



# Inverse Mapping Theory on Split Quaternions in Clifford Analysis

Ji Eun Kim<sup>a</sup>, Kwang Ho Shon<sup>a,\*</sup>

<sup>a</sup>Department of Mathematics, Pusan National University, Busan 609-735, Republic of Korea

**Abstract.** We give a split regular function that has a split Cauchy-Riemann system in split quaternions and research properties of split regular mappings with values in  $\mathcal{S}$ . Also, we investigate properties of an inverse mapping theory with values in split quaternions.

## 1. Introduction

The set of split quaternions introduced by Cockle [4] in 1849 is as follows:

$$\mathcal{S} = \{z = x_0 + x_1i + x_2j + x_3k \mid x_r \in \mathbb{R}, r = 0, 1, 2, 3\},$$

where  $i^2 = -1$ ,  $j^2 = k^2 = 1$  and  $ijk = 1$ . Unlike a quaternion algebra, the set of split quaternions is non-commutative and contains zero divisors, nilpotent elements and non-trivial idempotents [20]. Recently, since split quaternions are used to express Lorentzian rotations, there are studies on geometric and physical applications of split quaternions require solving split quaternionic equations such as [2, 9, 17, 20]. Kula et al. [17] defined dual split quaternions and gave the screw motion in  $\mathbb{R}_1^3$ , using the properties of the Hamilton operators. They showed that quaternions have a scalar product that allows as to identify it with semi Euclidean. Lian et al. [18] established the formulas of the maximal rank of a matrix with variant quaternion matrices subject to linear matrix equations. Song et al. [24] studied the iterative solution to the coupled quaternion matrix equations and determined the existence conditions of solution to the coupled quaternion matrix equations, by making use of a generalization of the classical complex conjugate graduate iterative algorithm.

Inverse function theorem is presented by a variety of views. The history of inverse function theorem is attributed to R. Descartes on algebraic geometry, J. L. Lagrange on real power series, A. L. Cauchy on complex power series, and U. Dini on functions of real variables and differential geometry. Further there are some developments of inverse function theorem in differentiable manifolds, Riemannian geometry, partial differential equations, and numerical analysis. Krantz et al. [16], Dontchev et al. [6], Hurwicz et al. [10], and Scarpello [23] intend to present many variants of the inverse function theorem, complete with

---

2010 *Mathematics Subject Classification.* Primary 32G35, 32W50; Secondary 32A99, 11E88

*Keywords.* split quaternion; split regular; inverse mapping; Clifford analysis

Received: 21 October 2014; Accepted: 26 February 2015

Communicated by Hari M. Srivastava

The second author was supported by the Research Fund Program of Research Institute for Basic Sciences, Pusan National University, Korea, 2014, Project No. RIBS-PNU-2014-201 and the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2013R1A1A2008978).

The corresponding author: Kwang Ho Shon

*Email addresses:* jeunkim@pusan.ac.kr (Ji Eun Kim), khshon@pusan.ac.kr (Kwang Ho Shon)

proofs and applications to algebra, differential geometry, functional analysis, and to many other branches of mathematics. Complex variables versions of the theorems can be seen in Krantz et al. [16] and Burckel [3]. Oliveira [19] presented simple proofs of inverse function theorem on a finite-dimensional Euclidean space that employ only the intermediate value theorem and the mean value theorem.

We [12–14] researched corresponding Cauchy-Riemann systems and properties of a regularity of functions with values in special quaternions on Clifford analysis. Also, we [15] gave a regular function with values in dual split quaternions and relations between a corresponding Cauchy-Riemann system and a regularity of functions with values in dual split quaternions.

In this paper, we give a split regular mapping that has a split Cauchy-Riemann system on split quaternions and research the inverse mapping of a regular mapping in split quaternions. And, we investigate properties of split regular mappings, defined on a bounded open set in  $\mathcal{S} \times \mathcal{S}$ .

## 2. Preliminaries

The split quaternionic field  $\mathcal{S}$  is a four dimensional non-commutative  $\mathbb{R}$ -field generated by four base elements  $e_0, e_1, e_2$  and  $e_3$  with the following non-commutative multiplication rules (see [17]):

$$e_1^2 = -1, \quad e_2^2 = e_3^2 = 1, \quad e_j e_k = -e_k e_j, \quad \bar{e}_j = -e_j \quad (j \neq k, j \neq 0, k \neq 0),$$

$$e_1 e_2 = e_3, \quad e_2 e_3 = -e_1, \quad e_3 e_1 = e_2.$$

The element  $e_0$  is the identity of  $\mathcal{S}$  and  $e_1$  identifies the imaginary unit  $i = \sqrt{-1}$  in the  $\mathbb{C}$ -field of complex numbers. Split quaternions  $z$  and  $w$  are given by

$$z = \sum_{r=0}^3 x_r e_r = z_1 + z_2 e_2, \quad w = \sum_{r=0}^3 y_r e_r = w_1 + w_2 e_2,$$

where  $x_r$  and  $y_r$  ( $r = 0, 1, 2, 3$ ) are real numbers and  $z_1 = x_0 + x_1 e_1$ ,  $z_2 = x_2 + x_3 e_1$ ,  $w_1 = y_0 + y_1 e_1$  and  $w_2 = y_2 + y_3 e_1$  are complex numbers in  $\mathbb{C}$ .

From the representations of  $z_k$  ( $k = 1, 2$ ), we have  $e_2 z_k = \bar{z}_k e_2$  ( $k = 1, 2$ ) and we identify  $\mathcal{S}$  with  $\mathbb{C}^2$ .

The split quaternionic conjugate  $z^*$ , the modulus  $N(z)$  and the inverse  $z^{-1}$  of  $z$  in  $\mathcal{S}$  are defined by the following:

$$z^* = \sum_{r=0}^3 \bar{e}_r x_r = \bar{z}_1 - z_2 e_2,$$

$$N(z) = z z^* = x_0^2 + x_1^2 - x_2^2 - x_3^2 = |z_1|^2 - |z_2|^2,$$

$$z^{-1} = \frac{z^*}{N(z)} \quad (N(z) \neq 0).$$

We use the following differential operators:

$$D_z := \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial \bar{z}_2}, \quad D_z^* = \frac{\partial}{\partial \bar{z}_1} + e_2 \frac{\partial}{\partial z_2},$$

$$D_w := \frac{\partial}{\partial w_1} - e_2 \frac{\partial}{\partial \bar{w}_2}, \quad D_w^* = \frac{\partial}{\partial \bar{w}_1} + e_2 \frac{\partial}{\partial w_2},$$

where  $\partial/\partial z_r$ ,  $\partial/\partial \bar{z}_r$  ( $r = 1, 2$ ),  $\partial/\partial w_r$  and  $\partial/\partial \bar{w}_r$  ( $r = 1, 2$ ) are usual differential operators used in complex analysis. Then

$$D_z D_z^* = D_z^* D_z = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} - \frac{\partial^2}{\partial z_2 \partial \bar{z}_2}, \quad D_w D_w^* = D_w^* D_w = \frac{\partial^2}{\partial w_1 \partial \bar{w}_1} - \frac{\partial^2}{\partial w_2 \partial \bar{w}_2}$$

are the Coulomb operators (see [5]).

Let  $\Omega = \Omega_1 \times \Omega_2$  be a bounded open set in  $\mathcal{S} \times \mathcal{S}$ . We consider a function  $f$ , defined on  $\Omega$  and with values in  $\mathcal{S}$ :

$$f = f_1 + f_2 e_2,$$

$$(z, w) \in \Omega \mapsto f(z, w) = f_1(z, w) + f_2(z, w) e_2 \in \mathcal{S},$$

where  $f_k(z, w)$  ( $k = 1, 2$ ) are complex-valued functions,  $z$  and  $w$  are elements of  $\mathcal{S}$ . For example,

$$f(z, w) = zw = f_1 + f_2 e_2,$$

where  $f_1 = z_1 w_1 + z_2 \overline{w_2}$  and  $f_2 = z_1 w_2 + z_2 \overline{w_1}$ .

**Remark 2.1.** Let  $\Omega = \Omega_1 \times \Omega_2$  be a bounded open set in  $\mathcal{S} \times \mathcal{S}$  and a function  $f$  be defined on  $\Omega$  and with values in  $\mathcal{S}$ . Then

$$D_z^* f(z, w) = \left( \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial z_2} \right) (f_1 + f_2 e_2) = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1} e_2 + e_2 \frac{\partial f_1}{\partial z_2} + e_2 \frac{\partial f_2}{\partial z_2} e_2 = \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial \overline{f_2}}{\partial z_2} \right) + \left( \frac{\partial f_2}{\partial z_1} + \frac{\partial \overline{f_1}}{\partial z_2} \right),$$

$$f(z, w) D_w^* = (f_1 + f_2 e_2) \left( \frac{\partial}{\partial w_1} + e_2 \frac{\partial}{\partial w_2} \right) = \frac{\partial f_1}{\partial w_1} + \frac{\partial f_2}{\partial w_1} e_2 + e_2 \frac{\partial \overline{f_1}}{\partial w_2} + \frac{\partial f_2}{\partial w_2} = \left( \frac{\partial f_1}{\partial w_1} + \frac{\partial f_2}{\partial w_2} \right) + \left( \frac{\partial f_2}{\partial w_1} + \frac{\partial f_1}{\partial w_2} \right).$$

**Definition 2.2.** Let  $\Omega = \Omega_1 \times \Omega_2$  be a bounded open set in  $\mathcal{S} \times \mathcal{S}$ . A function  $f(z, w) = f_1(z, w) + f_2(z, w) e_2$  is said to be split regular in  $\Omega$  if the following two conditions are satisfied:

- (i)  $f_1(z, w)$  and  $f_2(z, w)$  are continuously differential functions in  $\Omega$ , and
- (ii)  $D_z^* f(z, w) = 0$  and  $f(z, w) D_w^* = 0$  in  $\Omega$ .

**Remark 2.3.** From the equations  $D_z^* f(z) = 0$  and  $f(z) D_w^* = 0$ , we have

$$\begin{cases} \frac{\partial f_1}{\partial z_1} = -\frac{\partial \overline{f_2}}{\partial z_2}, & \frac{\partial f_2}{\partial z_1} = -\frac{\partial \overline{f_1}}{\partial z_2}, \\ \frac{\partial f_1}{\partial w_1} = -\frac{\partial f_2}{\partial w_2}, & \frac{\partial f_2}{\partial w_1} = -\frac{\partial f_1}{\partial w_2}. \end{cases} \tag{1}$$

The above system is called the split Cauchy-Riemann system for  $f(z)$  with values in  $\mathcal{S}$ .

**Proposition 2.4.** Let  $\Omega = \Omega_1 \times \Omega_2$  be a bounded open set in  $\mathcal{S} \times \mathcal{S}$ . If  $f(z, w) = f_1(z, w) + f_2(z, w) e_2$  and  $g(z, w) = g_1(z, w) + g_2(z, w) e_2$  are split regular functions in  $\Omega$ , where  $f_1, f_2, g_1$  and  $g_2$  are complex-valued functions and not real-valued functions, then

- (i)  $af + bg$  is a split regular function in  $\Omega$ , where  $a$  and  $b$  are real constants,
- (ii) There exists a constant  $c = c_1 + c_2 e_2$  in  $\mathcal{S}$  such that  $fc$  and  $cf$  are not split regular functions in  $\Omega$ ,
- (iii)  $fg$  is not necessarily a split regular function in  $\Omega$ .

Proof. (i) Since  $f$  and  $g$  are split regular functions in  $\Omega$ , we have

$$D_z^*(f \pm g) = D_z^* f \pm D_z^* g = 0, \quad (f \pm g) D_w^* = f D_w^* \pm g D_w^* = 0.$$

Therefore, by the definition of a split regular function in  $\Omega$ ,  $af + bg$  is a split regular function in  $\Omega$ .

(ii) By the multiplication rule of split quaternions, since

$$D_z^*(cf) = c_1 \frac{\partial f_1}{\partial \bar{z}_1} + c_2 \frac{\partial \bar{f}_2}{\partial \bar{z}_1} + \bar{c}_1 \frac{\partial \bar{f}_2}{\partial z_2} + \bar{c}_2 \frac{\partial f_1}{\partial z_2} + \left( c_1 \frac{\partial f_2}{\partial \bar{z}_1} + c_2 \frac{\partial \bar{f}_1}{\partial \bar{z}_1} + \bar{c}_1 \frac{\partial \bar{f}_1}{\partial z_2} + \bar{c}_2 \frac{\partial f_2}{\partial z_2} \right) e_2$$

and

$$(fc)D_w^* = \frac{\partial f_1}{\partial \bar{w}_1} c_1 + \frac{\partial f_2}{\partial \bar{w}_1} \bar{c}_2 + \frac{\partial \bar{f}_2}{\partial \bar{w}_2} \bar{c}_1 + \frac{\partial f_1}{\partial \bar{w}_2} c_2 + \left( \frac{\partial f_2}{\partial w_1} \bar{c}_1 + \frac{\partial f_1}{\partial w_1} c_2 + \frac{\partial \bar{f}_1}{\partial w_2} \bar{c}_1 + \frac{\partial \bar{f}_2}{\partial w_2} \bar{c}_2 \right) e_2,$$

where  $c_1 \neq 0$  or  $c_2 \neq 0$ , we have  $D_z^*(cf) \neq 0$  and  $(fc)D_w^* \neq 0$ . By the definition of a split regular function in  $\Omega$ ,  $cf$  and  $fc$  are not split regular functions in  $\Omega$ . For example, let  $c = e_2$ . Then

$$D_z^*(cf) = \left( \frac{\partial \bar{f}_1}{\partial \bar{z}_1} + \frac{\partial f_2}{\partial z_1} \right) e_2 + \left( \frac{\partial \bar{f}_2}{\partial \bar{z}_1} + \frac{\partial f_1}{\partial z_2} \right).$$

Since  $f_1$  and  $f_2$  are complex-valued functions, not real-valued functions, we have  $D_z^*(cf) \neq 0$ . Therefore,  $cf$  is not a split regular function in  $\Omega$ .

(iii) We give a count-example for the statement (iii). Let  $f(z, w) = z \neq 0$  and  $g(z, w) = z \neq 0$ . Then  $f$  and  $g$  are split regular functions in  $\Omega$ , but since

$$D_z^*(f(z, w)g(z, w)) = D_z^*(z^2) = \left( \frac{\partial}{\partial \bar{z}_1} + e_2 \frac{\partial}{\partial \bar{z}_2} \right) \{ (z_1^2 + z_2 \bar{z}_2) + (z_1 z_2 + z_2 \bar{z}_1) e_2 \} = z_2 e_2 + e_2 z_2 \neq 0,$$

$f(z, w)g(z, w)$  is not a split regular function in  $\Omega$ .  $\square$

### 3. Split Regular Mappings

**Definition 3.1.** Let  $\Omega = \Omega_1 \times \Omega_2$  be a bounded open set in  $\mathcal{S} \times \mathcal{S}$  and let  $F = (f, g) : \Omega \rightarrow \mathcal{S} \times \mathcal{S}$  such that

$$F(z, w) = (f(z, w), g(z, w)).$$

Then  $F$  is called a split regular mapping if all its components  $f$  and  $g$  are split regular functions in  $\Omega$ , where  $f = f_1 + f_2 e_2$  and  $g = g_1 + g_2 e_2$  with  $f_k$  and  $g_k$  ( $k = 1, 2$ ) are complex-valued functions in  $\Omega$ . That is, each components of  $F(z, w)$  are satisfied the equation  $D_z^* f = 0$ ,  $D_z^* g = 0$ ,  $f D_w^* = 0$  and  $g D_w^* = 0$  in  $\Omega$ .

**Example 3.2.** Let  $\Omega = \Omega_1 \times \Omega_2$  be a bounded open set in  $\mathcal{S} \times \mathcal{S}$  and let

$$F = (f, g) : \Omega \rightarrow \mathcal{S} \times \mathcal{S}.$$

Suppose

$$F(z, w) = (zw, z + w) = (f(z, w), g(z, w))$$

is a mapping, where  $f(z, w) = zw = f_1 + f_2 e_2$  and  $g(z, w) = z + w = g_1 + g_2 e_2$  with  $f_1 = z_1 w_1 + z_2 \bar{w}_2$ ,  $f_2 = z_1 w_2 + z_2 \bar{w}_1$ ,  $g_1 = z_1 + w_1$  and  $g_2 = z_2 + w_2$ . Since  $D_z^* f = 0$ ,  $D_z^* g = 0$ ,  $f D_w^* = 0$  and  $g D_w^* = 0$  in  $\Omega$ ,  $F$  is a split regular mapping in  $\Omega$ .

**Definition 3.3.** Let  $D_1$  and  $D_2$  be bounded domains in  $\mathcal{S} \times \mathcal{S}$ . If a split regular mapping  $F = (f, g) : D_1 \rightarrow D_2$  is bijective and the inverse mapping  $F^{-1} : D_2 \rightarrow D_1$  is split regular, then  $F$  is called a split biregular mapping in  $D_1$ . In this case, we say that the domains  $D_1$  and  $D_2$  are split biregularly equivalent.

**Theorem 3.4.** [Real version of inverse mapping theorem] [1]. Assume  $\lambda = (\lambda_1, \dots, \lambda_n) \in C$ , on an open set  $S$  in  $\mathbb{R}^n$ , where  $C$  is a set of differentiable and continuous functions, and let  $T = \lambda(S)$ . If the jacobian determinant  $J_\lambda(a) \neq 0$  for some point  $a$  in  $S$ , then there are two open sets  $X \subset S$  and  $Y \subset T$  and a uniquely determined function  $\mu$  such that

- (i)  $a \in X$  and  $\lambda(a) \in Y$ ,
- (ii)  $Y = \lambda(X)$ ,
- (iii)  $\lambda$  is one-to-one on  $X$ ,
- (iv)  $\mu$  is defined on  $Y$ ,  $\mu(Y) = X$ , and  $\mu(\lambda(x)) = x$  for every  $x$  in  $X$ ,
- (v)  $\mu \in C$  on  $Y$ .

**Theorem 3.5.** Let  $\Omega = \Omega_1 \times \Omega_2$  be a bounded open set in  $\mathcal{S} \times \mathcal{S}$  and  $F : \Omega \rightarrow \mathcal{S} \times \mathcal{S}$  be a split regular mapping defined in an open neighborhood of a point  $(z^0, w^0)$ . For a point  $(z^0, w^0) \in \Omega$  and its image  $F(z^0, w^0)$ , if  $\det J_F(z^0, w^0)$  does not vanish, then  $F$  is a split biregular mapping from an open neighborhood of  $(z^0, w^0)$  onto an open neighborhood of  $F(z^0, w^0)$ , where

$$\det J_F(z^0, w^0) = \begin{vmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \frac{\partial f_1}{\partial w_1} & \frac{\partial f_1}{\partial w_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \frac{\partial f_2}{\partial w_1} & \frac{\partial f_2}{\partial w_2} \\ \frac{\partial g_1}{\partial z_1} & \frac{\partial g_1}{\partial z_2} & \frac{\partial g_1}{\partial w_1} & \frac{\partial g_1}{\partial w_2} \\ \frac{\partial g_2}{\partial z_1} & \frac{\partial g_2}{\partial z_2} & \frac{\partial g_2}{\partial w_1} & \frac{\partial g_2}{\partial w_2} \end{vmatrix}_{(z^0, w^0)}$$

$$= \begin{vmatrix} \frac{\partial u_0}{\partial x_0} & \frac{\partial u_0}{\partial x_1} & \frac{\partial u_0}{\partial x_2} & \frac{\partial u_0}{\partial x_3} & \frac{\partial u_0}{\partial y_0} & \frac{\partial u_0}{\partial y_1} & \frac{\partial u_0}{\partial y_2} & \frac{\partial u_0}{\partial y_3} \\ \frac{\partial u_1}{\partial x_0} & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & \frac{\partial u_1}{\partial y_0} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \frac{\partial u_1}{\partial y_3} \\ \frac{\partial u_2}{\partial x_0} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_2}{\partial y_0} & \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \frac{\partial u_2}{\partial y_3} \\ \frac{\partial u_3}{\partial x_0} & \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & \frac{\partial u_3}{\partial y_0} & \frac{\partial u_3}{\partial y_1} & \frac{\partial u_3}{\partial y_2} & \frac{\partial u_3}{\partial y_3} \\ \frac{\partial v_0}{\partial x_0} & \frac{\partial v_0}{\partial x_1} & \frac{\partial v_0}{\partial x_2} & \frac{\partial v_0}{\partial x_3} & \frac{\partial v_0}{\partial y_0} & \frac{\partial v_0}{\partial y_1} & \frac{\partial v_0}{\partial y_2} & \frac{\partial v_0}{\partial y_3} \\ \frac{\partial v_1}{\partial x_0} & \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} & \frac{\partial v_1}{\partial y_0} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} & \frac{\partial v_1}{\partial y_3} \\ \frac{\partial v_2}{\partial x_0} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_2}{\partial y_0} & \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} & \frac{\partial v_2}{\partial y_3} \\ \frac{\partial v_3}{\partial x_0} & \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} & \frac{\partial v_3}{\partial y_0} & \frac{\partial v_3}{\partial y_1} & \frac{\partial v_3}{\partial y_2} & \frac{\partial v_3}{\partial y_3} \end{vmatrix}_{(x_0^0, x_1^0, x_2^0, x_3^0, y_0^0, y_1^0, y_2^0, y_3^0)}$$

Proof. We will refer processing proofs of Kaup [11] and Field [7]. From Theorem 3.4 and the definition of a split biregular mapping, there exist a local inverse mapping  $G = (p, q) : F(\Omega) \rightarrow \Omega$  defined in an open neighborhood of  $F(z^0, w^0)$ . Now, we show that  $G$  is a split regular mapping in an open neighborhood of  $F(z^0, w^0)$ . Since  $(z, w) = G(F(z, w))$  is a split regular mapping in  $\Omega$ , we have

$$\begin{aligned} 0 &= D_z^*(z, w) = D_z^*G(F(z, w)) = D_z^*G(f(z, w), g(z, w)) = D_z^*(p(f(z, w), g(z, w)), q(f(z, w), g(z, w))) \\ &= (D_s^*p(f(z, w), g(z, w)), D_t^*q(f(z, w), g(z, w))) = ((D_s^*p)(D_z^*f) + (D_t^*p)(D_z^*g), (D_s^*q)(D_z^*f) + (D_t^*q)(D_z^*g)). \end{aligned}$$

In detail,

$$\begin{aligned} (D_s^*p)(D_z^*f) &= \left\{ \left( \frac{\partial p_1}{\partial s_1} + \frac{\partial \bar{p}_2}{\partial s_2} \right) + \left( \frac{\partial p_2}{\partial s_1} + \frac{\partial \bar{p}_1}{\partial s_2} \right) e_2 \right\} \left\{ \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial \bar{f}_2}{\partial z_2} \right) + \left( \frac{\partial f_2}{\partial z_1} + \frac{\partial \bar{f}_1}{\partial z_2} \right) e_2 \right\} \\ &= \left( \frac{\partial p_1}{\partial s_1} + \frac{\partial \bar{p}_2}{\partial s_2} \right) \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial \bar{f}_2}{\partial z_2} \right) + \left( \frac{\partial p_1}{\partial s_1} + \frac{\partial \bar{p}_2}{\partial s_2} \right) \left( \frac{\partial f_2}{\partial z_1} + \frac{\partial \bar{f}_1}{\partial z_2} \right) e_2 \\ &\quad + \left( \frac{\partial p_2}{\partial s_1} + \frac{\partial \bar{p}_1}{\partial s_2} \right) \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial \bar{f}_2}{\partial z_2} \right) e_2 + \left( \frac{\partial p_2}{\partial s_1} + \frac{\partial \bar{p}_1}{\partial s_2} \right) \left( \frac{\partial f_2}{\partial z_1} + \frac{\partial \bar{f}_1}{\partial z_2} \right), \end{aligned}$$

$$\begin{aligned} (D_t^*p)(D_z^*g) &= \left\{ \left( \frac{\partial p_1}{\partial t_1} + \frac{\partial \bar{p}_2}{\partial t_2} \right) + \left( \frac{\partial p_2}{\partial t_1} + \frac{\partial \bar{p}_1}{\partial t_2} \right) e_2 \right\} \left\{ \left( \frac{\partial g_1}{\partial z_1} + \frac{\partial \bar{g}_2}{\partial z_2} \right) + \left( \frac{\partial g_2}{\partial z_1} + \frac{\partial \bar{g}_1}{\partial z_2} \right) e_2 \right\} \\ &= \left( \frac{\partial p_1}{\partial t_1} + \frac{\partial \bar{p}_2}{\partial t_2} \right) \left( \frac{\partial g_1}{\partial z_1} + \frac{\partial \bar{g}_2}{\partial z_2} \right) + \left( \frac{\partial p_1}{\partial t_1} + \frac{\partial \bar{p}_2}{\partial t_2} \right) \left( \frac{\partial g_2}{\partial z_1} + \frac{\partial \bar{g}_1}{\partial z_2} \right) e_2 \\ &\quad + \left( \frac{\partial p_2}{\partial t_1} + \frac{\partial \bar{p}_1}{\partial t_2} \right) \left( \frac{\partial g_1}{\partial z_1} + \frac{\partial \bar{g}_2}{\partial z_2} \right) e_2 + \left( \frac{\partial p_2}{\partial t_1} + \frac{\partial \bar{p}_1}{\partial t_2} \right) \left( \frac{\partial g_2}{\partial z_1} + \frac{\partial \bar{g}_1}{\partial z_2} \right), \end{aligned}$$

$$\begin{aligned} (D_s^*q)(D_z^*f) &= \left\{ \left( \frac{\partial q_1}{\partial s_1} + \frac{\partial \bar{q}_2}{\partial s_2} \right) + \left( \frac{\partial q_2}{\partial s_1} + \frac{\partial \bar{q}_1}{\partial s_2} \right) e_2 \right\} \left\{ \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial \bar{f}_2}{\partial z_2} \right) + \left( \frac{\partial f_2}{\partial z_1} + \frac{\partial \bar{f}_1}{\partial z_2} \right) e_2 \right\} \\ &= \left( \frac{\partial q_1}{\partial s_1} + \frac{\partial \bar{q}_2}{\partial s_2} \right) \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial \bar{f}_2}{\partial z_2} \right) + \left( \frac{\partial q_1}{\partial s_1} + \frac{\partial \bar{q}_2}{\partial s_2} \right) \left( \frac{\partial f_2}{\partial z_1} + \frac{\partial \bar{f}_1}{\partial z_2} \right) e_2 \\ &\quad + \left( \frac{\partial q_2}{\partial s_1} + \frac{\partial \bar{q}_1}{\partial s_2} \right) \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial \bar{f}_2}{\partial z_2} \right) e_2 + \left( \frac{\partial q_2}{\partial s_1} + \frac{\partial \bar{q}_1}{\partial s_2} \right) \left( \frac{\partial f_2}{\partial z_1} + \frac{\partial \bar{f}_1}{\partial z_2} \right), \end{aligned}$$

$$\begin{aligned} (D_t^*q)(D_z^*g) &= \left\{ \left( \frac{\partial q_1}{\partial t_1} + \frac{\partial \bar{q}_2}{\partial t_2} \right) + \left( \frac{\partial q_2}{\partial t_1} + \frac{\partial \bar{q}_1}{\partial t_2} \right) e_2 \right\} \left\{ \left( \frac{\partial g_1}{\partial z_1} + \frac{\partial \bar{g}_2}{\partial z_2} \right) + \left( \frac{\partial g_2}{\partial z_1} + \frac{\partial \bar{g}_1}{\partial z_2} \right) e_2 \right\} \\ &= \left( \frac{\partial q_1}{\partial t_1} + \frac{\partial \bar{q}_2}{\partial t_2} \right) \left( \frac{\partial g_1}{\partial z_1} + \frac{\partial \bar{g}_2}{\partial z_2} \right) + \left( \frac{\partial q_1}{\partial t_1} + \frac{\partial \bar{q}_2}{\partial t_2} \right) \left( \frac{\partial g_2}{\partial z_1} + \frac{\partial \bar{g}_1}{\partial z_2} \right) e_2 \\ &\quad + \left( \frac{\partial q_2}{\partial t_1} + \frac{\partial \bar{q}_1}{\partial t_2} \right) \left( \frac{\partial g_1}{\partial z_1} + \frac{\partial \bar{g}_2}{\partial z_2} \right) e_2 + \left( \frac{\partial q_2}{\partial t_1} + \frac{\partial \bar{q}_1}{\partial t_2} \right) \left( \frac{\partial g_2}{\partial z_1} + \frac{\partial \bar{g}_1}{\partial z_2} \right), \end{aligned}$$

where

$$\begin{aligned} p(f(z, w), g(z, w)) &= p(f, g) = p_1(f, g) + p_2(f, g)e_2, \quad q(f(z, w), g(z, w)) = q(f, g) = q_1(f, g) + q_2(f, g)e_2, \\ s &= f(z, w) = f_1(z, w) + f_2(z, w)e_2 \end{aligned}$$

with  $s_1 = f_1(z, w)$  and  $s_2 = f_2(z, w)$ ; Also,  $t = g(z, w) = g_1(z, w) + g_2(z, w)e_2$  with  $t_1 = g_1(z, w)$  and  $t_2 = g_2(z, w)$ . Also, we have

$$\begin{aligned} 0 &= (z, w)D_w^* = G(F(z, w))D_w^* = G(f(z, w), g(z, w))D_w^* = (p(f(z, w), g(z, w)), q(f(z, w), g(z, w)))D_w^* \\ &= (p(f(z, w), g(z, w))D_w^*, q(f(z, w), g(z, w))D_w^*) = (pD_s^*fD_w^* + pD_t^*gD_w^*, qD_s^*qD_w^* + qD_t^*gD_w^*). \end{aligned}$$

In detail,

$$\begin{aligned} (pD_s^*)(fD_w^*) &= \left\{ \left( \frac{\partial p_1}{\partial s_1} + \frac{\partial p_2}{\partial s_2} \right) + \left( \frac{\partial p_2}{\partial s_1} + \frac{\partial p_1}{\partial s_2} \right) e_2 \right\} \left\{ \left( \frac{\partial f_1}{\partial w_1} + \frac{\partial f_2}{\partial w_2} \right) + \left( \frac{\partial f_2}{\partial w_1} + \frac{\partial f_1}{\partial w_2} \right) e_2 \right\} \\ &= \left( \frac{\partial p_1}{\partial s_1} + \frac{\partial p_2}{\partial s_2} \right) \left( \frac{\partial f_1}{\partial w_1} + \frac{\partial f_2}{\partial w_2} \right) + \left( \frac{\partial p_1}{\partial s_1} + \frac{\partial p_2}{\partial s_2} \right) \left( \frac{\partial f_2}{\partial w_1} + \frac{\partial f_1}{\partial w_2} \right) e_2 \\ &\quad + \left( \frac{\partial p_2}{\partial s_1} + \frac{\partial p_1}{\partial s_2} \right) \left( \frac{\partial f_1}{\partial w_1} + \frac{\partial f_2}{\partial w_2} \right) e_2 + \left( \frac{\partial p_2}{\partial s_1} + \frac{\partial p_1}{\partial s_2} \right) \left( \frac{\partial f_2}{\partial w_1} + \frac{\partial f_1}{\partial w_2} \right), \end{aligned}$$

$$\begin{aligned} (pD_t^*)(gD_w^*) &= \left\{ \left( \frac{\partial p_1}{\partial t_1} + \frac{\partial p_2}{\partial t_2} \right) + \left( \frac{\partial p_2}{\partial t_1} + \frac{\partial p_1}{\partial t_2} \right) e_2 \right\} \left\{ \left( \frac{\partial g_1}{\partial w_1} + \frac{\partial g_2}{\partial w_2} \right) + \left( \frac{\partial g_2}{\partial w_1} + \frac{\partial g_1}{\partial w_2} \right) e_2 \right\} \\ &= \left( \frac{\partial p_1}{\partial t_1} + \frac{\partial p_2}{\partial t_2} \right) \left( \frac{\partial g_1}{\partial w_1} + \frac{\partial g_2}{\partial w_2} \right) + \left( \frac{\partial p_1}{\partial t_1} + \frac{\partial p_2}{\partial t_2} \right) \left( \frac{\partial g_2}{\partial w_1} + \frac{\partial g_1}{\partial w_2} \right) e_2 \\ &\quad + \left( \frac{\partial p_2}{\partial t_1} + \frac{\partial p_1}{\partial t_2} \right) \left( \frac{\partial g_1}{\partial w_1} + \frac{\partial g_2}{\partial w_2} \right) e_2 + \left( \frac{\partial p_2}{\partial t_1} + \frac{\partial p_1}{\partial t_2} \right) \left( \frac{\partial g_2}{\partial w_1} + \frac{\partial g_1}{\partial w_2} \right), \end{aligned}$$

$$\begin{aligned} (qD_s^*)(fD_w^*) &= \left\{ \left( \frac{\partial q_1}{\partial s_1} + \frac{\partial q_2}{\partial s_2} \right) + \left( \frac{\partial q_2}{\partial s_1} + \frac{\partial q_1}{\partial s_2} \right) e_2 \right\} \left\{ \left( \frac{\partial f_1}{\partial w_1} + \frac{\partial f_2}{\partial w_2} \right) + \left( \frac{\partial f_2}{\partial w_1} + \frac{\partial f_1}{\partial w_2} \right) e_2 \right\} \\ &= \left( \frac{\partial q_1}{\partial s_1} + \frac{\partial q_2}{\partial s_2} \right) \left( \frac{\partial f_1}{\partial w_1} + \frac{\partial f_2}{\partial w_2} \right) + \left( \frac{\partial q_1}{\partial s_1} + \frac{\partial q_2}{\partial s_2} \right) \left( \frac{\partial f_2}{\partial w_1} + \frac{\partial f_1}{\partial w_2} \right) e_2 \\ &\quad + \left( \frac{\partial q_2}{\partial s_1} + \frac{\partial q_1}{\partial s_2} \right) \left( \frac{\partial f_1}{\partial w_1} + \frac{\partial f_2}{\partial w_2} \right) e_2 + \left( \frac{\partial q_2}{\partial s_1} + \frac{\partial q_1}{\partial s_2} \right) \left( \frac{\partial f_2}{\partial w_1} + \frac{\partial f_1}{\partial w_2} \right), \end{aligned}$$

$$\begin{aligned} (qD_t^*)(gD_w^*) &= \left\{ \left( \frac{\partial q_1}{\partial t_1} + \frac{\partial q_2}{\partial t_2} \right) + \left( \frac{\partial q_2}{\partial t_1} + \frac{\partial q_1}{\partial t_2} \right) e_2 \right\} \left\{ \left( \frac{\partial g_1}{\partial w_1} + \frac{\partial g_2}{\partial w_2} \right) + \left( \frac{\partial g_2}{\partial w_1} + \frac{\partial g_1}{\partial w_2} \right) e_2 \right\} \\ &= \left( \frac{\partial q_1}{\partial t_1} + \frac{\partial q_2}{\partial t_2} \right) \left( \frac{\partial g_1}{\partial w_1} + \frac{\partial g_2}{\partial w_2} \right) + \left( \frac{\partial q_1}{\partial t_1} + \frac{\partial q_2}{\partial t_2} \right) \left( \frac{\partial g_2}{\partial w_1} + \frac{\partial g_1}{\partial w_2} \right) e_2 \\ &\quad + \left( \frac{\partial q_2}{\partial t_1} + \frac{\partial q_1}{\partial t_2} \right) \left( \frac{\partial g_1}{\partial w_1} + \frac{\partial g_2}{\partial w_2} \right) e_2 + \left( \frac{\partial q_2}{\partial t_1} + \frac{\partial q_1}{\partial t_2} \right) \left( \frac{\partial g_2}{\partial w_1} + \frac{\partial g_1}{\partial w_2} \right). \end{aligned}$$

Since  $f$  and  $g$  are split regular functions in  $\Omega$ , by the equation (1), we have

$$D_z^* p(f(z, w), g(z, w)) = p(f(z, w), g(z, w)) D_w^* = 0,$$

$$D_z^* q(f(z, w), g(z, w)) = q(f(z, w), g(z, w)) D_w^* = 0.$$

From the properties of split regular functions and Proposition 2.4, regardless of the split regularity of  $f$  and  $g$ ,  $D_z^* p$ ,  $D_z^* q$ ,  $pD_w^*$  and  $qD_w^*$  should be vanished in  $\Omega$ . Therefore,  $G$  is a split regular mapping in an open neighborhood of  $F(z^0, w^0)$ .  $\square$

**Example 3.6.** Let  $\Omega$  be a bounded open set in  $\mathcal{S} \times \mathcal{S}$  and let  $F : \Omega \rightarrow \mathcal{S} \times \mathcal{S}$ . Consider a mapping on  $\Omega$  and with values in  $\mathcal{S} \times \mathcal{S}$ :

$$F(z, w) = (z + w, z - w) = ((z_1 + w_1) + (z_2 + w_2)e_2, (z_1 - w_1) + (z_2 - w_2)e_2).$$

Then  $F$  is a split regular mapping defined in an open neighborhood of a point  $(0, 0)$ . For the point  $(0, 0) \in \Omega$  and its image  $F(0, 0)$ , we have

$$\det J_F(0, 0) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}_{(0,0)} \neq 0.$$

Then a local inverse mapping  $G = (p, q) : F(\Omega) \rightarrow \Omega$ , defined in an open neighborhood of  $F(0, 0)$ , such that  $G(r, t) = (p(r, t), q(r, t)) = \left(\frac{1}{2}(r + t), \frac{1}{2}(r - t)\right)$  is split regular in an open neighborhood of  $F(0, 0)$ , where  $(r, t) \in F(\Omega)$ . Therefore,  $F$  is a split biregular mapping from an open neighborhood of  $(0, 0)$  onto an open neighborhood of  $F(0, 0)$ .

## References

- [1] T. M. Apostol, *Mathematical Analysis*, Narosa Publishing House, New Delhi, India, 2002.
- [2] D. C. Brody, E. M. Graefe, On complexified mechanics and coquaternions, *Journal of Physics A: Mathematical and Theoretical* 44 (2011) 1–9.
- [3] R. B. Burckel, *An Introduction to Classical Complex Analysis*, vol. 1, Academic Press, New York, USA, 1980.
- [4] J. Cockle, On Systems of Algebra involving more than one Imaginary, *Philosophical Magazine (series3)* 35 (1849) 434–435.
- [5] J. Dolbeault, M. J. Esteban, E. Séré, On the eigenvalues of operators with gaps. Application to Dirac operators, *Journal of Functional Analysis* 174 (1) (2000) 208–226.
- [6] A. L. Dontchev, R. T. Rockafellar, *Implicit Functions and Solution Mappings*, Springer-Verlag, Berlin, Germany, 2009.
- [7] M. Field, *Several complex variables and complex manifolds I*, Cambridge University Press, Cambridge, United Kingdom, 1982.
- [8] J. Frauendiener, T. Vogel, Algebraic stability analysis of constraint propagation, *Classical and Quantum Gravity* 22 (9) (2005).
- [9] I. Frenkel, M. Libine, Split quaternionic analysis and separation of the series for  $SL(2, \mathbb{R})$  and  $SL(2, \mathbb{C})/SL(2, \mathbb{R})$ , *Advances in Mathematics* 228 (2011) 678–763.
- [10] L. Hurwicz, M. K. Richter, Implicit functions and diffeomorphisms without  $C^1$ , *Advanced Mathematical Economics* 5 (2006) 65–96.
- [11] L. Kaup, B. Kaup, *Holomorphic functions of several variables: an introduction to the fundamental theory*, vol. 3, Walter de Gruyter, Berlin, Germany, 1983.
- [12] J. E. Kim, S. J. Lim, K. H. Shon, Taylor series of functions with values in dual quaternion, *Journal of the Korean Society of Mathematical Education Series B: The Pure and Applied Mathematics* 20 (4) (2013) 251–258.
- [13] J. E. Kim, S. J. Lim, K. H. Shon, Regular functions with values in ternary number system on the complex Clifford analysis, *Abstract and applied analysis* 2013 (Article ID 136120) (2013) 7 pages.
- [14] J. E. Kim, S. J. Lim, K. H. Shon, Regularity of functions on the reduced quaternion field in Clifford analysis, *Abstract and applied analysis* 2014 (Article ID 654798) (2014) 8 pages.
- [15] J. E. Kim, K. H. Shon, The Regularity of functions on Dual split quaternions in Clifford analysis, *Abstract and applied analysis* 2014 (Article ID 369430) (2014) 8 pages.
- [16] S. G. Krantz, H. R. Parks, *The Implicit Function Theorem—History, Theory, and Applications*, Springer-Verlag, Berlin, Germany, 2002.
- [17] L. Kula, Y. Yayl, Split Quaternions and Rotations in Semi Euclidean Space, *Journal of Korean Mathematical Society* 44 (2007) 1313–1327.
- [18] D. Lian, J. Y. Chiang, The maximal rank of a kind of partial banded block matrix subject to linear equations, *Filomat* 27 (2)(2013) 381–389.
- [19] O. R. B. de Oliveira, The implicit and the inverse function theorems: easy proofs, arXiv preprint arXiv: 1212. 2066 (2012).
- [20] M. Özdemir, A. A. Ergin, Rotations with unit timelike quaternions in Minkowski 3-space, *Journal of Geometry and Physics* 56 (2006) 322–336.
- [21] W. Rudin, *Principles of Mathematical Analysis*, Mc. Graw Hill, New York, USA, 1976.
- [22] W. Rudin, *Function theory in the unit ball of  $\mathbb{C}^n$* , vol. 213, Springer-Verlag, Berlin, Germany, 2002.
- [23] G. M. Scarpello, A historical outline of the theorem of implicit functions, *Divulgaciones Matemáticas* 10 (2) (2002) 171–180.
- [24] C. Song, G. Chen, X. Zhang, An iterative solution to coupled quaternion matrix equations, *Filomat*, 26 (4) (2012) 809–826.
- [25] J. L. Weiner, G. R. Wilkens, Quaternions and Rotations in  $\mathbb{E}^4$ , *American Mathematical Monthly* 112 (1) (2005) 69–76.