenerate submanifolds. Moreover, the geometry of lightlike submanifolds is used in mathematical physics, in particular, in general relativity since lightlike submanifolds produce models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons). The universe can be represented as a four dimensional submanifold embedded in a $(4 + n)$-dimensional spacetime manifold. Lightlike hypersurfaces are also studied in the theory of electromagnetism [6]. Thus, large number of applications but limited information available, motivated us to do research on this subject matter. Kupeli [11] and Bejancu-Duggal [6] developed the general theory of degenerate (lightlike) submanifolds. They constructed a transversal vector bundle of lightlike submanifold and investigated various properties of these manifolds. Moreover, Sahin and Yildirim ([12]) studied both slant lightlike submanifolds and screen slant lightlike submanifolds of an indefinite Sasakian manifold. They obtained necessary and sufficient conditions for the existence of a slant lightlike submanifold.

The idea of Riemannian manifold of quasi-constant curvature was introduced by B. Y. Chen and K. Yano [2] as follows: A *Riemannian manifold of quasi-constant curvature* is a Riemannian manifold $(\widetilde{M}, \widetilde{g})$

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equipped with the curvature tensor $\widetilde{R}$ satisfying

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z,W) &= \alpha\{\widetilde{g}(Y,Z)g(X,W) - \widetilde{g}(X,Z)g(Y,W)\} \\ &+ \beta\{\widetilde{g}(X,W)\theta(Y)\theta(Z) - \widetilde{g}(X,Z)\theta(Y)\theta(W) \\ &\quad + \widetilde{g}(Y,Z)\theta(X)\theta(W) - \widetilde{g}(Y,W)\theta(X)\theta(Z)\}, \end{aligned} \tag{1.1}$$

where α, β are scalar functions and θ is a 1-form defined by

$$\theta(X) = \widetilde{g}(X,\zeta), \tag{1.2}$$

and ζ is a unit vector field on $\widetilde{M}$, which called the *curvature vector field* of $\widetilde{M}$. It is well-known that if the curvature tensor $\widetilde{R}$ is of the form (1.1), then $\widetilde{M}$ is conformally flat. If $\beta = 0$, then $\widetilde{M}$ is a space of constant curvature.

A non-flat Riemannian manifold $\widetilde{M}$ of dimension $n(> 2)$ is called a *quasi-Einstein manifold* [1] if its Ricci tensor $\widetilde{Ric}$ satisfies the condition

$$\widetilde{Ric}(X,Y) = a\,\widetilde{g}(X,Y) + b\,\phi(X)\phi(Y),$$

where a, b are scalar functions such that $b \neq 0$ and ϕ is a non-vanishing 1-form such that $\widetilde{g}(X,U) = \phi(X)$ for any vector field X, where U is a unit vector field. If $b = 0$, then $\widetilde{M}$ is an Einstein manifold. It can be easily seen that *every Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.* Moreover, A. Del et al ([3]) gave some geometric properties for generalized quasi-Einstenin manifolds. They contructed non-trivial examples to prove the existence of a generalized quasi-Einstein manifolds.

The subject of this paper is to study the geometry of half lightlike submanifolds of a semi-Riemannian manifold $(\widetilde{M},\widetilde{g})$ of quasi-constant curvature. We prove two characterization theorems for such a half lightlike submanifold $(M,g,S(TM),S(TM^{\perp}))$ as follows :

Theorem 1.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M},\widetilde{g})$ of quasi-constant curvature. If the curvature vector field ζ of $\widetilde{M}$ is tangent to M and $S(TM)$ is totally geodesic in M, then we have the following results :*

(1) *If $S(TM^{\perp})$ is a Killing distribution, then the functions α and β, defined by* (1.1)*, vanish identically. Furthermore, $\widetilde{M}$, M and the leaf M^{*} of $S(TM)$ are flat manifolds.*

(2) *If $S(TM^{\perp})$ is a conformal Killing distribution, then the function β vanishes identically. Furthermore, $\widetilde{M}$ and M^{*} are space of constant curvatures and M is an Einstein manifolds such that $Ric = (r/m)\,g$, where r is the induced scalar curvature of M.*

Theorem 1.2. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ of quasi-constant curvature. If ζ is tangent to M, $S(TM)$ is totally umbilical in M and $S(TM^{\perp})$ is a conformal Killing distribution with a non-constant conformal factor, then the function β vanishes identically. Moreover $\widetilde{M}$ and M^{*} are space of constant curvatures and M is a totally umbilical Einstein manifolds such that $Ric = (r/m)\,g$.*

2. Half Lightlike Submanifolds

It is well-known that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of half lightlike submanifolds M of a semi-Rimannian manifold $(\widetilde{M},\widetilde{g})$ of codimension 2 is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank 1. Thus there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, called the *screen* and *coscreen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\widetilde{M}$. Certainly $TM^{\perp}$ is a subbundle of $S(TM)^{\perp}$. As $S(TM^{\perp})$ is a non-degenerate subbundle of $S(TM)^{\perp}$, the orthogonal complementary distribution $S(TM^{\perp})^{\perp}$ of $S(TM^{\perp})$ in $S(TM)^{\perp}$ is also a non-degenerate distribution of rank 2 and satisfies the following orthogonal decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp}.$$

Clearly $Rad(TM)$ is a vector subbundle of $S(TM^{\perp})^{\perp}$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\widetilde{g}(L, L) = \epsilon = \pm 1$. For any null section ξ of $Rad(TM)$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ [5] satisfying

$$\widetilde{g}(\xi, N) = 1,\ \widetilde{g}(N, N) = \widetilde{g}(N, X) = \widetilde{g}(N, L) = 0,\ \ \forall\, X \in \Gamma(S(TM)). \tag{2.2}$$

Denote by $ltr(TM)$ the subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. Then we show that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution $S(TM)$ respectively. Then the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$ is decomposed as follow :

$$\begin{aligned} T\widetilde{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{2.3}$$

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ and P the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1). Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.4}$$

$$\widetilde{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.5}$$

$$\widetilde{\nabla}_X L = -A_L X + \phi(X)N, \tag{2.6}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.7}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \quad \forall\, X,\ Y \in \Gamma(TM), \tag{2.8}$$

where ∇ and ∇^* are induced connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_{ξ} and A_L are linear operators on TM and τ, ρ and ϕ are 1-forms on TM. Since $\widetilde{\nabla}$ is torsion-free, the induced connection ∇ of M is also torsion-free and both B and D are symmetric. From the facts $B(X, Y) = \widetilde{g}(\widetilde{\nabla}_X Y,\ \xi)$ and $D(X, Y) = \epsilon\widetilde{g}(\widetilde{\nabla}_X Y,\ L)$, we know that B and D are independent of the choice of a screen distribution and

$$B(X, \xi) = 0, \quad D(X, \xi) = -\epsilon\phi(X),\ \forall\, X \in \Gamma(TM). \tag{2.9}$$

The induced connection ∇ on M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y), \tag{2.10}$$

for all X, Y, $Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \widetilde{g}(X, N), \quad \forall\, X \in \Gamma(TM). \tag{2.11}$$

But the connection ∇^* on M^* is metric. The above three local second fundamental forms of M and M^* are related to their shape operators by

$$B(X, Y) = g(A^*_{\xi} X, Y), \qquad \widetilde{g}(A^*_{\xi} X, N) = 0, \tag{2.12}$$

$$C(X, PY) = g(A_N X, PY), \qquad \widetilde{g}(A_N X, N) = 0, \tag{2.13}$$

$$\epsilon D(X, Y) = g(A_L X, Y) - \phi(X)\eta(Y), \quad \widetilde{g}(A_L X, N) = \epsilon\rho(X), \tag{2.14}$$

for all $X,\ Y \in \Gamma(TM)$. By (2.12) and (2.13), we show that A^*_ξ and A_N are $\Gamma(S(TM))$-valued shape operators related to B and C respectively and A^*_ξ is self-adjoint on TM and

$$A^*_\xi \xi = 0. \tag{2.15}$$

Denote by $\widetilde{R},\ R$ and R^* the curvature tensors of the Levi-Civita connection $\widetilde{\nabla}$ on $\widetilde{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten equations (2.4)~(2.8) for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$: For all $X,\ Y,\ Z,\ W \in \Gamma(TM)$,

$$\begin{aligned}
\widetilde{g}(\widetilde{R}(X,Y)Z,\ PW) &= g(R(X,Y)Z,\ PW) \\
&\quad + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW) \\
&\quad + \epsilon\{D(X,Z)D(Y,PW) - D(Y,Z)D(X,PW)\}, && (2.16)\\
\widetilde{g}(\widetilde{R}(X,Y)Z,\ \xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\
&\quad + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) \\
&\quad + D(Y,Z)\phi(X) - D(X,Z)\phi(Y), && (2.17)\\
\widetilde{g}(\widetilde{R}(X,Y)Z,\ N) &= \widetilde{g}(R(X,Y)Z,\ N) \\
&\quad + \epsilon\{D(X,Z)\rho(Y) - D(Y,Z)\rho(X)\}, && (2.18)\\
\epsilon\, \widetilde{g}(\widetilde{R}(X,Y)Z,\ L) &= (\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) \\
&\quad + \rho(X)B(Y,Z) - \rho(Y)B(X,Z), && (2.19)\\
\widetilde{g}(\widetilde{R}(X,Y)\xi,\ N) &= g(A^*_\xi X,\ A_N Y) - g(A^*_\xi Y,\ A_N X) \\
&\quad - 2d\tau(X,Y) + \rho(X)\phi(Y) - \rho(Y)\phi(X), && (2.20)\\
\widetilde{g}(R(X,Y)PZ,\ PW) &= g(R^*(X,Y)PZ,\ PW) \\
&\quad + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), && (2.21)\\
g(R(X,Y)PZ,\ N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\
&\quad + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X). && (2.22)
\end{aligned}$$

The *Ricci curvature tensor*, denoted by $\widetilde{Ric}$, of $\widetilde{M}$ is defined by

$$\widetilde{Ric}(X,Y) = trace\{Z \to \widetilde{R}(Z,X)Y\},$$

for any $X,\ Y \in \Gamma(T\widetilde{M})$. Let $\dim \widetilde{M} = m+3$. Locally, $\widetilde{Ric}$ is given by

$$\widetilde{Ric}(X,Y) = \sum_{i=1}^{m+3} \epsilon_i\, \widetilde{g}(\widetilde{R}(E_i,X)Y,\ E_i), \tag{2.23}$$

where $\{E_1,\ \dots,\ E_{m+3}\}$ is an orthonormal frame field of $T\widetilde{M}$ and $\epsilon_i\,(=\pm 1)$ denotes the causal character of respective vector field E_i. If $\dim(\widetilde{M}) > 2$ and

$$\widetilde{Ric} = \widetilde{\kappa g}, \quad \widetilde{\kappa} \text{ is a constant}, \tag{2.24}$$

then $\widetilde{M}$ is an *Einstein manifold*. The *scalar curvature* $\widetilde{r}$ is defined by

$$\widetilde{r} = \sum_{i=1}^{m+3} \epsilon_i\, \widetilde{Ric}(E_i,\ E_i). \tag{2.25}$$

Putting (2.24) in (2.25) implies that $\widetilde{M}$ is Einstein if and only if

$$\widetilde{Ric} = \frac{\widetilde{r}}{m+3}\,\widetilde{g}.$$

A vector field X on $\widetilde{M}$ is said to be a *conformal Killing vector field* [10] if $\widetilde{\mathcal{L}}_X\widetilde{g} = -2\delta\,\widetilde{g}$ for any smooth function δ, where $\widetilde{\mathcal{L}}_X$ denotes the Lie derivative with respect to X, that is,

$$(\widetilde{\mathcal{L}}_X\widetilde{g})(Y, Z) = X(\widetilde{g}(Y, Z)) - \widetilde{g}([X, Y], Z) - \widetilde{g}(Y, [X, Z]), \quad \forall\, X, Y, Z \in \Gamma(T\widetilde{M}).$$

In particular, if $\delta = 0$, then X is called a *Killing vector field* [9]. A distribution $\mathcal{G}$ on $\widetilde{M}$ is called a *conformal Killing* (resp. *Killing*) *distribution* on $\widetilde{M}$ if each vector field belonging to $\mathcal{G}$ is a conformal Killing (resp. Killing) vector field on $\widetilde{M}$.

Theorem 1.1 [9, 10]. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. If the coscreen distribution $S(TM^{\perp})$ is a conformal Killing* (resp. *Killing*) *distribution, then there exists a smooth function δ such that*

$$D(X, Y) = \epsilon\delta\, g(X, Y), \;\; \{\text{resp.} \;\; D(X, Y) = 0,\} \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.26}$$

Proof. By straightforward calculations and use (2.6) and (2.14), we have

$$(\widetilde{\mathcal{L}}_L\widetilde{g})(X, Y) = \widetilde{g}(\widetilde{\nabla}_X L, Y) + \widetilde{g}(X, \widetilde{\nabla}_Y L),$$
$$\widetilde{g}(\widetilde{\nabla}_X L, Y) = -g(A_L X, Y) + \phi(X)\eta(Y) = -\epsilon D(X, Y),$$

for any $X,\ Y \in \Gamma(TM)$. From $(\widetilde{\mathcal{L}}_L\widetilde{g})(X, Y) = -2\epsilon D(X, Y)$ we deduce our assertion.

3. Proof of Theorems

Let $R^{(0,2)}$ denote the induced Ricci type tensor of type (0, 2) on M given by

$$R^{(0,2)}(X, Y) \; = \; trace\{Z \to R(Z, X)Y\}, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.1}$$

Consider an induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$, and let $E = \{\xi, W_a, N, L\}$ be the corresponding frame field on $\widetilde{M}$. By using (2.23) and (3.1), for all $X,\ Y \in \Gamma(TM)$, we get

$$\begin{aligned} \widetilde{Ric}(X, Y) \;&=\; \sum_{a=1}^{m} \epsilon_a\, \widetilde{g}(\widetilde{R}(W_a, X)Y,\, W_a) \;+\; \widetilde{g}(\widetilde{R}(\xi, X)Y,\, N) \\ &+\; \epsilon\, \widetilde{g}(\widetilde{R}(L, X)Y,\, L) \;+\; \widetilde{g}(\widetilde{R}(N, X)Y,\, \xi), \end{aligned} \tag{3.2}$$

$$R^{(0,2)}(X, Y) \;=\; \sum_{a=1}^{m} \epsilon_a\, g(R(W_a, X)Y,\, W_a) + \widetilde{g}(R(\xi, X)Y,\, N). \tag{3.3}$$

Substituting (2.16) and (2.18) in (3.2) and using (2.12)~(2.14) and (3.3), we obtain

$$\begin{aligned} R^{(0,2)}(X, Y) \;&=\; \widetilde{Ric}(X, Y) + B(X, Y)trA_N + D(X, Y)trA_L \\ &-\; g(A_N X,\, A^*_\xi Y) - \epsilon\, g(A_L X,\, A_L Y) + \rho(X)\phi(Y) \\ &-\; \widetilde{g}(\widetilde{R}(\xi, Y)X,\, N) - \epsilon\, \widetilde{g}(\widetilde{R}(L, Y)X,\, L), \end{aligned} \tag{3.4}$$

for any $X,\ Y \in \Gamma(TM)$ [9, 10]. This shows that $R^{(0,2)}$ is not symmetric. A tensor field $R^{(0,2)}$ of M, given by (3.1), is called its *induced Ricci tensor* [8–10] if it is symmetric. From now and in the sequel, a symmetric $R^{(0,2)}$ tensor will be denoted by Ric.

Using (2.20), (3.4) and the first Bianchi's identity, we obtain

$$R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = 2d\tau(X, Y).$$

Theorem 3.1 [7]. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then the tensor field $R^{(0,2)}$ is an induced symmetric Ricci tensor, Ric, if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on any coordinate neighborhood $\mathcal{U} \subset M$.*

For the rest of this paper, let M be a half lightlike submanifold of a semi-Riemannian manifold $\widetilde{M}$ of quasi-constant curvature. We may assume that the curvature vector field ζ of $\widetilde{M}$ is a unit spacelike tangent vector field of M and $\dim \widetilde{M} > 4$. Denote e by the smooth function such that $e = \theta(N)$. Using (1.1), (2.23) and the facts $\theta(\xi) = \theta(L) = 0$, we have

$$\widetilde{Ric}(X, Y) = \{(m+2)\alpha + \beta\}g(X, Y) + (m+1)\beta\, \theta(X)\theta(Y), \tag{3.5}$$

$$\widetilde{g}(\widetilde{R}(\xi, Y)X, N) = \alpha g(X, Y) + \beta\, \theta(X)\theta(Y), \tag{3.6}$$

$$\epsilon\widetilde{g}(\widetilde{R}(L, Y)X, L) = \alpha g(X, Y) + \beta\, \theta(X)\theta(Y), \tag{3.7}$$

for all $X, Y \in \Gamma(TM)$. Substituting (3.5)~(3.7) into (3.4), we have

$$\begin{aligned} R^{(0,2)}(X, Y) \;=\; & (m\alpha + \beta)g(X, Y) + (m-1)\beta\theta(X)\theta(Y) \\ & + B(X, Y)trA_N + D(X, Y)trA_L - g(A_N X, A^*_\xi Y) \\ & - \epsilon\, g(A_L X, A_L Y) + \rho(X)\phi(Y), \;\; \forall\, X, Y \in \Gamma(TM). \end{aligned} \tag{3.8}$$

Definition 1. We say that the screen distribution $S(TM)$ of M is *totally umbilical* [6] in M if, on any coordinate neighborhood $\mathcal{U} \subset M$, there is a smooth function γ such that $A_N X = \gamma PX$ for any $X \in \Gamma(TM)$, or equivalently,

$$C(X, PY) = \gamma\, g(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.9}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* in M.

Theorem 3.2. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of quasi-constant curvature. If the curvature vector field ζ of $\widetilde{M}$ is tangent to M, $S(TM)$ is totally umbilical in M and $S(TM^{\perp})$ is a conformal Killing distribution, then the tensor field $R^{(0,2)}$ is an induced symmetric Ricci tensor of M.*

Proof. From (2.9), (2.13), (2.14), (2.26) and (3.8), for all $X, Y \in \Gamma(TM)$, we have

$$D(X, Y) = \epsilon\delta\, g(X, Y), \qquad \phi(X) = 0, \qquad A_L X = \delta\, PX + \epsilon\rho(X)\xi\,; \tag{3.10}$$

$$\begin{aligned} R^{(0,2)}(X, Y) \;=\; & \{m\alpha + \beta + (m-1)\epsilon\delta^2 + \delta\rho(\xi)\}\, g(X, Y) + m\gamma\, B(X, Y) \\ & + (m-1)\beta\, \theta(X)\theta(Y) - \gamma\, g(X, A^*_\xi Y), \;\; \forall\, X, Y \in \Gamma(TM). \end{aligned} \tag{3.11}$$

Using (3.11) and the fact A^*_ξ is self-adjoint, we show that $R^{(0,2)}$ is symmetric.

3.1. Proof of Theorem 1.1

As $C = 0$, we have $\widetilde{g}(R(X, Y)PZ, N) = 0$ due to (2.22). From (2.18) and $(3.10)_1$, we have $\widetilde{g}(\widetilde{R}(X, Y)PZ, N) = \delta\{g(X, PZ)\rho(Y) - g(Y, PZ)\rho(X)\}$. By Theorem 3.1 and 3.2, we get $d\tau = 0$ on TM. Thus we have $\widetilde{g}(\widetilde{R}(X, Y)\xi, N) = 0$ due to (2.20). From the above results, we deduce the following equation

$$\widetilde{g}(\widetilde{R}(X, Y)Z, N) = \delta\{g(X, Z)\rho(Y) - g(Y, Z)\rho(X)\}, \;\; \forall\, X, Y, Z \in \Gamma(TM). \tag{3.12}$$

Replacing X by ξ and Z by X to (3.12) and then, comparing with (3.6), we have

$$\beta\, \theta(X)\theta(Y) \;=\; -\{\alpha + \delta\rho(\xi)\}\, g(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.13}$$

Case (1). If $S(TM^{\perp})$ is a Killing distribution, i.e., $\delta = 0$, then we have

$$\beta\, \theta(X)\theta(Y) \;=\; -\,\alpha\, g(X, Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.14}$$

Substituting (3.14) into (1.1) and using (2.16) and the facts $\widetilde{g}(\widetilde{R}(X,Y)Z,\ \xi) = 0$ and $\widetilde{g}(\widetilde{R}(X,Y)Z,\ L) = 0$ due to (1.1), we have

$$R(X,Y)Z = -\,\alpha\{g(Y,Z)X - g(X,Z)Y\}, \quad \forall\, X,\ Y,\ Z \in \Gamma(TM). \tag{3.15}$$

Thus M is a space of constant curvature $-\,\alpha$. Taking $X = Y = \zeta$ to (3.14), we have $\beta = -\,\alpha$. Substituting (3.14) into (3.11) with $\delta = \gamma = 0$, we have

$$Ric(X,Y) = 0, \quad \forall\, X,\ Y \in \Gamma(TM).$$

On the other hand, substituting (3.15) and $g(R(\xi,Y)X,\ N) = 0$ into (3.3), we have

$$Ric(X,Y) = -\,(m-1)\alpha\, g(X,Y), \quad \forall\, X,\ Y \in \Gamma(TM).$$

From the last two equations, we get $\alpha = 0$ as $m > 1$. Thus $\beta = 0$ and $\widetilde{M}$ and M are flat manifolds by (1.1) and (3.15). From this result and (2.21), we show that M^* is also flat.

Case (2). If $S(TM^{\perp})$ is a conformal Killing distribution. Assume that $\beta \neq 0$. Taking $X = Y = \zeta$ to (3.13), we have $\beta = -\{\alpha + \delta\rho(\xi)\}$. From this and (3.13), we show that

$$g(X,Y) = \theta(X)\theta(Y), \quad \forall\, X,\ Y \in \Gamma(TM). \tag{3.16}$$

Substituting (3.16) into (1.1) and using (2.16) with $C = 0$ and $(3.10)_1$, we have

$$g(R(X,Y)Z,\ W) = (\alpha + 2\beta + \epsilon\delta^2)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \tag{3.17}$$

for all $X,\ Y,\ Z,\ W \in \Gamma(TM)$. Substituting (3.16) into (3.11) with $\gamma = 0$, we have

$$Ric(X,Y) = (m-1)\{\alpha + \beta + \epsilon\delta^2\}g(X,Y), \quad \forall\, X,\ Y \in \Gamma(TM),$$

due to the fact $\delta\rho(\xi) = -(\alpha+\beta)$. On the other hand, from (2.18), (3.6) and (3.13), we have $g(R(\xi,Y)X,\ N) = 0$. Substituting this result and (3.17) into (3.3), we have

$$Ric(X,Y) = (m-1)\{\alpha + 2\beta + \epsilon\delta^2\}\, g(X,Y), \quad \forall\, X,\ Y \in \Gamma(TM).$$

The last two equations imply $\beta = 0$ as $m > 1$. It is a contradiction. Thus $\beta = 0$ and $\widetilde{M}$ is a space of constant curvature α. From (2.21) and (3.17), we show that M^* is a space of constant curvature $(\alpha + \epsilon\delta^2)$. But M is not a space of constant curvature by $(3.10)_3$.

Let $\kappa = (m-1)(\alpha + \epsilon\delta^2)$. Then the last two equations reduce to

$$R^{(0,2)}(X,Y) = Ric(X,Y) = \kappa g(X,Y), \ \ \forall\, X,\ Y \in \Gamma(TM). \tag{3.18}$$

Thus M is an Einstein manifold. The scalar quantity r of M [4], obtained from $R^{(0,2)}$ by the method of (2.25), is given by

$$r = R^{(0,2)}(\xi,\xi) + \sum_{a=1}^{m} \epsilon_a\, R^{(0,2)}(W_a, W_a).$$

Since M is an Einstein manifold satisfying (3.18), we obtain

$$r = \kappa g(\xi,\ \xi) + \kappa \sum_{a=1}^{m} \epsilon_a\, g(W_a,\ W_a) = \kappa\, m.$$

Thus we have

$$Ric(X,Y) = (r/m)g(X,Y)$$

which provides a geometric interpretation of half lightlike Einstein submanifold (same as in Riemannian case) as we have shown that the constant $\kappa = r/m$.

3.2. Proof of Theorem 1.2

Assume that ζ is tangent to M, $S(TM)$ is totally umbilical and $S(TM^{\perp})$ is a conformal Killing vector field. Using (1.1), the equation (2.19) reduce to

$$(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) = B(X,Z)\rho(Y) - B(Y,Z)\rho(X), \tag{3.19}$$

for all $X,\ Y,\ Z \in \Gamma(TM)$. Replacing W by N to (1.1), we have

$$\begin{aligned}\widetilde{g}(\widetilde{R}(X,Y)Z,\,N) &= \{\alpha\eta(X) + e\beta\theta(X)\}g(Y,Z)\\ &\quad - \{\alpha\eta(Y) + e\beta\theta(Y)\}g(X,Z) + \beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z),\end{aligned} \tag{3.20}$$

for all $X,\ Y,\ Z \in \Gamma(TM)$. Applying ∇_X to (3.9) and using (2.10), we have

$$(\nabla_X C)(Y,PZ) = (X\gamma)g(Y,PZ) + \gamma B(X,PZ)\eta(Y), \quad \forall\, X,\ Y,\ Z \in \Gamma(TM).$$

Substituting this equation into (2.22), we obtain

$$\begin{aligned}\widetilde{g}(R(X,Y)PZ,\,N) \;=\; &\{X\gamma - \gamma\tau(X)\}g(Y,PZ) - \{Y\gamma - \gamma\tau(Y)\}g(X,PZ)\\ &+ \gamma B(X,PZ)\eta(Y) - \gamma B(Y,PZ)\eta(X), \quad \forall\, X,\ Y,\ Z \in \Gamma(TM).\end{aligned}$$

Substituting this equation and (3.20) into (2.18) and using $\theta(\xi) = 0$, we obtain

$$\begin{aligned}&\gamma\, B(X,Z)\eta(Y) - \{Y[\gamma] - \gamma\tau(Y) - \alpha\eta(Y) - e\beta\theta(Y) - \delta\rho(Y)\}g(X,Z)\\ &= \;\gamma\, B(Y,Z)\eta(X) - \{X[\gamma] - \gamma\tau(X) - \alpha\eta(X) - e\beta\theta(X) - \delta\rho(X)\}g(Y,Z)\\ &\quad + \beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z), \quad \forall\, X,\ Y,\ Z \in \Gamma(TM).\end{aligned}$$

Replacing Y by ξ to this and using $(2.9)_1$ and the fact $\theta(\xi) = 0$, we have

$$\gamma B(X,Y) = \{\xi[\gamma] - \gamma\tau(\xi) - \alpha - \delta\rho(\xi)\}g(X,Y) - \beta\theta(X)\theta(Y), \tag{3.21}$$

for all $X,\ Y \in \Gamma(TM)$. Differentiating (2.26) and using (3.19), we have

$$\begin{aligned}&\{\delta\eta(X) - \epsilon\rho(X)\}B(Y,Z) - \{\delta\eta(Y) - \epsilon\rho(Y)\}B(X,Z)\\ &\qquad = \;(X\delta)\, g(Y,Z) - (Y\delta)\, g(X,Z).\end{aligned}$$

Replacing Y by ξ in the last equation and using $(2.9)_1$, we obtain

$$\{\delta - \epsilon\rho(\xi)\}B(X,Z) = (\xi\delta)\, g(X,Z).$$

As the conformal factor δ is non-constant, we show that $\delta - \epsilon\rho(\xi) \neq 0$. Thus we have

$$B(X,Y) = \sigma g(X,Y), \quad \forall\, X,\ Y \in \Gamma(TM), \tag{3.22}$$

where $\sigma = (\xi\delta)(\delta - \epsilon\rho(\xi))^{-1}$. From $(3.10)_1$ and (3.22), we show that the second fundamental form tensor h, given by $h(X,Y) = B(X,Y)N + D(X,Y)L$, satisfies

$$h(X,Y) = \mathcal{H}\, g(X,Y), \quad \forall\, X,\ Y \in \Gamma(TM).$$

Thus M is totally umbilical. Substituting (3.22) into (3.21), we have

$$\{\xi[\gamma] - \gamma\tau(\xi) - \sigma\gamma - \alpha - \delta\rho(\xi)\}g(X,Y) = \beta\theta(X)\theta(Y), \quad \forall\, X,\ Y \in \Gamma(TM).$$

Taking $X = Y = \zeta$ to this equation, we have $\beta = \xi[\gamma] - \gamma\tau(\xi) - \sigma\gamma - \alpha - \delta\rho(\xi)$.

Assume that $\beta \neq 0$. Then we have

$$g(X,Y) = \theta(X)\theta(Y), \quad \forall\, X,\ Y \in \Gamma(TM). \tag{3.23}$$

Substituting (3.23) into (1.1) and using (2.16), (3.9) and $(3.10)_1$, we have

$$g(R(X,Y)Z,\,W) = (\alpha + 2\beta + \sigma\gamma + \epsilon\delta^2)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \tag{3.24}$$

for all $X,\ Y,\ Z,\ W \in \Gamma(TM)$. Substituting (3.22) and (3.23) into (3.11), we have

$$Ric(X,Y) = \{m(\alpha+\beta) + (m-1)(\sigma\gamma + \epsilon\delta^2) + \delta\rho(\xi)\}g(X,Y). \tag{3.25}$$

On the other hand, substituting (3.24) and the fact

$$\widetilde{g}(R(\xi,Y)X,\,N) = \{\alpha + \beta + \delta\rho(\xi)\}g(X,Y)$$

into (3.3), we have

$$Ric(X,Y) = \{m\alpha + (2m-1)\beta + (m-1)(\sigma\gamma + \epsilon\delta^2) + \delta\rho(\xi)\}g(X,Y). \tag{3.26}$$

Comparing (3.25) and (3.26), we obtain $(m-1)\beta = 0$. As $m > 1$, we have $\beta = 0$. It is a contradiction. Thus we have $\beta = 0$. Consequently, by (1.1), (2.21) and (3.24), we show that $\widetilde{M}$ and M^* are spaces of constant curvatures α and $(\alpha + 2\sigma\gamma + \epsilon\delta^2)$ respectively.

Let $\kappa = m\alpha + (m-1)(\sigma\gamma + \epsilon\delta^2) + \delta\rho(\xi)$. Then (3.25) and (3.26) reduce to

$$R^{(0,2)}(X,Y) = Ric(X,Y) = \kappa g(X,Y), \quad \forall\, X,\ Y \in \Gamma(TM).$$

Thus M is an Einstein manifold. The scalar quantity r of M is given by

$$\begin{aligned} r &= R^{(0,2)}(\xi,\xi) + \sum_{a=1}^{m} \epsilon_a\, R^{(0,2)}(W_a, W_a) \\ &= \kappa g(\xi,\,\xi) + \kappa \sum_{a=1}^{m} \epsilon_a\, g(W_a,\,W_a) \;=\; \kappa\, m. \end{aligned}$$

Thus we have

$$Ric(X,Y) = (r/m)g(X,Y).$$

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