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A Study of Approximate Normal Distribution Derived from Combinatoric Convolution Sums of Divisor Functions

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Abstract. In this paper, we consider the relations between Bernoulli polynomials, Legendre polynomials and combinatoric convolution sums of divisor functions. In addition, we give examples of approximate normal distribution derived from combinatoric convolution sums of divisor functions.

1. Introduction

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ will denote the sets of positive integers and the field of real numbers, respectively. Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \qquad (|t|<1).$$

It is well known that (see [4, (3.132)-(3.133)], [10, pp.228-232])

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k} = \frac{1}{2^n\cdot n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n$$

and $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$, where $[x]$ is the greatest integer not exceeding x, and satisfy the following relation of orthogonality

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{mn},$$

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or equivalently,

$$\int_0^1 P_n(2x-1)P_m(2x-1)dx = \frac{1}{2n+1}\delta_{mn} \quad ([15]),$$

where

$$\delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

is the "Kronecker delta".

It should be mentioned that Bernoulli or Legendre identities associated with the generalization of Bernoulli polynomials, Luo-Srivastava generalizations of Apostol-Bernoulli polynomials have been studied by Srivastava [5] and Luo [9].

Computing Bernoulli numbers, Euler numbers, Genocchi numbers and their applications have been studied by Qi [12] and Srivastava [6]. A few more related references are [3], [7], [8], [9], [12].

The binomial distribution is the basis for the popular binomial test of statistical significance. The binomial distribution is frequently used to model the number of successes in a sample of size n drawn with replacement from a population of size N. In this article, we introduce a generalized binomial distribution derived from combinatoric convolution sum of divisor functions.

This paper consists of 5 sections. In section 2, we reconstruct relations of Legendre polynomials and Bernoulli polynomials. In section 3, we drive the convolution sums of Bernoulli polynomials and express Legendre polynomials by convolution sums of divisor functions. In section 4, we can see an applied model of combinatoric convolution sums. And in section 5, we suggest more examples of example 2.6.

2. Preliminaries

Proposition 2.1. (*[14, Lemma 2.1]*) *For $n \in \mathbb{N}$, we have*

$$P_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\left(\frac{x-1}{2}\right)^k.$$

Proposition 2.2. (*[11, Theorem 2.4]i*) *Let $n \in \mathbb{N}$. Then*

$$\frac{1}{2}P_{2n+1}(2x-1) = \sum_{k=0}^{n} \frac{(2n+2k+1)!}{(2k)!(2k+1)!(2n-2k+1)!} B_{2k+1}(x). \tag{1}$$

We can easily prove that Legendre polynomials and Bernoulli polynomials have the following properties.

$$P_n(1) = 1, \quad \textit{and} \tag{2}$$

$$B_{2n+1}(1) = 0, \tag{3}$$

where $n \in \mathbb{R}$.

Lemma 2.3. *Let $n \in \mathbb{N} \cup \{0\}$ and p be a prime. Then we have*

$$B_{2n+1}\left(\frac{p+1}{2}\right) = \sum_{k=0}^{n} a_{2k+1}P_{2k+1}(p)$$

with $\sum_{k=0}^{n} a_{2k+1} = 0$.

Proof. Consider the Bernoulli polynomials and Legendre Polynomials. Then we obtain

$$B_1\Big(\frac{p+1}{2}\Big) = \alpha_1 P_1(p), \quad \textit{with} \quad \alpha_1 = \frac{1}{2}. \tag{4}$$

In (1), let $x = \frac{p+1}{2}$. Then

$$\frac{1}{2}P_{2n+1}(p) = \sum_{k=0}^{n} \frac{(2n+2k+1)!}{(2k)!(2k+1)!(2n-2k+1)!} B_{2k+1}\Big(\frac{p+1}{2}\Big).$$

And if $n = 1$, then

$$\begin{aligned} \frac{1}{2}P_3(p) &= \sum_{k=0}^{1} \frac{(3+2k)!}{(2k)!(2k+1)!(3-2k)!} B_{2k+1}\Big(\frac{p+1}{2}\Big) \\ &= \beta_1 B_1\Big(\frac{p+1}{2}\Big) + \beta_3 B_3\Big(\frac{p+1}{2}\Big), \\ \textit{where} \quad \beta_i &= \sum_{k=0}^{i} \frac{(3+2k)!}{(2k)!(2k+1)!(3-2k)!} \in \mathbb{R}. \end{aligned} \tag{5}$$

So, by (4) and (5), we get

$$\beta_3 B_3\Big(\frac{p+1}{2}\Big) = -\beta_1 P_1(p) + \frac{1}{2}P_3(p),$$

and

$$B_3\Big(\frac{p+1}{2}\Big) = -\frac{\beta_1}{\beta_3}P_1(p) + \frac{1}{2\beta_3}P_3(p).$$

There exist γ_1 and γ_3, such that

$$B_3\Big(\frac{p+1}{2}\Big) = \gamma_1 P_1(p) + \gamma_3 P_3(p). \tag{6}$$

Since, by continuous calculation of $B_{2n+1}\big(\frac{p+1}{2}\big)$, we easily see that can be represented by $P_{2k+1}(p)$. So, all of coefficients of $P_i(p)$ are a_i, respectively, then

$$B_{2n+1}\Big(\frac{p+1}{2}\Big) = \sum_{k=0}^{n} a_{2k+1} P_{2k+1}(p). \tag{7}$$

And, by (2) and (3), we get $P_n(1) = 1$ *and* $B_{2n+1}(1) = 0$. So, if $p = 1$, then the left side calculation is 0 and the right side calculation is $\sum_{k=0}^{n} a_{2k+1}$. Therefore, we see that $\sum_{k=0}^{n} a_{2k+1} = 0$, i.e, the sum of the coefficients of (7) are zero. □

By (1), we drive that

$$\frac{1}{2}P_n = A_n B_n,$$

$$\textit{where} \quad \frac{1}{2}P_n = \begin{pmatrix} P_1(p) \\ P_3(p) \\ \vdots \\ P_{2n-1}(p) \end{pmatrix}, \quad B_n = \begin{pmatrix} B_1(\frac{p+1}{2}) \\ B_3(\frac{p+1}{2}) \\ \vdots \\ B_{2n-1}(\frac{p+1}{2}) \end{pmatrix}, \quad \textit{and}$$

$$A_n := \begin{pmatrix} \frac{1!}{0!1!1!} & 0 & 0 & 0 & \cdots & 0 \\ \frac{3!}{0!1!3!} & \frac{5!}{2!3!1!} & 0 & 0 & \cdots & 0 \\ \frac{5!}{0!1!5!} & \frac{7!}{2!3!3!} & \frac{9!}{4!5!1!} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{(2n-1)!}{0!1!(2n-1)!} & \frac{(2n+1)!}{2!3!(2n-3)!} & \frac{(2n+3)!}{4!5!(2n-5)!} & \cdots & \frac{(4n-5)!}{(2n-4)!(2n-3)!3!} & \frac{(4n-3)!}{(2n-2)!(2n-1)!1!} \end{pmatrix}. \tag{8}$$

Then we know that

$$P'_n = A_n B'_n.$$

with

$$P'_n = \frac{dP_n}{dp}.$$

So, we can obtain the following lemma.

Lemma 2.4. *Let $n \geq 2$. Then,*

$$Det(A_{n-1}) = \frac{\prod_{k=0}^{n-1}(4k+1)!}{\prod_{k=0}^{2n-1} k!} = \prod_{k=0}^{n-1} \binom{4k+1}{2k}.$$

Proof. Then A_n is a lower triangular matrix. So by definition of determinant of triangular matrix,

$$\begin{aligned} Det(A_n) &= \frac{1!5!9! \cdots (4n-7)!(4n-3)!}{0!1!2!3! \cdots (2n-2)!(2n-1)!} \\ &= \frac{\prod_{k=0}^{n-1}(4k+1)!}{\prod_{k=0}^{2n-1} k!} = \prod_{k=0}^{n-1} \binom{4k+1}{2k}. \end{aligned}$$

□

Theorem 2.5. *Let $a_{i,i},\ i \geq 1$ be the diagonal elements and let $b_{i,j},\ i, j \geq 1$ of (8). Then,*

$$A_n^{-1} = \begin{pmatrix} a_{1,1}^{-1} & 0 & 0 & \cdots & 0 \\ b_{2,1} & a_{2,2}^{-1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n+1,1} & b_{n+1,2} & b_{n+1,3} & \cdots & a_{n+1,n+1}^{-1} \end{pmatrix},$$

and

$$b_{i+1,i+1} = a_{i+1,i+1}^{-1} = \frac{det(A_{i-1})}{det(A_i)} = \frac{(2i-2)!(2i-1)!}{(4i-3)!}, \quad for \quad 1 \leq i \leq n.$$

Proof. A_n is triangular matrix. Then, we easily check that the diagonal elements of A_n^{-1} are the inverse of the corresponding diagonal elements of A_n.

And by [13], we can easily prove that

$$b_{i+1,i+1} = a_{i+1,i+1}^{-1} = (-1)^{(i+1)+(i+1)} \frac{det(A_{i-1})}{det(A_i)} = \frac{det(A_{i-1})}{det(A_i)} = \frac{(2i-2)!(2i-1)!}{(4i-3)!},$$

for $1 \leq i \leq n$. □

Example 2.6. *Using (8), Lemma 2.4 and Theorem 2.5, we suggest examples.*

n	A_n	A_n^{-1}	$Det(A_n)$
0	$\begin{pmatrix}\frac{1!}{0!1!1!}\end{pmatrix}$	$\begin{pmatrix}\frac{1!}{0!1!1!}\end{pmatrix}$	1
1	$\begin{pmatrix}1 & 0\\ 1 & 10\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ -\frac{1}{10} & \frac{1}{10}\end{pmatrix}$	$\frac{1!5!}{0!1!2!3!}$
2	$\begin{pmatrix}1 & 0 & 0\\ 1 & 10 & 0\\ 1 & 70 & 126\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ -\frac{1}{10} & \frac{1}{10} & 0\\ \frac{1}{21} & -\frac{1}{18} & \frac{1}{126}\end{pmatrix}$	$\frac{1!5!9!}{0!1!2!3!4!5!}$
3	$\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 10 & 0 & 0\\ 1 & 70 & 126 & 0\\ 1 & 252 & 2310 & 1716\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0 & 0\\ -\frac{1}{10} & \frac{1}{10} & 0 & 0\\ \frac{1}{21} & -\frac{1}{18} & \frac{1}{126} & 0\\ -\frac{1}{20} & \frac{119}{1980} & -\frac{5}{468} & \frac{1}{1716}\end{pmatrix}$	$\frac{1!5!9!13!}{0!1!2!3!4!5!6!7!}$

In appendix, we suggest more examples ($n = 4, 5, 6, 7, 8$).

Remark 2.7. *It is also interesting to note that the binomial identities involving harmonic numbers and their applications have been studied by W. Chu [3].*

3. Legendre Polynomials, Bernoulli Polynomials and Divisor Functions

Proposition 3.1. ([1, Proposition 1]) *Let $n \geq 2$ and $k \geq 1$. Then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(n-m;2) = \frac{1}{4}\sigma_{2k+1,0}(n;2) + \frac{2^{2k}}{2k+1}\sum_{\substack{d\mid n\\ d\ odd}} B_{2k+1}\left(\frac{d+1}{2}\right).$$

In particular, if n is an odd integer, then

$$(2k+1)\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(n-m;2) = 2^{2k}\sum_{d\mid n} B_{2k+1}\left(\frac{d+1}{2}\right). \tag{9}$$

If $n = 2^\alpha$, then

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(n-m;2) = \frac{1}{4}\left(\frac{(2^{2k+1})^{\alpha+1}-1}{2^{2k}-1}\right).$$

Lemma 3.2. *Let p be an odd prime integer and $x = \frac{p+1}{2}$. Then*

$$\frac{1}{2}P_{2n+1}(p) - \frac{p}{2} = \sum_{k=1}^{n}\binom{2n+2k+1}{2k\quad 2k\quad 2n-2k+1}\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{p-1}\frac{\sigma_{2k-2s-1,1}(m;2)}{2^{2k-2s-1}}\cdot\frac{\sigma_{2s+1,1}(p-m;2)}{2^{2s+1}}.$$

Proof. By Proposition 3.1, in particular, n is an odd prime. Then

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(p-m;2) = \frac{2^{2k}}{2k+1}B_{2k+1}\Big(\frac{p+1}{2}\Big).$$

Dividing both sides by 2^{2k}, then

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{n-1}\frac{\sigma_{2k-2s-1,1}(m;2)}{2^{2k-2s-1}}\cdot\frac{\sigma_{2s+1,1}(p-m;2)}{2^{2s+1}} = \frac{1}{2k+1}B_{2k+1}\Big(\frac{p+1}{2}\Big). \tag{10}$$

And, by Proposition 2.2,

$$\begin{aligned}\frac{1}{2}P_{2n+1}(2x-1) &= \sum_{k=0}^{n} \frac{(2n+2k+1)!}{(2k)!(2k+1)!(2n-2k+1)!} B_{2k+1}(x) \\ &= \sum_{k=0}^{n} \frac{(2n+2k+1)!}{(2k)!(2k)!(2n-2k+1)!} \frac{1}{2k+1} B_{2k+1}(x) \\ &= \sum_{k=1}^{n} \frac{(2n+2k+1)!}{(2k)!(2k)!(2n-2k+1)!} \frac{1}{2k+1} B_{2k+1}(x) - B_1(x).\end{aligned}$$

So,

$$\frac{1}{2}P_{2n+1}(2x-1) - B_1(x) = \sum_{k=1}^{n} \binom{2n+2k+1}{2k \quad 2k \quad 2n-2k+1} \frac{1}{2k+1} B_{2k+1}(x).$$

In particular, n is an odd prime and $x = \frac{p+1}{2}$. Then

$$\frac{1}{2}P_{2n+1}(p) - \frac{p}{2} = \sum_{k=1}^{n} \binom{2n+2k+1}{2k \quad 2k \quad 2n-2k+1} \frac{1}{2k+1} B_{2k+1}\Big(\frac{p+1}{2}\Big). \tag{11}$$

Equating (10) and (11),

$$\frac{1}{2}P_{2n+1}(p) - \frac{p}{2} = \sum_{k=1}^{n} \binom{2n+2k+1}{2k \quad 2k \quad 2n-2k+1} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{p-1} \frac{\sigma_{2k-2s-1,1}(m;2)}{2^{2k-2s-1}} \cdot \frac{\sigma_{2s+1,1}(p-m;2)}{2^{2s+1}}.$$

Hence we obtain our result. □

Theorem 3.3. *For all $n \geq 1$ and any positive prime integer p. We obtain an equation,*

$$P_{2n+1}(p) \equiv \sum_{k=1}^{n} \binom{2n+2k+1}{2k \quad 2k \quad 2n-2k+1} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{p-1} \frac{\sigma_{2k-2s-1,1}(m;2)}{2^{2k-2s-1}} \cdot \frac{\sigma_{2s+1,1}(p-m;2)}{2^{2s}} \pmod{p}.$$

4. Application of Combinatoric Convolution Sums

If the random variable X follows the binomial distribution with parameters n and p, we write $X \sim B(n, p)$. The probability of getting exactly k successes in n trials is given by the probability mass function:

$$f(k; n, p) = P_r(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \ldots, n$.

The formula can be understood as follows: we want k successes (p^k) and $n-k$ failures $(1-p)^{n-k}$. However, the k successes can occur anywhere among the n trials, and there are $\binom{n}{k}$ different ways of distributing k successes in a sequence of n trials.

In this section, we suggest a new type binomial distribution derived from combinatoric convolution sum of divisor functions.

Let

$$DB(n,k,t) := \frac{\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1,1}(m;2)\sigma_{2s+1,1}(n-m;2)}{\frac{1}{4}\sigma_{2k+1,0}(n;2) + \frac{2^{2k}}{2k+1} \sum_{\substack{d|n \\ d \text{ odd}}} B_{2k+1}\left(\frac{d+1}{2}\right)}.$$

Consider the values of $DB(n,k,t)$. Out of the result from $k = 30, n = 2 \sim 25$, we can see that the values of $DB(n,k,t)$ approximate normal distribution. When the form is $s = 14$, $s = 15$, the maximum value appears 0.198538 and shows the result of bilateral symmetry. Forms of bilateral symmetry are almost same, but in case that they are much distant from the maximum, the result will be slightly different. So we have 6 different shapes of forms that are similar to normal distribution (They are almost same, in fact.). Considering the differences of each point among the 6 cases, the result is as follows when $ND1$ is the criteria. In particular, if $n = 2^\alpha$ or $n = p$, then the formula $DB(n,k,t)$ is very simple.

We use the following notations:

$$\begin{aligned}
&ND1 := DB(2,30,1), \quad ND2 := DB(6,30,3), \quad ND3 := DB(10,30,5), \quad ND4 := DB(14,30,7),\\
&ND5 := DB(18,30,9), \quad ND6 := DB(22,30,11),\\
&\widetilde{ND2} := ND2 - ND1, \quad \widetilde{ND3} := ND3 - ND1, \quad \widetilde{ND4} := ND4 - ND1, \quad \widetilde{ND5} := ND5 - ND1,\\
&\widetilde{ND6} := ND6 - ND1.
\end{aligned}$$

values	$ND1$	$\widetilde{ND2}$	$\widetilde{ND3}$
$s = 1$	5.93622×10^{-14}	$+0.21986 \times 10^{-14}$	$+0.04749 \times 10^{-14}$
$s = 2$	9.47421×10^{-12}	$+0.03899 \times 10^{-12}$	$+0.00303 \times 10^{-12}$
$s = 3$	6.69962×10^{-10}	$+0.00307 \times 10^{-10}$	$+0.00009 \times 10^{-10}$
$s = 4$	2.56447×10^{-8}	$+0.00013 \times 10^{-8}$	0
$s = 5$	5.9449×10^{-7}	$+0.00003 \times 10^{-7}$	0
$s = 6$	8.96308×10^{-6}	0	0
$s = 7$	0.000092277	0	0
$s = 8$	0.000671723	0	0
$s = 9$	0.00354717	0	0
$s = 10$	0.0138508	0	0
$s = 11$	0.0405671	0	0
$s = 12$	0.0900589	0	0
$s = 13$	0.152664	0	0
$s = 14$	0.198538	0	0
$s = 15$	0.198538	0	0
$s = 16$	0.152664	0	0
$s = 17$	0.0900589	0	0
$s = 18$	0.0405671	0	0
$s = 19$	0.0138508	0	0
$s = 20$	0.00354717	0	0
$s = 21$	0.000671723	0	0
$s = 22$	0.000092277	0	0
$s = 23$	8.96308×10^{-6}	0	0
$s = 24$	5.9449×10^{-7}	$+0.00003 \times 10^{-7}$	0
$s = 25$	2.56447×10^{-8}	$+0.00013 \times 10^{-8}$	0
$s = 26$	6.69962×10^{-10}	$+0.00307 \times 10^{-10}$	$+0.00009 \times 10^{-10}$
$s = 27$	9.47421×10^{-12}	$+0.03899 \times 10^{-12}$	$+0.00303 \times 10^{-12}$
$s = 28$	5.93622×10^{-14}	$+0.21986 \times 10^{-14}$	$+0.04749 \times 10^{-14}$
$s = 29$	1.04083×10^{-16}	$+0.34695 \times 10^{-16}$	$+0.20817 \times 10^{-16}$

Table 1. The values of $ND1, \widetilde{ND2}$ and $\widetilde{ND3}$.

values	$\widetilde{ND4}$	$\widetilde{ND5}$	$\widetilde{ND6}$
$s = 1$	$+0.01731 \times 10^{-14}$	$+0.21986 \times 10^{-14}$	$+0.04749 \times 10^{-14}$
$s = 2$	$+0.00057 \times 10^{-12}$	$+0.03899 \times 10^{-12}$	$+0.00303 \times 10^{-12}$
$s = 3$	$+0.00001 \times 10^{-10}$	$+0.00307 \times 10^{-10}$	$+0.00009 \times 10^{-10}$
$s = 4$	0	$+0.00013 \times 10^{-8}$	0
$s = 5$	0	$+0.00003 \times 10^{-7}$	0
$s = 6$	0	0	0
$s = 7$	0	0	0
$s = 8$	0	0	0
$s = 9$	0	0	0
$s = 10$	0	0	0
$s = 11$	0	0	0
$s = 12$	0	0	0
$s = 13$	0	0	0
$s = 14$	0	0	0
$s = 15$	0	0	0
$s = 16$	0	0	0
$s = 17$	0	0	0
$s = 18$	0	0	0
$s = 19$	0	0	0
$s = 20$	0	0	0
$s = 21$	0	0	0
$s = 22$	0	0	0
$s = 23$	0	0	0
$s = 24$	0	$+0.00003 \times 10^{-7}$	0
$s = 25$	0	$+0.00013 \times 10^{-8}$	0
$s = 26$	$+0.00001 \times 10^{-10}$	$+0.00307 \times 10^{-10}$	0
$s = 27$	$+0.00057 \times 10^{-12}$	$+0.03915 \times 10^{-12}$	$+0.00006 \times 10^{-12}$
$s = 28$	$+0.01731 \times 10^{-14}$	$+0.22801 \times 10^{-14}$	$+0.00446 \times 10^{-14}$
$s = 29$	$+0.14869 \times 10^{-16}$	$+0.4626 \times 10^{-16}$	$+0.09463 \times 10^{-16}$

Table 2. The values of $\widetilde{ND4}, \widetilde{ND5}$ and $\widetilde{ND6}$.

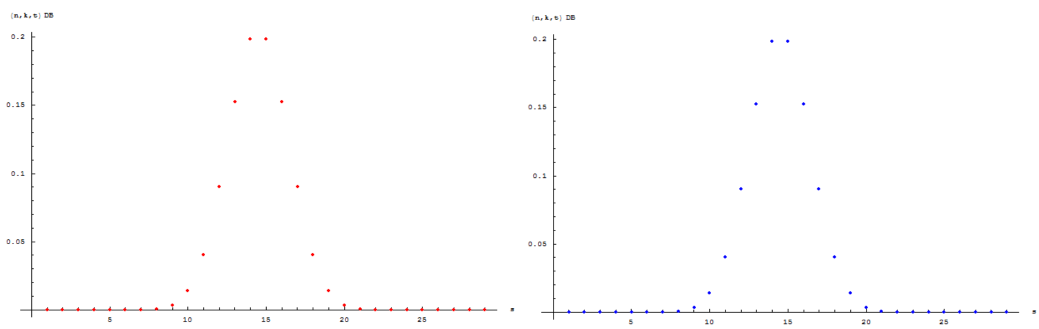

Figure 1: The graphs of ND1 and ND2

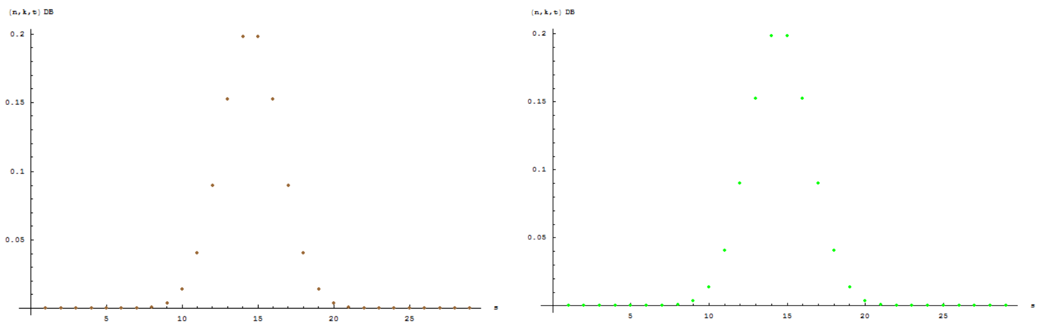

Figure 2: The graphs of ND3 and ND4

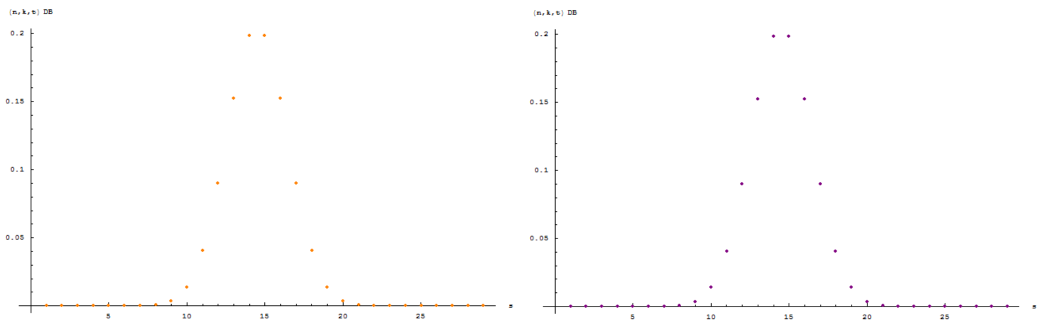

Figure 3: The graphs of ND5 and ND6

Remark 4.1. *Let μ and Var is a mean and standard deviation, respectively.* (*See [2, pp.192].*) *Using Lemma (3.2), we can find the generalized binomial distribution derived from combinatoric convolution sum of divisor functions. So we can easily find that following formula of $ND1 := DB(2, 30, 1)$. We note that*

$$f_{DB(n,k,t)}(s) = \frac{1}{\sqrt{2\pi}Var} exp[-\frac{(s-\mu)^2}{2Var^2}],$$

where

$$\mu := 0.0344827278547721, \quad Var := 0.061835972, \quad \pi := 3.1416, \quad e := 2.7183.$$

5. Appendix

Using (8), Lemma 2.4 and Theorem 2.5, we suggest more examples($n = 4, 5, 6, 7, 8$).

A_4	$\begin{pmatrix} 1 & 0 & 0 & 0 & \\ 1 & 10 & 0 & 0 & 0 \\ 1 & 70 & 126 & 0 & 0 \\ 1 & 252 & 2310 & 1716 & 0 \\ 1 & 660 & 18018 & 60060 & 24310 \end{pmatrix}$
A_4^{-1}	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 \\ \frac{1}{21} & -\frac{1}{18} & \frac{1}{126} & 0 & 0 \\ -\frac{1}{20} & \frac{119}{1980} & -\frac{5}{468} & \frac{1}{1716} & 0 \\ \frac{1}{11} & -\frac{236}{2145} & \frac{4}{195} & -\frac{7}{4862} & \frac{1}{24310} \end{pmatrix}$
$Det(A_4)$	$\frac{1!5!9!13!17!}{0!1!2!3!4!5!6!7!8!9!}$
A_5	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 10 & 0 & 0 & 0 & 0 \\ 1 & 70 & 126 & 0 & 0 & 0 \\ 1 & 252 & 2310 & 1716 & 0 & 0 \\ 1 & 660 & 18018 & 60060 & 24310 & 0 \\ 1 & 1430 & 90090 & 816816 & 1385670 & 352716 \end{pmatrix}$
A_5^{-1}	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 \\ \frac{1}{21} & -\frac{1}{18} & \frac{1}{126} & 0 & 0 & 0 \\ -\frac{1}{20} & \frac{119}{1980} & -\frac{5}{468} & \frac{1}{1716} & 0 & 0 \\ \frac{1}{11} & -\frac{236}{2145} & \frac{4}{195} & -\frac{7}{4862} & \frac{1}{24310} & 0 \\ -\frac{691}{2730} & \frac{359}{1170} & -\frac{3223}{55692} & \frac{217}{50388} & -\frac{1}{6188} & \frac{1}{352716} \end{pmatrix}$
$Det(A_5)$	$\frac{1!5!9!13!17!21!}{0!1!2!3!4!5!6!7!8!9!10!11!}$
A_6	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 10 & 0 & 0 & 0 & 0 & 0 \\ 1 & 70 & 126 & 0 & 0 & 0 & 0 \\ 1 & 252 & 2310 & 1716 & 0 & 0 & 0 \\ 1 & 660 & 18018 & 60060 & 24310 & 0 & 0 \\ 1 & 1430 & 90090 & 816816 & 1385670 & 352716 & 0 \\ 1 & 2730 & 340340 & 6651216 & 29099070 & 29745716 & 5200300 \end{pmatrix}$
A_6^{-1}	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{21} & -\frac{1}{18} & \frac{1}{126} & 0 & 0 & 0 & 0 \\ -\frac{1}{20} & \frac{119}{1980} & -\frac{5}{468} & \frac{1}{1716} & 0 & 0 & 0 \\ \frac{1}{11} & -\frac{236}{2145} & \frac{4}{195} & -\frac{7}{4862} & \frac{1}{24310} & 0 & 0 \\ -\frac{691}{2730} & \frac{359}{1170} & -\frac{3223}{55692} & \frac{217}{50388} & -\frac{1}{6188} & \frac{1}{352716} & 0 \\ 1 & -\frac{1237}{1020} & \frac{10373}{45220} & -\frac{235}{13566} & \frac{19}{27370} & -\frac{11}{678300} & \frac{1}{5200300} \end{pmatrix}$
$Det(A_6)$	$\frac{1!5!9!13!17!21!25!}{0!1!2!3!4!5!6!7!8!9!10!11!12!13!}$
A_7	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 70 & 126 & 0 & 0 & 0 & 0 & 0 \\ 1 & 252 & 2310 & 1716 & 0 & 0 & 0 & 0 \\ 1 & 660 & 18018 & 60060 & 24310 & 0 & 0 & 0 \\ 1 & 1430 & 90090 & 816816 & 1385670 & 352716 & 0 & 0 \\ 1 & 2730 & 340340 & 6651216 & 29099070 & 29745716 & 5200300 & 0 \\ 1 & 4760 & 1058148 & 38798760 & 350574510 & 892371480 & 608435100 & 77558760 \end{pmatrix}$
A_7^{-1}	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{21} & -\frac{1}{18} & \frac{1}{126} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{20} & \frac{119}{1980} & -\frac{5}{468} & \frac{1}{1716} & 0 & 0 & 0 & 0 \\ \frac{1}{11} & -\frac{236}{2145} & \frac{4}{195} & -\frac{7}{4862} & \frac{1}{24310} & 0 & 0 & 0 \\ -\frac{691}{2730} & \frac{359}{1170} & -\frac{3223}{55692} & \frac{217}{50388} & -\frac{1}{6188} & \frac{1}{352716} & 0 & 0 \\ 1 & -\frac{1237}{1020} & \frac{10373}{45220} & -\frac{235}{13566} & \frac{19}{27370} & -\frac{11}{678300} & \frac{1}{5200300} & 0 \\ -\frac{3617}{680} & \frac{750167}{116280} & -\frac{28399}{23256} & \frac{5501}{59432} & -\frac{59}{15640} & \frac{11}{116280} & -\frac{13}{8617640} & \frac{1}{77558760} \end{pmatrix}$
$Det(A_7)$	$\frac{1!5!9!13!17!21!25!29!}{0!1!2!3!4!5!6!7!8!9!10!11!12!13!14!15!}$

A_8	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 70 & 126 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 252 & 2310 & 1716 & 0 & 0 & 0 & 0 & 0 \\ 1 & 660 & 18018 & 60060 & 24310 & 0 & 0 & 0 & 0 \\ 1 & 1430 & 90090 & 816816 & 1385670 & 352716 & 0 & 0 & 0 \\ 1 & 2730 & 340340 & 6651216 & 29099070 & 29745716 & 5200300 & 0 & 0 \\ 1 & 4760 & 1058148 & 38798760 & 350574510 & 892371480 & 608435100 & 77558760 & 0 \\ 1 & 7752 & 2848860 & 178474296 & 2921454250 & 14915351880 & 24702465060 & 12021607800 & 1166803110 \end{pmatrix}$
A_8^{-1}	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{21} & -\frac{1}{18} & \frac{1}{126} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{20} & \frac{119}{1980} & -\frac{5}{468} & \frac{1}{1716} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{11} & -\frac{236}{2145} & \frac{4}{195} & -\frac{7}{4862} & \frac{1}{24310} & 0 & 0 & 0 & 0 \\ -\frac{691}{2730} & \frac{359}{1170} & -\frac{3223}{55692} & \frac{217}{50388} & -\frac{1}{6188} & \frac{1}{352716} & 0 & 0 & 0 \\ 1 & -\frac{1237}{1020} & \frac{10373}{45220} & -\frac{235}{13566} & \frac{19}{27370} & -\frac{11}{678300} & \frac{1}{5200300} & 0 & 0 \\ -\frac{3617}{680} & \frac{750167}{116280} & -\frac{28399}{23256} & \frac{5501}{59432} & -\frac{59}{15640} & \frac{11}{116280} & -\frac{13}{8617640} & \frac{1}{77558760} & \\ \frac{43867}{1197} & -\frac{1254146}{28215} & \frac{3475154}{412965} & -\frac{138104}{216315} & \frac{568}{21735} & -\frac{1738}{2603475} & \frac{338}{29464725} & -\frac{1}{7527762} & \frac{1}{1166803110} \end{pmatrix}$
$Det(A_8)$	$\frac{1!5!9!13!17!21!25!29!33!}{0!1!2!3!4!5!6!7!8!9!10!11!12!13!14!15!16!17!}$

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