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# The Families of $L$-Series Associated with Decomposition of the Generating Functions

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**Abstract.** By using periodic functions from the nonnegative integers to the complex numbers, we generalize the generating function of the $q$–Apostol type Eulerian polynomials and numbers attached the character defined in [1]. Then using this generating function, we a construct new $L$–type series. By using periodic functions, we derive decomposition of the generating functions for the $q$–Euler numbers and polynomials. Applying the Mellin transformation to the decomposition of the generating functions, we introduce and investigate the various properties of a certain new family of the Dirichlet type $L$–series and the Dirichlet $L$–function. Finally, we derive many potentially useful results involving these functions polynomials and numbers.

## 1. Introduction and Main Definition

A Dirichlet $L_k$-series is defined by the following the form

$$L_k(s) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s},$$

where $\chi_k$, the number theoretic character, is an integer function with period $k$ and $s$ a complex variable with real part greater than 1. If $\chi_k$ is a Dirichlet character, then $L_k(s)$ is reduces to the well-known the Dirichlet $L$–function. These series are used in many branches of Mathematics. These series especially are very important in additive number theory and in analytic number theory. These series were used to prove Dirichlet's theorem and also related to the modular forms, the automorphic form, the Dirichlet $L$–functions, the Lerch transcendental function, the Riemann zeta function, and the other special functions. All of these functions are fundamentally important in Analytic Number Theory and in Complex Analysis. The family of zeta functions are also appeared in quantum statistics (the Fermi–Dirac and the Bose-Einstein integral functions) and quantum interference and entanglement (cf. [7], [24]).

In this paper, replacing a Dirichlet character with a periodic function from the nonnegative integers to the complex numbers, we modify the generating function of the $q$–Apostol type Eulerian polynomials and

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numbers attached the character defined in [1]. Using the generating function of the $q$–Eulerian polynomials attached a periodic function $\chi$, we construct a new $L$–type series. We get some fundamental properties of these series and also decompose the generating functions for the $q$–Eulerian polynomials attached to the periodic function. This decomposition provided us to compute the $q$–Apostol–type Eulerian polynomials more easily. By using this inspiration, we derive some new decompositions for the $q$–Apostol–type Eulerian polynomials and $L$–type series attached to the periodic function with the period $f$. These decompositions are related to the period of the periodic function. We summarize our results in detail as follows;

In Section 1, we give generating function for the $q$–Apostol type Eulerian polynomials and numbers attached a periodic function $\chi$. By applying the Mellin transformation to these generating functions, we define interpolation functions for the $q$–Apostol type Eulerian numbers and polynomials. We also define a family of Dirichlet type $L$–series related to the family of Dirichlet type zeta function.

In Section 3, we give some algebraic concepts which are related to the sets and some properties of subgroups. By using these concepts, we decompose a family of Dirichlet type $L$–series which interpolate generalized Eulerian numbers and polynomials at negative integers. We also give one example which are related to our decomposition theorem.

Throughout this paper, we use the following standard notions:

$\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathbb{N} \cup \{0\}$ and also, as usual, $\mathbb{R}$ denotes the set of real number, $\mathbb{R}^+$ denotes the set of positive real number and $\mathbb{C}$ denotes the set of complex numbers.

If $q \in \mathbb{R}$ then we assume that $0 < q < 1$. If $q \in \mathbb{C}$ then we assume that $\left|q\right| < 1$. Then

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q}, \text{ if } q \neq 1 \\ x, \text{ if } q = 1. \end{cases}$$

## 2. The $q$-Eulerian Polynomials and Numbers

Let $\chi$ be a function from $\mathbb{N}_0$ to $\mathbb{C}$. If there is a positive integer $f$ such that $\chi(fm + x) = \chi(x)$ and for $m, x \in \mathbb{N}_0$, then $\chi$ is called a periodic function with the period $f$. It is clear that any character with the conductor $f$ is a periodic function with the period $f$. Then by using the periodic function, we modify our definition of the $q$-Apostol type Eulerian polynomials and numbers attached the character $\chi$ in [1]:

**Definition 2.1.** *Let $a$, $b \in \mathbb{R}^+$ ($a \neq b$ and $a \geq 1$), $u \in \mathbb{C} \setminus \{1\}$, $\lambda, q \in \mathbb{C}$ with $\left|q\right| < 1$. Let $\chi$ be a function from $\mathbb{N}_0$ to $\mathbb{C}$ with the period $f$.*

*i) The $q$–Eulerian numbers attached the character $\chi$:*

$$\mathcal{H}_{n,\chi}(u; a, b; \lambda; q)$$

*are defined by means of the following generating function:*

$$F_{\lambda,q,\chi}(t, u, a, b) = \left(1 - \frac{a^{[f]t}}{u^f}\right) \sum_{m=0}^{\infty} \left(\frac{\lambda}{u}\right)^m b^{[m]t} \chi(m) = \sum_{n=0}^{\infty} H_{n,\chi}(u; a, b; \lambda; q) \frac{t^n}{n!}, \tag{1}$$

*ii) The $q$–Eulerian polynomials attached the character $\chi$:*

$$\mathcal{H}_{n,\chi}(x; u; a, b; \lambda; q)$$

*are defined by means of the following generating function:*

$$F_{\lambda,q,\chi}(t, x, u, a, b) = \left(1 - \frac{a^{[f]t}}{u^f}\right) \sum_{m=0}^{\infty} \left(\frac{\lambda}{u}\right)^m b^{[m+x]t} \chi(m) = \sum_{n=0}^{\infty} H_{n,\chi}(x; u; a, b; \lambda; q) \frac{t^n}{n!} \tag{2}$$

*where*

$$\left|\frac{\lambda}{u} b^t\right| < 1.$$

Observe that

$$\mathcal{H}_{n,\chi}(0;u;a,b;\lambda;q)=\mathcal{H}_{n,\chi}(u;a,b;\lambda;q),$$

which denotes the $q$–Eulerian numbers. When $q\to 1$ and $\chi(m)\equiv 1$ for all $m\in\mathbb{N}_0$ into (2), we have

$$\left(1-\frac{a^t}{u}\right)\sum_{m=0}^{\infty}\left(\frac{\lambda}{u}\right)^m b^{(m+x)t}=\sum_{n=0}^{\infty}H_n(x;u;a,b;\lambda)\frac{t^n}{n!}$$

(*cf.* [22], [18], [19]). Setting $a=\lambda=1$ and $b=e$ into the above equation, we arrive at the generating function for the Frobenious-Euler polynomials, $H_n(x;u)=H_n(x;u;1,e;1)$ :

$$\frac{1-u}{e^t-u}e^{tx}=\sum_{n=0}^{\infty}H_n(x;u)\frac{t^n}{n!}.$$

Setting $u=-1$ into the above equation, we have generating function for the classical Euler polynomials

$$\frac{2}{e^t+1}e^{tx}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}$$

which of course $E_n(0)=E_n$, denotes the classical Euler numbers (*cf.* [1]-[23]).

By using the period $f$ of $\chi$ with $\chi(f)=1$ and combining with $F_{\lambda,q}(t,x,u,a,b)$, we modify (1) and (2), respectively as follows:

$$F_{\lambda,q,\chi}(t,u,a,b)=\sum_{i=0}^{f-1}\left(\frac{\lambda}{u}\right)^i\chi(i)F_{\lambda^f,q^f}\left(t[f],\frac{i}{f},u^f,a,b\right)$$

and

$$F_{\lambda,q,\chi}(t,x,u,a,b)=\sum_{i=0}^{f-1}\left(\frac{\lambda}{u}\right)^i\chi(i)F_{\lambda^f,q^f}\left([f]t,\frac{i+x}{f},u^f,a,b\right). \tag{3}$$

We also note that Equation (1) is the unique solution of the following a $q$–difference equation:

$$F_{\lambda,q,\chi}(t,u,a,b)=\left(1-\frac{a^{[f]t}}{u^f}\right)\sum_{m=0}^{f-1}\left(\frac{\lambda}{u}\right)^m\chi(m)b^{[m]t}+\left(\frac{\lambda}{u}\right)^f b^{[f]t}F_{\lambda,q,\chi}\left(t,u,a,b^{q^f}\right).$$

On the other hand, we get

$$F_{\lambda,q,\chi}(t,x,u,a,b)=\sum_{m=0}^{\infty}\left(\frac{\lambda}{u}\right)^m\chi(m)\left(e^{([m+x]\ln b)t}-\frac{1}{u^f}e^{([f]\ln a+[m+x]\ln b)t}\right).$$

Therefore, by using the expression of expansional function in the above equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty}H_{n,\chi}(x;u;a,b;\lambda;q)\frac{t^n}{n!} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^m\chi(m)\\ &\quad\left(([m+x]\ln b)^n-\frac{1}{u^f}\left([f]\ln a+[m+x]\ln b\right)^n\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

**Theorem 2.2.** *Let $a$, $b \in \mathbb{R}^+$, $u \in \mathbb{C}\setminus\{1\}$, $\lambda, q \in \mathbb{C}$ such that $\left|q\right| < 1, \left|\frac{\lambda \ln b}{u}\right| < 1, a \neq b$ and $a \geq 1$. For positive integer $n$,*

$$H_{n,\chi}(x;u;a,b;\lambda;q) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{u}\right)^m \chi(m) \left(([m+x]\ln b)^n - \frac{1}{u^f}\left([f]\ln a + [m+x]\ln b\right)^n\right). \tag{4}$$

Let $a$, $b \in \mathbb{R}^+$, $u \in \mathbb{C}\setminus\{1\}$, $\lambda, q \in \mathbb{C}$ such that $\left|q\right| < 1, \left|\frac{\lambda b^t}{u}\right| < 1, a \neq b$ and $a \geq 1$. From [18], we recall the following generating function:

$$F_{\lambda,q}(t,x,u,a,b) = \left(1 - \frac{a^t}{u}\right)\sum_{m=0}^{\infty}\left(\frac{\lambda}{u}\right)^m b^{[m+x]t} = \sum_{m=0}^{\infty} H_{m,\lambda,q}(x;u;a;b)\frac{t^m}{m!}. \tag{5}$$

By applying the Mellin transformation to the generating function in Equation (5), we get the following integral representation of the family zeta functions:

$$\zeta_{\lambda,q}(s,x,u,a,b) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}F_{\lambda,q}(-t,x,u,a,b)dt \;\; (\min\{\mathcal{R}(s),\mathcal{R}(x)\} > 0), \tag{6}$$

where the additional constraint $\mathcal{R}(x) > 0$ is required for the convergence of the infinite integral occurring on the right-hand side at its upper terminal.

By using the above integral representation, we are ready to define the following definition of the Hurwitz-type zeta function:

**Definition 2.3.** *Let $a, b \in \mathbb{R}^+$ with $a \neq b (a \geq 1), u \in \mathbb{C}\setminus\{1\}$, $\lambda, q \in \mathbb{C}$ such that $\left|q\right| < 1$ and $\left|\frac{\lambda}{u}\right| < 1$. We define the Hurwitz-Lerch type zeta function*

$$\zeta_{\lambda,q}(s,x,u,a,b) = \sum_{m=0}^{\infty}\left(\frac{\lambda}{u}\right)^m\left(\frac{1}{([m+x]\ln b)^s} - \frac{1}{u\left(\ln a + [m+x]\ln b\right)^s}\right). \tag{7}$$

Setting $x = 0$ in (7), we obtain the Riemann type zeta function as follows;

$$\zeta_{\lambda,q}(s,u,a,b) = \sum_{m=1}^{\infty}\left(\frac{\lambda}{u}\right)^m\left(\frac{1}{([m]\ln b)^s} - \frac{1}{u\left(\ln a + [m]\ln b\right)^s}\right). \tag{8}$$

By applying the elementary series identity (cf. [23]):

$$\sum_{n=0}^{\infty}\Lambda(n) = \sum_{j=0}^{f-1}\sum_{m=0}^{\infty}\Lambda(j+fm) \quad (f \in \mathbb{N}). \tag{9}$$

It is not difficult to derive the following alternative form of the definition (8). That is, by the following theorem, we give a relation between the function $\zeta_{\lambda,q}(s,x,u,a,b)$ and the function $\zeta_{\lambda,q}(s,u,a,b)$:

**Theorem 2.4.** *For an integer $d$ and $s \in \mathbb{C}$, we have*

$$\zeta_{\lambda,q}(s,u,a,b) = \left(\frac{1}{[d]}\right)^s\sum_{i=0}^{d}\left(\frac{\lambda}{u}\right)^i\zeta_{\lambda^d,q^d}\left(s,\frac{i}{d},u^d,a^{\frac{u^{1-d}}{[d]}},b^{u^{1-d}}\right).$$

*Proof.* By applying (9) to (8), we get

$$\begin{aligned}
\zeta_{\lambda,q}(s,u,a,b) &= \sum_{n=1}^{\infty}\sum_{i=0}^{d-1}\left(\frac{\lambda}{u}\right)^{nd+i}\left(\frac{1}{([nd+i]\ln b)^s}-\frac{1}{u\,(\ln a+[nd+i]\ln b)^s}\right)\\
&= \left(\frac{1}{[d]}\right)^s\sum_{n=1}^{\infty}\sum_{i=0}^{d-1}\left(\frac{\lambda}{u}\right)^{i}\left(\frac{\lambda^d}{u^d}\right)^{n}\\
&\left(\frac{1}{\left(\left[\left(n+\frac{i}{d}\right):q^d\right]\ln b\right)^s}-\frac{1}{u^d\left(\ln a^{\frac{u^{1-d}}{[d]}}+\left[\left(n+\frac{i}{d}\right):q^d\right]\ln b^{u^{1-d}}\right)^s}\right)\\
&= \left(\frac{1}{[d]}\right)^s\sum_{i=0}^{d}\left(\frac{\lambda}{u}\right)^{i}\zeta_{\lambda^d,q^d}\left(s,\frac{i}{d},u^d,a^{\frac{u^{1-d}}{[d]}},b^{u^{1-d}}\right).
\end{aligned}$$

Thus we complete the proof. □

When $\chi(f)=1$, by applying the Mellin transformation to (3) and using (6), we obtain

$$L_{\lambda,q,\chi}(s,x,u,a,b) = \sum_{i=0}^{f-1}\left(\frac{\lambda}{u}\right)^{i}\chi(i)\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}F_{\lambda^f,q^f}\left([f]t,\frac{i+x}{f},u^f,a,b\right)dt \tag{10}$$

$(\min\{\mathcal{R}(s),\mathcal{R}(x)\} > 0)$.

By using the above integral representation, a relationship between the functions $L_{\lambda,q,\chi}(s,x,u,a,b)$ and $\zeta_{\lambda,q}(s,x,u,a,b)$ is provided by Theorem 2.5 below.

**Theorem 2.5.** *Let $s\in\mathbb{C}$. Also let $\chi$ be a periodic function from $\mathbb{N}_0$ to $\mathbb{C}$ with the period $f$ and $\chi(f)=1$. Then*

$$L_{\lambda,q,\chi}(s,x,u,a,b) = \frac{1}{[f]^s}\sum_{i=0}^{f-1}\left(\frac{\lambda}{u}\right)^{i}\chi(i)\zeta_{\lambda^f,q^f}\left(s,\frac{i+x}{f},u^f,a,b\right). \tag{11}$$

Consequently, combining Equation (7) with Equation (11), we are ready to define a two-variable $L$–series as follows:

**Definition 2.6.** *Let $a,\ b\in\mathbb{R}^+$, $u\in\mathbb{C}\setminus\{1\}$, $\lambda,q\in\mathbb{C}$ such that $|q|<1,\left|\frac{\lambda}{u}\right|<1, a\neq b$ and $a\geq 1$. We define the following two-variable L-series*

$$L_{\lambda,q,\chi}(s,x,u,a,b) = \sum_{m=0}^{\infty}\left(\frac{\lambda}{u}\right)^{m}\chi(m)\left(\frac{1}{([m+x]\ln b)^s}-\frac{1}{u^f\,([f]\ln a+[m+x]\ln b)^s}\right). \tag{12}$$

Substituting $x=1$ into Equation (12), we get one-variable $L$–series

$$L_{\lambda,q,\chi}(s,u,a,b) = L_{\lambda,q,\chi}(s,1,u,a,b)$$

by

$$L_{\lambda,q,\chi}(s,u,a,b) = \sum_{m=1}^{\infty}\left(\frac{\lambda}{u}\right)^{m}\chi(m)\left(\frac{1}{([m]\ln b)^s}-\frac{1}{u^f\,([f]\ln a+[m]\ln b)^s}\right).$$

As asserted by Theorem 2.7 below, the $L$–series $L_{\lambda,q,\chi}(s,x,u,a,b)$ can be used to interpolate the generalized Eulerian polynomials $\mathcal{H}_{n,\chi}(x;u;a,b;\lambda;q)$ defined by the generating function in (3) attached to any periodic function from $\mathbb{N}_0$ to $\mathbb{C}$ with the period $f$.

Substituting $s=-n$ ($n\in\mathbb{N}$) into Equation (12), we get Theorem 4. This result gives us the proof of the following Theorem.

**Theorem 2.7.** *Let n be an positive integer. Then we have*

$$L_{\lambda,q,\chi}(-n,x,u,a,b) = \mathcal{H}_{n,\chi}(x;u;a,b;\lambda;q).$$

## 3. Decomposition of the Generating Function and $L$–Functions

In this section, our aim is to decompose $L$–function attached to any periodic function $\chi$ from $\mathbb{N}_0$ to $\mathbb{C}$ with the period $f$. Here we use the notations $lcm(x,y)$ for the least common multiple ($gcd(x,y)$ for the greatest common divisor) of $x$ and $y$ and we also need the following notations otherwise stated;

$i)xA = \{xa : a \in A\}$ for any subset $A \subseteq \mathbb{N}_0$ and $x \in \mathbb{N}_0$.

$ii)A_n = \cup_{i=1}^{n}(dp_i\mathbb{N}_0)$ where $d \in \mathbb{N}$, $p_i$ is a prime number such that $gcd(dp_i, dp_j) = d$ for all $i, j \in \{1, ..., n\}$ and $i \neq j$,

$iii)A_{n0} = \{dp_1, ..., dp_n\}$,

$iv)A_{ni} = \{lcm(a,b) : \ a, b \in A_{n(i-1)}\}$.

We start to recall the fact that

$$x\mathbb{N}_0 \cap y\mathbb{N}_0 = lcm(x,y)\mathbb{N}_0$$

for a positive integers $x$ and $y$. In particular, we get $x\mathbb{N}_0 \cap y\mathbb{N}_0 = xy\mathbb{N}_0$ whenever $x$ and $y$ are distinct prime numbers. We recall the following result from [1, Theorem 3.1] with the modified proof .

**Theorem 3.1.** *With the above notations, we get*

$$\sum_{i\in A_n} F(i) = \sum_{j=0}^{n-1}(-1)^j \sum_{l\in A_{nj}} \sum_{i\in l\mathbb{N}_0} F(i). \tag{13}$$

*Proof.* For the proof of this Theorem, we use the induction method.

For $i, j \in \{1, ..., n\}$, it is clear that $lcm(dp_i, dp_j) = dp_ip_j$ since $\gcd(p_i, p_j) = 1$.

If $n = 2$ in (13), then $A_{20} = \{dp_1, dp_2\}$ and $A_{21} = \{dp_1p_2\}$. We have

$$\sum_{i\in A_2} F(i) = \sum_{j=0}^{1}(-1)^j \sum_{l\in A_{2j}} \sum_{i\in l\mathbb{N}_0} F(i). \tag{14}$$

If $n = 3$ in (13), then $A_{30} = \{dp_1, dp_2, dp_3\} = A_{20} \cup \{dp_3\}$, $A_{31} = p_3A_{20} \cup A_{21} = \{dp_1p_2, dp_1p_3, dp_2p_3\}$ and $A_{32} = p_3A_{21} = \{dp_1p_2p_3\}$. By using the De-Morgan's law of sets, we get the following equality

$$(dp_3\mathbb{N}_0) \cap A_2 = \cup_{i=1}^{2}(dp_3\mathbb{N}_0 \cap dp_i\mathbb{N}_0) = p_3A_2.$$

Then

$$\sum_{i\in\cup_{i=1}^{2} dp_3p_i\mathbb{N}_0} F(i) = \sum_{j=0}^{1}(-1)^j \sum_{l\in p_3A_{2j}} \sum_{i\in l\mathbb{N}_0} F(i). \tag{15}$$

By combining Equation (14) with (15), we get

$$\begin{aligned}
\sum_{i\in A_3} F(i) &= \sum_{i\in p_3d\mathbb{N}_0} F(i) + \sum_{i\in A_2} F(i) - \sum_{i\in\cup_{i=1}^{2} dp_3p_i\mathbb{N}_0} F(i) \\
&= \sum_{i\in p_3d\mathbb{N}_0} F(i) + \left(\sum_{j=0}^{1}(-1)^j \sum_{l\in A_{2j}} \sum_{i\in l\mathbb{N}_0} F(i)\right) - \left(\sum_{j=0}^{1}(-1)^j \sum_{l\in p_3A_{2j}} \sum_{i\in l\mathbb{N}_0} F(i)\right) \\
&= \sum_{l\in A_{30}} \sum_{i\in l\mathbb{N}_0} F(i) - \left(\sum_{l\in A_{21}} \sum_{i\in l\mathbb{N}_0} F(i) + \sum_{l\in p_3A_{20}} \sum_{i\in l\mathbb{N}_0} F(i)\right) + \sum_{l\in p_3A_{21}} \sum_{i\in l\mathbb{N}_0} F(i) \\
&= \sum_{l\in A_{30}} \sum_{i\in l\mathbb{N}_0} F(i) - \sum_{l\in A_{31}} \sum_{i\in l\mathbb{N}_0} F(i) + \sum_{l\in A_{32}} \sum_{i\in l\mathbb{N}_0} F(i)
\end{aligned}$$

Now we assume the hypothesis is hold for $n-1$, and construct the following sets

$$\begin{aligned} A_{n0} &= \{dp_1, ..., dp_n\} = \{dp_n\} \cup A_{(n-1)0} \\ A_{ni} &= A_{(n-1)i} \cup p_n A_{(n-1)(i-1)} \end{aligned}$$

for all $0 < i < n-1$ and $A_{n(n-1)} = p_n A_{(n-1)(i-2)}$. It is clear that $p_n A_{(n-1)(i-1)} \cap A_{(n-1)i} = \emptyset$ for all $i$. Moreover, we have

$$(dp_n \mathbb{N}_0) \cap A_{n-1} = p_n A_{n-1}.$$

By the induction on $n$, we get

$$\sum_{i \in A_{n-1}} F(i) = \sum_{j=0}^{n-2} (-1)^j \sum_{l \in A_{(n-1)j}} \sum_{i \in l\mathbb{N}_0} F(i), \tag{16}$$

$$\sum_{i \in p_n A_{n-1}} F(i) = \sum_{j=0}^{n-2} (-1)^j \sum_{l \in p_n A_{(n-1)j}} \sum_{i \in l\mathbb{N}_0} F(i). \tag{17}$$

By using Equation (16) and (17), we have

$$\begin{aligned} \sum_{i \in A_n} F(i) &= \sum_{i \in p_n d\mathbb{N}_0} F(i) + \left( \sum_{j=0}^{n-2} (-1)^j \sum_{l \in A_{(n-1)j}} \sum_{i \in l\mathbb{N}_0} F(i) \right) - \sum_{i \in p_n A_{n-1}} F(i) \\ &= \sum_{l \in A_{n0}} \sum_{i \in l\mathbb{N}_0} F(i) - \left( \sum_{l \in A_{(n-1)1}} \sum_{i \in l\mathbb{N}_0} F(i) + \sum_{l \in p_n A_{(n-1)0}} \sum_{i \in l\mathbb{N}_0} F(i) \right) + \\ &\quad \left( \sum_{j=2}^{n-2} (-1)^j \sum_{l \in A_{(n-1)j}} \sum_{i \in l\mathbb{N}_0} F(i) \right) - \left( \sum_{j=1}^{n-2} (-1)^j \sum_{l \in p_n A_{(n-1)j}} \sum_{i \in l\mathbb{N}_0} F(i) \right) \\ &= \sum_{j=0}^{n-1} (-1)^j \sum_{l \in A_{nj}} \sum_{i \in l\mathbb{N}_0} F(i). \end{aligned}$$

The proof is completed. $\square$

Let $f = \prod_{i=1}^{n} p_i^{t_i}$ where $p_i$ is a prime number for $i \in \mathbb{N}$ and $t_i, n \in \mathbb{N}$. Then it is easy to prove that $\mathbb{N}_0 = C \cup A_n$ and $C \cap A_n = \emptyset$ where $A_n = \cup_{i=1}^{n} (p_i \mathbb{N}_0)$ and

$$C = \{l \in \mathbb{N}_0 : \gcd(f, l) = 1\}.$$

Therefore, we get the following equation in [1, Theorem 3.4];

$$\sum_{i \in \mathbb{N}_0} F(i) = \sum_{i \in C} F(i) + \sum_{i \in A_n} F(i). \tag{18}$$

Now to decompose our generating function, we define the following two periodic functions.
*i*) Let $f = df_d$ for some positive integer $d$ and $f_d$. Then we define the function

$$\chi_d(n) := \chi(dn).$$

Then for $m \in \mathbb{N}_0$,

$$\chi_d(x) = \chi(dx) = \chi(fm + dx) = \chi(f_d dm + dx) = \chi(d(f_d m + x)) = \chi_d(f_d m + x).$$

Thus its period is $f_d$.

*ii*) We define the function

$$\chi_D(n) = \begin{cases} \chi(n) & \text{if } \gcd(n, f) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that its period is $f$.

We are ready to obtain two decompositions of the generating function for the $q$–Eulerian polynomials attached to the periodic function $\chi$ by the following theorem.

**Theorem 3.2.** *With the above notations. Then*

$$F_{\lambda,q,\chi}(t,x,u,a,b) = \sum_{j=0}^{n-1}(-1)^j \sum_{d\in A_{nj}} F_{\lambda^d,q^d,\chi_d}\left(t,\frac{x}{d},u^d,a^{[d]},b^{[d]}\right)+F_{\lambda,q,\chi_D}(t,x,u,a,b), \tag{19}$$

$$F_{\lambda,q,\chi}(t,x,u,a,b) = \sum_{j=0}^{n-1}(-1)^j \sum_{d\in A_{nj}} F_{\lambda^d,q^d,\chi_d}\left([d]t,\frac{x}{d},u^d,a,b\right)+F_{\lambda,q,\chi_D}(t,x,u,a,b). \tag{20}$$

*Proof.* We note that $f = df_d$ for some positive integer $d$ and $f_d$ and observe that

$$\begin{aligned}
\sum_{n\in d\mathbb{N}_0}\left(\frac{\lambda}{u}\right)^n b^{[n+x]t}\chi(n) &= \sum_{n\in\mathbb{N}_0}\left(\frac{\lambda}{u}\right)^{dn} b^{[dn+x]t}\chi(dn) \\
&= \sum_{n\in\mathbb{N}_0}\sum_{i=0}^{f_d-1}\left(\frac{\lambda}{u}\right)^{d(nf_d+i)} b^{\left[d(nf_d+i)+x\right]t}\chi(d(nf_d+i)) \\
&= \sum_{n\in\mathbb{N}_0}\sum_{i=0}^{f_d-1}\left(\frac{\lambda^d}{u^d}\right)^{nf_d+i} b^{t[d][nf_d+i+\frac{x}{d}]_{q^d}}\chi_d(i) \\
&= \sum_{i\in\mathbb{N}_0}\left(\frac{\lambda^d}{u^d}\right)^{i} b^{t[d][i+\frac{x}{d}]_{q^d}}\chi_d(i).
\end{aligned}$$

By using the above equation, we get

$$\left(1-\frac{a^{[f]t}}{u^f}\right)\sum_{n\in(d\mathbb{N}_0)}\left(\frac{\lambda}{u}\right)^n b^{[n+x]t}\chi(n)=\left(1-\frac{a^{[d][f_d]_{q^d}t}}{(u^d)^{f_d}}\right)\sum_{i\in\mathbb{N}_0}\left(\frac{\lambda^d}{u^d}\right)^i b^{[d][i+\frac{x}{d}]_{q^d}t}\chi_d(i). \tag{21}$$

Then the Equation (21) equal to both $F_{\lambda^d,q^d,\chi_d}(t,\frac{x}{d},u^d,a^{[d]},b^{[d]})$ and $F_{\lambda^d,q^d,\chi_d}([d]t,\frac{x}{d},u^d,a,b)$. Hence combining Equation (18), Equation (13) and Equation (21), we obtain

$$\begin{aligned}
F_{\lambda,q,\chi}(t,x,u,a,b) &= \left(1-\frac{a^{[f]t}}{u^f}\right)\sum_{m\in\mathbb{N}_0}\left(\frac{\lambda}{u}\right)^m b^{[m+x]t}\chi(m) \\
&= \left(1-\frac{a^{[f]t}}{u^f}\right)\left(\sum_{m\in C}\left(\frac{\lambda}{u}\right)^m b^{[m+x]t}\chi(m)+\sum_{m\in A_n}\left(\frac{\lambda}{u}\right)^m b^{[m+x]t}\chi(m)\right) \\
&= F_{\lambda,q,\chi_D}(t,x,u,a,b)+\sum_{j=0}^{n-1}(-1)^j\sum_{q\in A_{nj}}\left(\left(1-\frac{a^{[f]t}}{u^f}\right)\sum_{i\in q\mathbb{N}_0}\left(\frac{\lambda}{u}\right)^i b^{[i+x]t}\chi(i)\right).
\end{aligned}$$

Therefore, we obtain Equation (19). Proof of Equation (20) is similar to that of Equation (19), we omit it. □

We also obtain the decomposition of the generating function $F_{\lambda,q,\chi}(t,u,a,b)$ by substituting $x = 0$ in to Equation (19). Moreover, using Equation (19), we have

$$\begin{aligned} F_{\lambda,q,\chi}(t,x,u,a,b) &= \sum_{m=0}^{\infty}\sum_{j=0}^{n-1}(-1)^j \sum_{d\in A_{nj}} H_{m,\chi_d}\left(\frac{x}{d};u^d;a^{[d]},b^{[d]};\lambda^d;q^d\right)\frac{t^m}{m!} \\ &+\sum_{m=0}^{\infty} H_{m,\chi_D}(x;u;a,b;\lambda;q)\frac{t^m}{m!}. \end{aligned}$$

After some elementary calculations in the above equation, we arrive at Equation (23). Similarly from the Equation (0), we arrive at Equation (22). Therefore we get the following theorem:

**Theorem 3.3.** *Let $a,b\in\mathbb{R}^+$, $u\in\mathbb{C}\setminus\{1\}$, $\lambda,q\in\mathbb{C}$ such that $\left|q\right|<1, \left|\frac{\lambda b}{u}\right|<1, a\neq b$ and $a\geq 1$. For a positive integer $m$, we have that*

$$H_{m,\chi}(x;u;a,b;\lambda;q) = \sum_{j=0}^{n-1}(-1)^j \sum_{d\in A_{nj}} [d]^m H_{m,\chi_d}\left(\frac{x}{d},u^d;a,b;\lambda^d;q^d\right) + H_{m,\chi_D}(x;u;a,b;\lambda;q) \tag{22}$$

*and*

$$H_{m,\chi}(x;u;a,b;\lambda;q) = \sum_{j=0}^{n-1}(-1)^j \sum_{d\in A_{nj}} H_{m,\chi_d}\left(\frac{x}{d},u^d;a^{[d]},b^{[d]};\lambda^d;q^d\right) + H_{m,\chi_D}(x;u;a,b;\lambda;q). \tag{23}$$

**Theorem 3.4.** *For $s\in\mathbb{C}$, we have*

$$L_{\lambda,q,\chi}(s,x,u,a,b) = \sum_{j=0}^{n-1}(-1)^j \sum_{d\in A_{nj}} L_{\lambda^d,q^d,\chi_d}\left(s,\frac{x}{d},u^d,a^{[d]},b^{[d]}\right) + L_{\lambda,q,\chi_D}(s,x,u,a,b) \tag{24}$$

*and*

$$L_{\lambda,q,\chi}(s,x,u,a,b) = \sum_{j=0}^{n-1}(-1)^j \sum_{d\in A_{nj}} \frac{1}{[d]^s} L_{\lambda^d,q^d,\chi_d}\left(s,\frac{x}{d},u^d,a,b\right) + L_{\lambda,q,\chi_D}(s,x,u,a,b). \tag{25}$$

*Proof.* By applying the Mellin transformation to Equation (19) and Equation (20), respectively and using similar method in (6) and in (10), we easily arrive at the desired results (24) and (25) asserted by Theorem 3.4. So we complete the proof. □

If the period of $\chi$ is prime then we obtain the following theorem.

**Theorem 3.5.** *Let period of $\chi$ be a prime integer $p$ with $\chi(p) = 1$. Then we have*

$$L_{\lambda,q,\chi}(s,x,u,a,b) = L_{\lambda,q,\chi_D}(s,x,u,a,b) + \frac{1}{[p]^s}\zeta_{\lambda^p,q^p}\left(s,\frac{x}{p},u^p,a,b\right). \tag{26}$$

*Proof.* We note that $n = 1$ in Equation (24) and $\chi_D(m) = \chi(m)$ whenever $\gcd(m,f) = 1$ and $\chi(mp) = \chi_p(m) = 1$ for all positive integer $m$. Therefore, we get that

$$\begin{aligned} L_{\lambda,q,\chi}(s,x,u,a,b) &= \sum_{d\in A_{n0}} L_{\lambda^d,q^d,\chi_d}\left(s,\frac{x}{d},u^d,a^{[d]},b^{[d]}\right) + L_{\lambda,q,\chi_D}(s,x,u,a,b) \\ &= L_{\lambda^p,q^p,1}\left(s,\frac{x}{p},u^p,a^{[p]},b^{[p]}\right) + L_{\lambda,q,\chi_D}(s,x,u,a,b). \end{aligned}$$

By Equation (12), we have

$$\begin{aligned} L_{\lambda,q,\chi}(s,x,u,a,b) &= L_{\lambda,q,\chi_D}(s,x,u,a,b)+\sum_{m=0}^{\infty}\left(\frac{\lambda^p}{u^p}\right)^m \\ &\quad \left(\frac{1}{\left(\left[m+\frac{x}{p}\right]_{q^p}\ln b^{[p]}\right)^s}-\frac{1}{u^p\left(\ln a^{[p]}+\left[m+\frac{x}{p}\right]_{q^p}\ln b^{[p]}\right)^s}\right) \\ &= L_{\lambda,q,\chi_D}(s,x,u,a,b)+\frac{1}{[p]^s}\sum_{m=0}^{\infty}\left(\frac{\lambda^p}{u^p}\right)^m \\ &\quad \left(\frac{1}{\left(\left[m+\frac{x}{p}\right]_{q^p}\ln b\right)^s}-\frac{1}{u^p\left(\ln a+\left[m+\frac{x}{p}\right]_{q^p}\ln b\right)^s}\right) \\ &= L_{\lambda,q,\chi_D}(s,x,u,a,b)+\frac{1}{[p]^s}\zeta_{\lambda^p,q^p}\left(s,\frac{x}{p},u^p,a,b\right). \end{aligned}$$

□

Let $\chi$ be a periodic function with a period $f$ and $\chi(f)=1$ where $f=p_1p_2$ for some prime integers $p_1$ and $p_2$. Then the period of $\chi_{p_1}$ is $p_2$ and $\chi_{p_1}(i)=\chi(p_1 i)$ for all $i\in\{1,...,p_2-1\}$ and $\chi_{p_1}(p_2)=\chi(p_{12})$ where $p_{12}=p_1p_2$. Therefore, it follows that

$$L_{\lambda^{p_{12}},q^{p_{12}},\chi_{p_{12}}}\left(s,\frac{x}{p_{12}},u^{p_{12}},a^{[p_{12}]},b^{[p_{12}]}\right)=\zeta_{\lambda^{p_{12}},q^{p_{12}}}\left(s,\frac{x}{p_{12}},u^{p_{12}},a^{[p_{12}]},b^{[p_{12}]}\right)$$

since $\chi(p_1p_2)=1$ . On the other hand, by using Equation (24), we get

$$\begin{aligned} L_{\lambda,q,\chi}(s,x,u,a,b) &= L_{\lambda^{p_1},q^{p_1},\chi_{p_1}}\left(s,\frac{x}{p_1},u^{p_1},a^{[p_1]},b^{[p_1]}\right)+L_{\lambda^{p_2},q^{p_2},\chi_{p2}}\left(s,\frac{x}{p_2},u^{p_2},a^{[p_2]},b^{[p_2]}\right) \\ &\quad -\zeta_{\lambda^{p_{12}},q^{p_{12}}}\left(s,\frac{x}{p_2},u^{p_{12}},a^{[p_{12}]},b^{[p_{12}]}\right)+L_{\lambda,q,\chi_D}(s,x,u,a,b). \end{aligned}$$

Moreover, by using the same argument, one may obtain some different decompositions for the $L$–type function.

**Example 3.6.** *Let the period of $\chi$ be $p_1^t p_2^{t_2} p_3^{t_3}$ for prime integers $p_i$ and $t_i\in\mathbb{N}$. Then we construct the following sets $A_{30}=\{p_1,p_2,p_3\}$, $A_{31}=\{p_{12}=p_1p_2,p_{13}=p_1p_3,p_{23}=p_2p_3\}$ and $A_{32}=\{p_{123}=p_1p_2p_3\}$. Then by using Equation (23), we decompose the q–Eulerian number attached the periodic function $\chi$ :*

$$\begin{aligned} H_{m,\chi}(u;a,b;\lambda;q) &= H_{m,\chi_D}(u;a,b;\lambda;q)+\sum_{i=1}^{3}H_{m,\chi_{p_i}}\left(u^{p_i};a^{[p_i]},b^{[p_i]};\lambda^{p_i};q^{p_i}\right) \\ &\quad -H_{m,\chi_{p_{23}}}\left(u^{p_{23}};a^{[p_{23}]},b^{[p_{23}]};\lambda^{p_{23}};q^{p_{23}}\right)-\sum_{i=2}^{3}H_{m,\chi_{p_{1i}}}\left(u^{p_{1i}};a^{[p_{1i}]},b^{[p_{1i}]};\lambda^{p_{1i}};q^{p_{1i}}\right) \\ &\quad +H_{m,\chi_{p_{123}}}(u^{p_{123}};a^{[p_{123}]},b^{[p_{123}]};\lambda^{p_{123}};q^{p_{123}}). \end{aligned}$$

*By using Equation (22), we obtain a different decomposition for the q–Eulerian number attached the periodic function* $\chi$ *:*

$$\begin{aligned} H_{m,\chi}(u;a,b;\lambda;q) &= H_{m,\chi_D}(u;a,b;\lambda;q)+\sum_{i=1}^{3}[p_i]^m H_{m,\chi_{p_i}}(u^{p_i};a,b;\lambda^{p_i};q^{p_i}) \\ &\quad -[p_{23}]^m H_{m,\chi_{p_{23}}}(u^{p_{23}};a,b;\lambda^{p_{23}};q^{p_{23}})-\sum_{i=2}^{3}[p_{1i}]^m H_{m,\chi_{p_{1i}}}(u^{p_{1i}};a,b;\lambda^{p_{1i}};q^{p_{1i}}) \\ &\quad +[p_{123}]^m H_{m,\chi_{p_{123}}}(u^{p_{123}};a,b;\lambda^{p_{123}};q^{p_{123}}). \end{aligned}$$

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