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Approximation Fixed Theorems for α-Partial Weakly Zamfirescu Mappings with Application to Homotopy Invariance

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Abstract. In this paper, we introduce the concept of α-partial weakly Zamfirescu mappings and give some approximate fixed point results for this mapping in α-complete metric spaces. We also give some approximate fixed point results in α-complete metric space endowed with an arbitrary binary relation and approximate fixed point results in α-complete metric space endowed with graph. As application, we give homotopy results for α-partial weakly Zamfirescu mapping.

1. Introduction and Preliminaries

Fixed point theory is one of the outstanding subfields of nonlinear functional analysis. It has been used in the research areas of mathematics and nonlinear sciences. Many authors have some detailed discussions and applications of a fixed point theorems (see example, [1], [2], [3], [7], [17], [20], [16], [18], and references there in). In 2012, Samet et al. [15] introduced the concepts of α-ψ-contraction mappings and α-admissible mappings and established various fixed point theorems for such mappings in complete metric spaces.

Definition 1.1 ([15]). *Let T be a self mapping on a nonempty set X and $\alpha : X \times X \to [0, \infty)$ be a mapping. We say that T is* α-adminssible *if the following condition holds:*

$$x, y \in X \text{ with } \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

By using this concept, several authors proved fixed point results (see in [8, 10, 11] and references therein). Afterward, Sintunavarat [19] (see also [12]) introduced the useful concept of transitivity for mappings as follows:

Definition 1.2 ([12, 19]). *Let X be a nonempty set. A mapping $\alpha : X \times X \to [0, \infty)$ is said to be* transitive *if the following condition holds:*

$$x, y, z \in X \text{ with } \alpha(x, y) \geq 1 \text{ and } \alpha(y, z) \geq 1 \Longrightarrow \alpha(x, z) \geq 1.$$

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They also proved fixed point results for new generalized contraction mapping by using this concept and established generalized Ulam-Hyers stability, well-posedness, and limit shadowing of fixed point problems for such mapping in metric spaces. In 2014, Hussain et al. [8] introduced concepts of α-η-complete metric space and α-η-continuous mapping and establish fixed point results for modified α-η-ψ-rational contraction mappings in α-η-complete metric spaces. By using these idea, we give the following concepts:

Definition 1.3 ([8]). *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. The metric space X is said to be* α-complete *if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in X.*

Remark 1.4. *If X is complete metric space, then X is also α-complete metric space. But the converse is not true in general case.*

Now we introduce the concept of α-continuous for (self and non-self) mapping.

Definition 1.5. *Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ be a mapping and A be a subset of X. We say that $T : A \to X$ is an* α-continuous mapping *on A if for each sequence $\{x_n\}$ in A, the following condition holds:*

$$x_n \to x \text{ as } n \to \infty \text{ for some } x \in \overline{A} \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \quad \Longrightarrow \quad Tx_n \to Tx \text{ as } n \to \infty.$$

If $A = X$, then T is called α-continuous mapping on X (due to Hussain et al. [8]).

Remark 1.6. *If T is a continuous mapping, then T is an α-continuous mapping, where $\alpha : X \times X \to [0, \infty)$ is an arbitrary mappings.*

Example 1.7. *Let $X = (0, \infty)$ and $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define mappings $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ by*

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [1, 6], \\ 0, & otherwise \end{cases}$$

and

$$Tx = \begin{cases} \frac{x}{2}, & x \in [1, 6], \\ x^2 + 3x + 5, & x \in (0, 1) \cup (6, \infty). \end{cases}$$

It is easy to see that T is not continuous at $x = 6$. So, T is not continuous on X, but T is α-continuous on X. Indeed, let $\{x_n\}$ be a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we have $x_n \in [1, 6]$ and then $Tx_n = \frac{x_n}{2}$ for all $n \in \mathbb{N}$. If $x_n \to x$ as $n \to \infty$ for some $x \in X$, we have $Tx_n = \frac{x_n}{2} \to \frac{x}{2} = Tx$ as $n \to \infty$. Therefore, T is α-continuous on X.

Next, we give some detail of approximate fixed point property and some useful lemma.

Definition 1.8. *Let (X, d) be a metric space. For a give $\epsilon > 0$, a point $x \in X$ is said to be an ϵ-*fixed points *of $T : X \to X$ if $d(x, Tx) < \epsilon$. The set of all ϵ-fixed points of T is denoted by $F_\epsilon(T)$, that is,*

$$F_\epsilon(T) := \{x \in X : d(x, Tx) < \epsilon\}.$$

Definition 1.9. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. We say that T has the* approximate fixed point property *if for all $\varepsilon > 0$, there exists an ε-fixed point of T, that is,*

$$\forall \varepsilon > 0, \qquad F_\varepsilon(T) \neq \emptyset$$

or, equivalently,

$$\inf_{x \in X} d(x, Tx) = 0.$$

Definition 1.10 ([6]). *Let (X, d) be a metric space and $T : X \to X$ be a mapping. A self mapping T on a metric space (X, d) is said to be* asymptotically regular *at a point $x \in X$, if*

$$d(T^n x, T^{n+1} x) \to 0 \text{ as } n \to \infty,$$

where $T^n x$ denotes the $n-th$ iterate of T at x.

Lemma 1.11. *Let (X, d) be a metric space and $T : X \to X$ be an asymptotically regular at a point $z \in X$, then T has the approximate fixed point property.*

On the other hand, Zamfirescu [21] introduced some generalized contraction mapping in 1972 as follows:

Definition 1.12 ([21]). *Let (X, d) be a metric space and T be a selfmap on X. Then T is called* Zamfirescu *whenever there exists $\xi \in [0, 1)$ with*

$$d(Tx, Ty) \leq \xi \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}$$

for all $x, y \in X$.

Afterward, Ariza-Ruiza et al. [4] introduced the notion of weakly Zamfirescu mappings as follows:

Definition 1.13 ([21]). *Let (X, d) be a metric space and T be a selfmap on X. Then T is called* weakly Zamfirescu *whenever there exists $\gamma : X \times X \to [0, 1]$ with*

$$\theta(a, b) := \sup\{\gamma(x, y) : a \leq d(x, y) \leq b\} < 1$$

for all $0 < a \leq b$, such that,

$$d(Tx, Ty) \leq \gamma(x, y) M_T(x, y)$$

for all $x, y \in X$, where

$$M_T(x, y) = \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

Recently, Miandaragh et. al. [13] introduced the concept of α-weakly Zamfirescu mappings as follows:

Definition 1.14 ([13]). *Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ be a mapping and T be a selfmap on X. Then T is called* α-weakly Zamfirescu *whenever there exists $\gamma : X \times X \to [0, 1]$ with*

$$\theta(a, b) := \sup\{\gamma(x, y) : a \leq d(x, y) \leq b\} < 1$$

for all $0 < a \leq b$, such that,

$$\alpha(x, y) d(Tx, Ty) \leq \gamma(x, y) M_T(x, y)$$

for all $x, y \in X$, where

$$M_T(x, y) = \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

They also give an approximate fixed point theorems for α-weakly Zamfirescu mappings in metric spaces.

In this paper, we introduce the concept of α-partial weakly Zamfirescu mappings and prove approximate fixed point results for such mappings in metric spaces. We also establish the approximate fixed point results in metric space endowed with an arbitrary binary relation and approximate fixed point results in metric space endowed with graph by using our main results. As application, we study the homotopy results for α-partial weakly Zamfirescu mapping.

2. Main Results

In this section, we introduce the concept of α-partial weakly Zamfirescu mappings and prove approximate fixed point results for such mappings in metric spaces. We also give the consequently of our results to another approximate fixed point results.

2.1. *Approximate fixed point for α-partial weakly Zamfirescu mappings*

In this subsection, we give the concept of new nonlinear mapping so called α-partial weakly Zamfirescu mapping which is a generalization of α-weakly Zamfirescu mapping and weakly Zamfirescu mapping. We also establish the approximate fixed point results for such mappings.

Definition 2.1. *Let (X,d) be a metric space, $\alpha : X\times X \to [0,\infty)$ be a mapping and T be a selfmap on X. Then T is called α-*partial weakly Zamfirescu mapping *whenever there exists a mapping $\gamma : X\times X \to [0,1]$ with*

$$\theta(a,b) := \sup\{\gamma(x,y) : a \le d(x,y) \le b\} < 1 \quad \text{for all } 0 < a \le b, \tag{1}$$

and it satisfies the following condition:

$$\text{for all } x,y \in X \text{ with} \quad \alpha(x,y) \ge 1 \quad \Longrightarrow \quad d(Tx,Ty) \le \gamma(x,y)M_T(x,y), \tag{2}$$

where $M_T(x,y) = \max\left\{d(x,y), \frac{1}{2}[d(x,Ty)+d(y,Tx)], \frac{1}{2}[d(x,Tx)+d(y,Ty)]\right\}$.

Now, we establish new approximate fixed point theorem for α-partial weakly Zamfirescu mappings in metric spaces.

Theorem 2.2. *Let (X,d) be a metric space, $\alpha : X\times X \to [0,\infty)$ be a mapping and T be an α-partial weakly Zamfirescu selfmap on X. If T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$, then T has the approximate fixed point property.*

Moreover, T has a fixed point provides that the following conditions hold:

(i) *T is α-continuous on X,*

(ii) *(X,d) is α-complete metric space,*

(iii) *α is transitive.*

Proof. Let $x_0 \in X$ be such that $\alpha(x_0,Tx_0) \ge 1$. Define the sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Now, we show that $d(x_n,x_{n+1}) \le \gamma(x_{n-1},x_n)d(x_{n-1},x_n)$ for all $n \in \mathbb{N}$. Since T is α-admissible, it is easy to check that $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$. From (2), for $n \in \mathbb{N}$, we get

$$d(x_n,x_{n+1}) \le \gamma(x_{n-1},x_n)M_T(x_{n-1},x_n),$$

where

$$\begin{aligned} M_T(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n), \frac{1}{2}[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)], \frac{1}{2}[d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})]\right\} \\ &= \max\left\{d(x_{n-1},x_n), \frac{1}{2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})], \frac{1}{2}[d(x_{n-1},x_{n+1})+d(x_n,x_n)]\right\} \\ &= \max\left\{d(x_{x-1},x_n), \frac{1}{2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})], \frac{1}{2}d(x_{n-1},x_{n+1})\right\} \\ &= \max\left\{d(x_{x-1},x_n), \frac{1}{2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})]\right\}. \end{aligned}$$

If $M_T(x_{n-1},x_n) = d(x_{n-1},x_n)$, we get

$$d(x_n,x_{n+1}) \le \gamma(x_{n-1},x_n)d(x_{n-1},x_n).$$

If $M_T(x_{n-1}, x_n) = \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$, we get

$$d(x_n, x_{n+1}) \leq \gamma(x_{n-1}, x_n)\frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{2},$$

that is,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{\gamma(x_{n-1}, x_n)}{2-\gamma(x_{n-1}, x_n)} d(x_{n-1}, x_n) \\ &\leq \gamma(x_{n-1}, x_n) d(x_{n-1}, x_n). \end{aligned} \tag{3}$$

Therefore $d(x_n, x_{n+1}) \leq \gamma(x_{n-1}, x_n)d(x_{n-1}, x_n) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. This implies that the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing and so it converges to the real number $d := \inf_{n\in\mathbb{N}} d(x_{n-1}, x_n)$.

Next, we show that $d = 0$. On the contrary, we get $d > 0$. Since $0 < d \leq d(x_n, x_{n+1}) \leq d(x_0, x_1)$ for all $n \in \mathbb{N}$, we have $\gamma(x_{n-1}, x_n) \leq \theta$ for all $n \in \mathbb{N}$, where $\theta := \theta(d, d(x_0, x_1))$. Hence, for all $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} d &\leq d(x_n, x_{n+1}) \\ &\leq \gamma(x_{n-1}, x_n)d(x_{n-1}, x_n) \\ &\leq \theta d(x_{n-1}, x_n) \\ &\leq \theta^2 d(x_{n-2}, x_{n-1}) \\ &\ \ \vdots \\ &\leq \theta^n d(x_0, x_1). \end{aligned}$$

But this is impossible because $d > 0$ and $0 \leq \theta < 1$. Therefore, we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4}$$

This implies that T is asymptotically regular at $x_0 \in X$. By Lemma 1.11, we can conclude that T has the approximate fixed point property.

Now, suppose that (X, d) is an α-complete metric space and T is α-continuous on X. We will prove that $\{x_n\}$ is a Cauchy sequence and that its limit is a fixed point for T. To do this, let us prove that

$$d(x_{n+1}, x_{n+k+1}) \leq \gamma(x_n, x_{n+k})d(x_n, x_{n+k}) + 2d(x_n, x_{n+1}), \quad \text{for all } n, k \in \mathbb{N}. \tag{5}$$

By transitivity of α, we get

$$\alpha(x_n, x_{n+k}) \geq 1, \quad \text{for all } n, k \in \mathbb{N}.$$

Since $\alpha(x_n, x_{n+k}) \geq 1$ for all $n, k \in \mathbb{N}$, we get

$$\begin{aligned} d(x_{n+1}, x_{n+k+1}) &= d(Tx_n, Tx_{n+k}) \\ &\leq \gamma(x_n, x_{n+k})M_T(x_n, x_{n+k}), \end{aligned}$$

where

$$\begin{aligned} M_T(x_n, x_{n+k}) &= \max\left\{d(x_n, x_{n+k}), \frac{1}{2}[d(x_n, Tx_n) + d(x_{n+k}, Tx_{n+k})], \frac{1}{2}[d(x_n, Tx_{n+k}) + d(x_{n+k}, Tx_n)]\right\} \\ &= \max\left\{d(x_n, x_{n+k}), \frac{1}{2}[d(x_n, x_{n+1}) + d(x_{n+k}, x_{n+k+1})], \frac{1}{2}[d(x_n, x_{n+k+1}) + d(x_{n+k}, x_{n+1})]\right\}. \end{aligned}$$

We consider the following three cases.

Case 1: If $M_T(x_n, x_{n+k}) = d(x_n, x_{n+k})$, then (5) is obvious.

Case 2: If $M_T(x_n, x_{n+k}) = \frac{1}{2}[d(x_n, x_{n+1}) + d(x_{n+k}, x_{n+k+1})]$, then

$$d(x_{n+1}, x_{n+k+1}) \leq \frac{\gamma(x_n, x_{n+k})}{2}[d(x_n, x_{n+1}) + d(x_{n+k}, x_{n+k+1})].$$

Applying (3),

$$d(x_{n+k}, x_{n+k+1}) \leq d(x_n, x_{n+1}).$$

So,

$$d(x_{n+1}, x_{n+k+1}) \leq \gamma(x_n, x_{n+k})d(x_n, x_{n+k}) + 2d(x_n, x_{n+1}).$$

Case 3: If $M_T(x_n, x_{n+k}) = \frac{1}{2}[d(x_n, x_{n+k+1}) + d(x_{n+k}, x_{n+1})]$, then

$$\begin{aligned} d(x_{n+1}, x_{n+k+1}) &\leq \frac{\gamma(x_n, x_{n+k})}{2}[d(x_n, x_{n+k+1}) + d(x_{n+k}, x_{n+1})] \\ &\leq \frac{\gamma(x_n, x_{n+k})}{2}[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+k+1}) + d(x_{n+k}, x_{n+1})]. \end{aligned}$$

This implies that,

$$\left(1 - \frac{\gamma(x_n, x_{n+k})}{2}\right)d(x_{n+1}, x_{n+k+1}) \leq \frac{\gamma(x_n, x_{n+k})}{2}[d(x_n, x_{n+1}) + d(x_{n+k}, x_{n+1})],$$

that is,

$$\begin{aligned} d(x_{n+1}, x_{n+k+1}) &\leq \frac{\gamma(x_n, x_{n+k})}{2 - \gamma(x_n, x_{n+k})}[d(x_{n+k}, x_{n+1}) + d(x_n, x_{n+1})] \\ &\leq \gamma(x_n, x_{n+k})[d(x_{n+k}, x_{n+1}) + d(x_n, x_{n+1})] \\ &\leq \gamma(x_n, x_{n+k})d(x_n, x_{n+k}) + 2d(x_n, x_{n+1}). \end{aligned}$$

To prove that $\{x_n\}$ is a Cauchy sequence, suppose that $\epsilon > 0$ and use (4) to obtain $N \in \mathbb{N}$ such that

$$d(x_N, x_{N+1}) < \frac{1}{6}\left(1 - \theta\left(\frac{\epsilon}{2}, \epsilon\right)\right) \cdot \epsilon. \tag{6}$$

We will prove inductively that $d(x_N, x_{N+k}) < \epsilon$ for all $k \in \mathbb{N}$. It is obvious for $k = 1$, and assuming $d(x_N, x_{N+k}) < \epsilon$, let us see that $d(x_N, x_{N+k+1}) < \epsilon$. Note that, using (5), we have that

$$\begin{aligned} d(x_N, x_{N+k+1}) &\leq d(x_{N+1}, x_{N+k+1}) + d(x_N, x_{N+1}) \\ &\leq \gamma(x_N, x_{N+k})d(x_N, x_{N+k}) + 3d(x_N, x_{N+1}). \end{aligned} \tag{7}$$

Thus, if $d(x_N, x_{N+k}) < \frac{\epsilon}{2}$, it follows from (6) and (7) that

$$\begin{aligned} d(x_N, x_{N+k+1}) &\leq d(x_N, x_{N+k}) + 3d(x_n, x_{N+1}) \\ &< \frac{\epsilon}{2} + 3 \cdot \frac{1}{6}\left(1 - \theta\left(\frac{\epsilon}{2}, \epsilon\right)\right) \cdot \epsilon \\ &< \epsilon. \end{aligned} \tag{8}$$

In the other hand, if $d(x_N, x_{N+k}) \geq \frac{\epsilon}{2}$, applying the induction hypothesis, we have that $\theta(x_N, x_{N+k}) \leq \theta(\frac{\epsilon}{2}, \epsilon)$. Then, from (6) and (7), we conclude that

$$\begin{aligned} d(x_N, x_{N+k+1}) &\leq \theta(x_N, x_{N+k})d(x_N, x_{N+k}) + 3d(x_N, x_{N+1}) \\ &< \theta\Big(\frac{\epsilon}{2}, \epsilon\Big) \cdot \epsilon + 3 \cdot \frac{1}{6}\Big(1 - \theta\Big(\frac{\epsilon}{2}, \epsilon\Big)\Big) \cdot \epsilon \\ &\leq \epsilon. \end{aligned} \tag{9}$$

From (8) and (9), we get $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is α-complete, we get $\{x_n\}$ is convergent, say to $u \in X$. By α-continuity of T, we get

$$\begin{aligned} u &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} Tx_n \\ &= Tu. \end{aligned}$$

Thus T has a fixed point. This completes the proof. □

In the next theorem, we replace the α-continuity condition of the mapping T in Theorem 2.2 by using the following condition:

Definition 2.3. *Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ be a mapping and A be a subset of X. We say that A* satisfies condition ($\star$) *if $\{x_n\}$ is sequence in A such that $x_n \to x$ as $n \to \infty$ for some $x \in \overline{A}$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Theorem 2.4. *Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ be a mapping and T be an α-partial weakly Zamfirescu selfmap on X. If T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has approximate fixed point property.*

Moreover, T has a fixed point provides that the following conditions hold:

(i) *X satisfies condition ($\star$),*

(ii) *(X, d) is α-complete metric space,*

(iii) *α is transitive.*

Proof. Follows from the proof of Theorem 2.2, we can construct the sequence $\{x_n\}$ in X such that $x_n \to u$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. By condition ($\star$), we have $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N}$.
From α-partial weakly Zamfirescu contractive condition, we get

$$\begin{aligned} d(Tx_n, Tu) &\leq \gamma(x_n, u)M_T(x_n, u) \\ &\leq M_T(x_n, u) \end{aligned}$$

for all $n \in \mathbb{N}$ Now, we obtain that

$$\begin{aligned} d(u, Tu) &= \lim_{n\to\infty} d(x_{n+1}, Tu) \\ &= \lim_{n\to\infty} d(Tx_n, Tu) \\ &\leq \lim_{n\to\infty} M_T(x_n, u) \\ &= \lim_{n\to\infty} \max\Big\{d(x_n, u), \frac{1}{2}[d(x_n, Tx_n) + d(u, Tu)], \frac{1}{2}[d(x_n, Tu) + d(u, Tx_n)]\Big\} \\ &= \lim_{n\to\infty} \max\Big\{d(x_n, u), \frac{1}{2}[d(x_n, x_{n+1}) + d(u, Tu)], \frac{1}{2}[d(x_n, Tu) + d(u, x_{n+1})]\Big\} \\ &= \frac{1}{2}d(u, Tu). \end{aligned}$$

This implies that $d(u, Tu) = 0$, that is $Tu = u$ and then T has a fixed point. This completes the proof. □

Corollary 2.5. *Let (X,d) be a metric space, $\alpha : X \times X \to [0,\infty)$ be a mapping and T be an α-weakly Zamfirescu selfmap on X. Suppose that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has the approximate fixed point property.*

Moreover, T has a fixed point provides that the following conditions hold:

(i) *T is α-continuous on X (or X satisfies condition $(\star)$),*

(ii) *(X,d) is an α-complete metric space,*

(iii) *α is transitive.*

Corollary 2.6. *Let (X,d) be a metric space, $\alpha : X \times X \to [0,\infty)$ be a mapping and $T : X \to X$ satisfies the following condition:*

$$[d(Tx,Ty) + \lambda]^{\alpha(x,y)} \leq \gamma(x,y) M_T(x,y) + \lambda \quad \text{for all } x,y \in X, \tag{10}$$

where $\lambda > 1$, $M_T(x,y) = \max\left\{d(x,y), \frac{1}{2}[d(x,Tx) + d(y,Ty)], \frac{1}{2}[d(x,Ty) + d(y,Tx)]\right\}$ and $\gamma : X \times X \to [0,1]$ is mapping with

$$\theta(a,b) := \sup\{\gamma(x,y) : a \leq d(x,y) \leq b\} < 1 \quad \text{for all } 0 < a \leq b.$$

Suppose that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has the approximate fixed point property.

Moreover, T has a fixed point provides that the following conditions hold:

(i) *T is α-continuous on X (or X satisfies condition $(\star)$),*

(ii) *(X,d) is α-complete metric space,*

(iii) *α is transitive.*

Corollary 2.7. *Let (X,d) be a metric space, $\alpha : X \times X \to [0,\infty)$ be a mapping and $T : X \to X$ satisfies the following condition:*

$$[\lambda - 1 + \alpha(x,y)]^{d(Tx,Ty)} \leq \lambda^{\gamma(x,y) M_T(x,y)} \quad \text{for all } x,y \in X, \tag{11}$$

where $\lambda > 1$, $M_T(x,y) = \max\left\{d(x,y), \frac{1}{2}[d(x,Tx) + d(y,Ty)], \frac{1}{2}[d(x,Ty) + d(y,Tx)]\right\}$ and $\gamma : X \times X \to [0,1]$ is mapping with

$$\theta(a,b) := \sup\{\gamma(x,y) : a \leq d(x,y) \leq b\} < 1 \quad \text{for all } 0 < a \leq b.$$

Suppose that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has the approximate fixed point property.

Moreover, T has a fixed point provides that the following conditions hold:

(i) *T is α-continuous on X (or X satisfies condition $(\star)$),*

(ii) *(X,d) is an α-complete metric space,*

(iii) *α is transitive.*

Remark 2.8. *We can see that α-weakly Zamfirescu contractive condition and weakly Zamfirescu contractive condition are special cases of α-partial weakly Zamfirescu contractive condition. Moreover, concepts of α-completeness and α-continuity are weaker that concepts of completeness and continuity. Therefore, Theorem 3.3 of Miandaragh et al. [13] and Proposition 26 and Theorem 28 of Ariza-Ruiza et al. [4] are consequently of Theorem 2.2 and Theorem 2.4.*

2.2. *Approximate fixed point theorems in metric spaces endowed with an arbitrary binary relations*

In this subsection, we present approximate fixed point theorems in metric spaces endowed with an arbitrary binary relations. The following notions and definitions are needed.

Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote

$$\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1}.$$

Clearly,

$$x, y \in X, \quad x\mathcal{S}y \quad \Longleftrightarrow \quad x\mathcal{R}y \quad \text{or} \quad y\mathcal{R}x.$$

It is easy to see that $\mathcal{S}$ is the symmetric relation attached to $\mathcal{R}$.

Definition 2.9. *Let T be a self mapping on a nonempty set X and $\mathcal{R}$ be a binary relation over X. We say that T is* comparative mapping *if*

$$x, y \in X, \quad x\mathcal{S}y \Longrightarrow (Tx)\mathcal{S}(Ty).$$

Definition 2.10. *Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. The metric space X is said to be* $\mathcal{S}$-complete *if and only if every Cauchy sequence $\{x_n\}$ in X with $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N}$, converges in X.*

Definition 2.11. *Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. We say that $T : X \to X$ is an* $\mathcal{S}$-continuous mapping *on (X, d) if for each sequence $\{x_n\}$ in X, we have*

$$x_n \to x \text{ as } n \to \infty \text{ for some } x \in X \text{ and } x_n\mathcal{S}x_{n+1} \text{ for all } n \in \mathbb{N} \quad \Longrightarrow \quad Tx_n \to Tx \text{ as } n \to \infty.$$

Definition 2.12. *Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. The space X has the* $\mathcal{S}$-transitive *property if the following condition holds:*

$$x, y, z \in X \text{ with } x\mathcal{S}y \text{ and } y\mathcal{S}z \Longrightarrow x\mathcal{S}z.$$

Definition 2.13. *Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. The mapping $T : X \to X$ is called a* partial weakly Zamfirescu mapping with respect to $\mathcal{S}$ *if there exists a mapping $\gamma : X \times X \to [0, 1]$ with*

$$\theta(a, b) := \sup\{\gamma(x, y) : a \leq d(x, y) \leq b\} < 1 \quad \text{for all } 0 < a \leq b,$$

and it satisfies the following condition:

$$\text{for all } x, y \in X, \quad x\mathcal{S}y \quad \Longrightarrow \quad d(Tx, Ty) \leq \gamma(x, y)M_T(x, y), \tag{12}$$

where $M_T(x, y) = \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}$.

Theorem 2.14. *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to X$ be a partial weakly Zamfirescu mapping with respect to $\mathcal{S}$. If T is comparative mapping and there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$, then T has the approximate fixed point property.*

Moreover, T has a fixed point provides that the following conditions hold:

(i) *T is $\mathcal{S}$-continuous,*

(ii) *(X, d) is $\mathcal{S}$-complete metric space,*

(iii) *X has the $\mathcal{S}$-transitive property.*

Proof. Consider a mapping $\alpha : X \times X \to [0,\infty)$ defined by

$$\alpha(x,y) = \begin{cases} 1, & x\mathcal{S}y, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

From that there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$, we get $\alpha(x_0, Tx_0) = 1$. It follows from T is comparative mapping that T is α-admissible mapping. Since T is a partial weakly Zamfirescu mapping with respect to $\mathcal{S}$, we have, for all $x, y \in X$, we get

$$x\mathcal{S}y \implies d(Tx,Ty) \le \gamma(x,y)M_T(x,y), \tag{14}$$

and thus

$$\alpha(x,y) \ge 1 \implies d(Tx,Ty) \le \gamma(x,y)M_T(x,y).$$

This implies that T is an α-partial weakly Zamfirescu mapping. Now all the hypotheses of Theorem 2.2 are satisfied. So T has the approximate fixed point property. Furthermore, $\mathcal{S}$-continuity of T, the $\mathcal{S}$-completeness of X and the $\mathcal{S}$-transitivity of X yield the existence of fixed point of T. This completes the proof. □

Definition 2.15. *Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a mapping. We say that X* satisfies condition $(\star_{\mathcal{S}})$ *if $\{x_n\}$ is sequence in X such that $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N}$, then $x_n\mathcal{S}x$ for all $n \in \mathbb{N}$.*

Theorem 2.16. *Let (X,d) be a metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to X$ be a partial weakly Zamfirescu mapping with respect to $\mathcal{S}$. If T is comparative mapping and there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$, then T has the approximate fixed point property.*

Moreover, T has a fixed point provides that the following conditions hold:

(i) *X satisfies condition $(\star_{\mathcal{S}})$,*

(ii) *(X,d) is $\mathcal{S}$-complete metric space,*

(iii) *X has the $\mathcal{S}$-transitive property.*

Proof. The result follows from Theorem 2.4 by considering the mappings α given by (13) and by observing that condition $(\star_{\mathcal{S}})$ implies condition $(\star)$. □

2.3. *Approximate fixed point in metric space endowed with graph*

Throughout this section, let (X,d) be a metric space. A set $\{(x,x) : x \in X\}$ is called a diagonal of the Cartesian product $X \times X$ and is denoted by Δ. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

In this subsection, we give the existence of approximate fixed point theorems on a metric space endowed with graph. Before presenting our results, we give the following notions and definitions.

Definition 2.17. *Let X be a nonempty set endowed with a graph G. We say that $T : X \to X$* preserve edge *if*

$$\text{for } x,y \in X, \quad (x,y) \in E(G) \implies (Tx,Ty) \in E(G).$$

Definition 2.18. *Let (X,d) be a metric space endowed with a graph G. The metric space X is said to be* $E(G)$-complete *if and only if every Cauchy sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, converges in X.*

Definition 2.19. *Let (X,d) be a metric space endowed with a graph G and $T : X \to X$ be a mapping. We say that T is an* $E(G)$-continuous mapping *on (X,d) if for each sequence $\{x_n\}$ in X with*

$$x_n \to x \text{ as } n \to \infty \text{ for some } x \in X \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N} \quad \Longrightarrow \quad Tx_n \to Tx \text{ as } n \to \infty.$$

Definition 2.20. *Let (X, d) be a metric space endowed with a graph G. The space X has the $E(G)$-*transitive *property if the following condition holds:*

$$x, y, z \in X \text{ with } (x, y) \in E(G) \text{ and } (y, z) \in E(G) \Longrightarrow (x, z) \in E(G).$$

Remark 2.21. *It is easy to see that if G is a connected graph, then X has a $E(G)$-transitive property.*

Definition 2.22. *Let (X, d) be a metric space endowed with a graph G. The mapping $T : X \to X$ is called a* partial weakly Zamfirescu mapping with respect to $E(G)$ *if there exists a mapping $\gamma : X \times X \to [0, 1]$ with*

$$\text{for all } 0 < a \leq b, \quad \theta(a, b) := \sup\{\gamma(x, y) : a \leq d(x, y) \leq b\} < 1,$$

and it satisfies the following condition:

$$\text{for all } x, y \in X \quad (x, y) \in E(G) \quad \Longrightarrow \quad d(Tx, Ty) \leq \gamma(x, y) M_T(x, y), \tag{15}$$

where $M_T(x, y) = \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}$.

Theorem 2.23. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a partial weakly Zamfirescu mapping with respect to $E(G)$. If T preserve edge and there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then T has the approximate fixed point property.*

Moreover, T has a fixed point provides that the following conditions hold:

(i) *T is $E(G)$-continuous,*

(ii) *(X, d) is $E(G)$-complete metric space,*

(iii) *X has $E(G)$-transitive property.*

Proof. Consider a mapping $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in E(G), \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

From that there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, we get $\alpha(x_0, Tx_0) = 1$. It follows from T preserve edge that T is α-admissible mapping. Since T is a partial weakly Zamfirescu mapping with respect to $E(G)$, we have, for all $x, y \in X$, we obtain that

$$(x, y) \in E(G) \quad \Longrightarrow \quad d(Tx, Ty) \leq \gamma(x, y) M_T(x, y) \tag{17}$$

that is,

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad d(Tx, Ty) \leq \gamma(x, y) M_T(x, y). \tag{18}$$

This implies that T is a α-partial weakly Zamfirescu selfmap on X. Now all the hypotheses of Theorem 2.2 are satisfied. So T has the approximate fixed point property. Furthermore, $E(G)$-continuity of T, the $E(G)$-completeness of X and $E(G)$-transitivity of X yield the existence of fixed point of T. This completes the proof. □

Definition 2.24. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. We say that X* satisfies condition $(\star_{\mathcal{E}})$ *if $\{x_n\}$ is sequence in X such that $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Theorem 2.25. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a partial weakly Zamfirescu mapping with respect to $E(G)$. If T is preserve edge and there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then T has the approximate fixed point property.*

Moreover, T has a fixed point provides that the following conditions hold:

(i) *X satisfies condition $(\star_{\mathcal{E}})$,*

(ii) *(X, d) is $E(G)$-complete metric space,*

(iii) *X has $E(G)$-transitive property.*

Proof. The result follows from Theorem 2.4 by considering the mappings α given by (16) and by observing that condition $(\star_{\mathcal{E}})$ implies condition $(\star)$. □

From Remark 2.21, we get the following result on connected graph.

Corollary 2.26. *Let (X, d) be a metric space endowed with a graph G and $T : X \to X$ be a partial weakly Zamfirescu mapping with respect to $E(G)$. If T is preserve edge and there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, then T has the approximate fixed point property.*
Moreover, T has a fixed point provides that the following conditions hold:

(i) *T is $E(G)$-continuous (or X satisfies condition $(\star_{\mathcal{E}})$),*

(ii) *(X, d) is $E(G)$-complete metric space,*

(iii) *G is connected graph.*

3. Homotopy Invariance

In this section, we will study the homotopy results for α-partial weakly Zamfirescu mappings. By using Theorem 2.2, we obtain the following local results, which will be used in the homotopy result (Theorem 3.2).

Theorem 3.1. *Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ be a mapping, $x_0 \in X$, $r > 0$ and $\overline{B(x_0, r)}$ is α-complete. Suppose that the following condition holds:*

1. *$T : \overline{B(x_0, r)} \to X$ is α-continuous on $\overline{B(x_0, r)}$ (or $\overline{B(x_0, r)}$satisfies condition $(\star)$),*
2. *there exists a mapping $\gamma : \overline{B(x_0, r)} \times \overline{B(x_0, r)} \to [0, 1]$ with*
$$\text{for all } 0 < a \leq b, \quad \theta(a, b) := \sup\{\gamma(x, y) : a \leq d(x, y) \leq b\} < 1,$$
and it satisfies the following condition:
$$\text{for all } x, y \in \overline{B(x_0, r)} \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad d(Tx, Ty) \leq \gamma(x, y) M_T(x, y), \tag{19}$$
where $M_T(x, y) = \max\left\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}$,
3. *α has transitive property.*

If $\alpha(x_0, x) \geq 1$ for all $x \in \overline{B(x_0, r)}$ and
$$d(x_0, Tx_0) < \frac{1}{3} \min\left\{\frac{r}{2}, r\left[1 - \theta\left(\frac{r}{2}, r\right)\right]\right\},$$
then T has a fixed point.

Proof. By using Theorem 2.2 (or Theorem 2.4), it suffices to show that the closed ball $\overline{B(x_0, r)}$ is invariant under T. Consider any $x \in \overline{B(x_0, r)}$, and obtain the relation
$$d(x_0, Tx) \leq d(x_0, Tx_0) + d(Tx_0, Tx).$$
Since $\alpha(x_0, x) \geq 1$ for all $x \in \overline{B(x_0, r)}$, we obtain that
$$d(x_0, Tx) \leq d(x_0, Tx_0) + \gamma(x_0, x) M_T(x_0, x),$$

where

$$M_T(x_0, x) = \max\left\{d(x_0, x), \frac{1}{2}[d(x_0, Tx_0) + d(x, Tx)], \frac{1}{2}[d(x_0, Tx) + d(x, Tx_0)]\right\}.$$

We now consider three cases.

Case 1: If $M_T(x_0, x) = d(x_0, x)$, then we have

$$d(x_0, Tx) \le d(x_0, Tx_0) + \gamma(x_0, x)d(x_0, x).$$

Case 2: If $M_T(x_0, x) = \frac{1}{2}[d(x_0, Tx_0) + d(x, Tx)]$, then we have

$$\begin{aligned} d(x_0, Tx) &\le d(x_0, Tx_0) + \frac{\gamma(x_0, x)}{2}[d(x_0, Tx_0) + d(x, Tx)] \\ &\le d(x_0, Tx_0) + \frac{\gamma(x_0, x)}{2}[d(x_0, Tx_0) + d(x, x_0) + d(x_0, Tx)], \end{aligned}$$

from which, having in mind that $\gamma(x_0, x) \le 1$,

$$d(x_0, Tx) \le 3d(x_0, Tx_0) + \gamma(x_0, x)d(x_0, x).$$

Case 3: If $M_T(x_0, x) = \frac{1}{2}[d(x_0, Tx) + d(x, Tx_0)]$, then we obtain that

$$\begin{aligned} d(x_0, Tx) &\le d(x_0, Tx_0) + \frac{\gamma(x_0, x)}{2}[d(x_0, Tx) + d(x, Tx_0)] \\ &\le d(x_0, Tx_0) + \frac{\gamma(x_0, x)}{2}[d(x_0, Tx) + d(x, x_0) + d(x_0, Tx_0)], \end{aligned}$$

form which, having in mind that $\gamma(x_0, x) \le 1$,

$$d(x_0, Tx) \le 3d(x_0, Tx_0) + \gamma(x_0, x)d(x_0, x).$$

Therefore, in any case, we get

$$d(x_0, Tx) \le 3d(x_0, Tx_0) + \gamma(x_0, x)d(x_0, x).$$

To end the proof, obtain that $d(x_0, Tx) \le r$ through the above inequality by considering two cases. If $d(x_0, x) \le \frac{r}{2}$, then $d(x_0, Tx) \le r$. Otherwise, we have $\frac{r}{2} \le d(x_0, x) \le r$ and hence $\gamma(x_0, x) \le \theta(\frac{r}{2}, r)$. Consequently, we get

$$d(x_0, Tx) \le r\left[1 - \theta\left(\frac{r}{2}, r\right)\right] + r\theta\left(\frac{r}{2}, r\right) = r.$$

This completes the proof. □

Theorem 3.2. *Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ be a mapping, U be a bounded open subset of X such that $\overline{U}$ is α-complete, and $H : \overline{U} \times [0, 1] \to X$ satisfying the following properties:*

(P1) $H(x, \lambda) \ne x$ for all $x \in \partial U$ and $\lambda \in [0, 1]$;

(P2) there exists $\gamma : \overline{U} \times \overline{U} \to [0, 1]$ satisfying

$$\theta(a, b) := \sup\{\gamma(x, y) : a \le d(x, y) \le b\} < 1 \quad \textit{for all } 0 < a \le b,$$

such that, for all $x, y \in \overline{U}$ and $\lambda \in [0, 1]$, we have the following condition holds:

$$\alpha(x, y) \ge 1 \implies d(H(x, \lambda), H(y, \lambda)) \le \gamma(x, y)M_H^\lambda(x, y), \tag{20}$$

where

$$M_H^\lambda(x, y) := \max\left\{d(x, y), \frac{1}{2}[d(x, H(x, \lambda)) + d(y, H(y, \lambda))], \frac{1}{2}[d(x, H(y, \lambda)) + d(y, H(x, \lambda))]\right\};$$

(P3) *$H(x,\lambda)$ is continuous in λ, uniformly for $x \in \overline{U}$, that is, for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$d(H(x,t), H(x,s)) \leq \epsilon$$

for all $x \in \overline{U}$ and $t, s \in [0,1]$ with $|t-s| < \delta$, where δ is independent of x.

(P4) *If $x \in \overline{U}$ such that $H(x,\lambda) = x$ for some $\lambda \in [0,1]$, then $\alpha(x,y) \geq 1$ for all $y \in \overline{U}$.*

(P5) *$H(\cdot,\lambda) : \overline{U} \to X$ is α-continuous, where $\lambda \in [0,1]$.*

(P6) *$\overline{U}$ satisfies condition $(\star)$).*

(P7) *α is transitive.*

If $H(\cdot,0)$ has a fixed point in U, then $H(\cdot,\lambda)$ also has a fixed point in U for all $\lambda \in [0,1]$.

Proof. Consider the nonempty set

$$A = \{\lambda \in [0,1] : H(x,\lambda) = x \quad \text{for some } x \in U\}.$$

We just need to prove that $A = [0,1]$, and for this it suffices to show that A is both closed and open in $[0,1]$.

We first prove that A is closed in $[0,1]$. Suppose that $\{\lambda_n\}$ is a sequence in A converging to $\lambda \in [0,1]$, and let us show that $\lambda \in A$. By definition of A, there exists a sequence $\{x_n\}$ in U with $x_n = H(x_n, \lambda_n)$ for all $n \in \mathbb{N}$. We shall prove that $\{x_n\}$ converges to a point $x_0 \in U$ with $H(x_0,\lambda) = x_0$and thus $\lambda \in A$.

In the first place, we shall prove that, for all $n, m \in \mathbb{N}$,

$$d(x_n,x_m) \leq \gamma(x_n,x_m)d(x_n,x_m) + \left(1 + \frac{\gamma(x_n,x_m)}{2}\right)[d(H(x_m,\lambda_m), H(x_m,\lambda)) + d(H(x_n,\lambda_n)H(x_n,\lambda))]. \tag{21}$$

From (P4), we get $\alpha(x_n,x_m) \geq 1$ for all $n,m \in \mathbb{N}$. To prove (21), observe that, if $n,m \in \mathbb{N}$, we get

$$\begin{aligned} d(x_n,x_m) &= d(H(x_n,\lambda_n), H(x_m,\lambda_m)) \\ &\leq d(H(x_n,\lambda_n), H(x_n,\lambda)) + d(H(x_n,\lambda), H(x_m,\lambda)) + d(H(x_m,\lambda), H(x_m,\lambda_m)) \\ &\leq d(H(x_n,\lambda_n), H(x_n,\lambda)) + \gamma(x_n,x_m)M_H^{\lambda}(x_n,x_m) + d(H(x_m,\lambda), H(x_m,\lambda_m)), \end{aligned}$$

where

$$M_H^{\lambda}(x_n,x_m) = \max\left\{d(x_n,x_m), \frac{1}{2}[d(x_n,H(x_n,\lambda)) + d(x_m,H(x_m,\lambda))], \frac{1}{2}[d(x_n,H(x_m,\lambda)) + d(x_m,H(x_n,\lambda))]\right\}.$$

To continue with the above chain of inequalities, just consider the following three possibilities for $M_H^{\lambda}(x_n,x_m)$.

Case 1: If $M_H^{\lambda}(x_n,x_m) = d(x_n,x_m)$, then (21) is obvious.

Case 2: If $M_H^{\lambda}(x_n,x_m) = \frac{1}{2}[d(x_n,H(x_n,\lambda)) + d(x_m,H(x_m,\lambda))]$, then we have

$$\begin{aligned} d(x_n.x_m) &\leq d(H(x_n,\lambda_n), H(x_n,\lambda)) + \frac{\gamma(x_n,x_m)}{2}[d(x_n,H(x_n,\lambda)) + d(x_m,H(x_m,\lambda))] + d(H(x_m,\lambda), H(x_m,\lambda_m)) \\ &= \left(1 + \frac{\gamma(x_n,x_m)}{2}\right)[d(H(x_m,\lambda_m), H(x_m,\lambda)) + d(H(x_n,\lambda_n), H(x_n,\lambda))]. \end{aligned}$$

Case 3: If $M_H^{\lambda}(x_n,x_m) = \frac{1}{2}[d(x_n,H(x_m,\lambda)) + d(x_m,H(x_n,\lambda))]$, then we get

$$\begin{aligned} d(x_n,x_m) &\leq d(H(x_n,\lambda_n), H(x_n,\lambda)) + \frac{\gamma(x_n.x_m)}{2}[d(x_n,H(x_m,\lambda)) + d(x_m,H(x_n,\lambda))] + d(H(x_m,\lambda), H(x_m,\lambda_m)) \\ &\leq d(H(x_n,\lambda_n), H(x_n,\lambda)) + \frac{\gamma(x_n,x_m)}{2}[2d(x_n,x_m) + d(H(x_m,\lambda_m), H(x_m,\lambda)) + d(H(x_n,\lambda_n), H(x_n,\lambda))] \\ &\quad + d(H(x_m,\lambda), H(x_m,\lambda_m)) \\ &= \gamma(x_n,x_m)d(x_n,x_m) + \left(1 + \frac{\gamma(x_n,x_m)}{2}\right)[d(H(x_m,\lambda_m), H(x_m,\lambda)) + d(H(x_n,\lambda_n), H(x_n,\lambda))]. \end{aligned}$$

Hence, (21) is proved.

Next we prove that $\{x_n\}$ is a Cauchy sequence. Otherwise, there exist a positive constant δ and two subsequences of $\{x_n\}$, $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that $d(x_{x_k}, x_{m_k}) \geq \delta$ for all $k \in \mathbb{N}$. Now let $M :=$ diam U. Consequently, we have $\gamma(x_{n_k}, x_{m_k}) \leq \theta(\delta, M)$ and then (21) leads to

$$d(x_{n_k}, x_{m_k}) \leq \theta(\delta, M) d(x_{n_k}, x_{m_k}) + \left(1 + \frac{\theta(\delta, M)}{2}\right)[d(H(x_{m_k}, \lambda_{m_k}), H(x_{m_k}, \lambda)) + d(H(x_{n_k}, \lambda_{n_k}), H(x_{n_k}, \lambda))]$$

and so

$$\delta \leq d(x_{n_k}, x_{m_k}) \leq \frac{2 + \theta(\delta, M)}{2(1 - \theta(\delta, M))}[d(H(x_{m_k}, \lambda_{m_k}), H(x_{m_k}, \lambda)) + d(H(x_{n_k}, \lambda_{n_k}), H(x_{n_k}, \lambda))]. \quad (22)$$

Since, by (P3), $d(H(x_{m_k}, \lambda_{m_k}), H(x_{m_k}, \lambda)) \to 0$ as $k \to \infty$, we reach a contradiction from (22). Hence, $\{x_n\}$ is a Cauchy sequence. Write $x_0 = \lim_{n\to\infty} x_n$ and let us see that $x_0 \in U$ and also that $x_0 = H(x_0, \lambda)$. That $x_0 = H(x_0, \lambda)$ is a consequence of the following relation:

$$\begin{aligned} d(x_n, H(x_0, \lambda)) &\leq d(x_n, H(x_n, \lambda)) + d(H(x_n, \lambda), H(x_0, \lambda)) \\ &\leq d(H(x_n, \lambda_n), H(x_n, \lambda)) + \max\left\{d(x_n, x_0), \frac{1}{2}[d(x_0, H(x_0, \lambda)) + d(x_n, H(x_n, \lambda))],\right. \\ &\quad \left.\frac{1}{2}[d(x_0, H(x_n, \lambda)) + d(x_0, H(x_0, \lambda))]\right\} \\ &\leq d(H(x_n, \lambda_n), H(x_n, \lambda)) + \max\left\{d(x_n, x_0), \frac{1}{2}[d(x_0, H(x_0, \lambda)) + d(x_n, H(x_n, \lambda))],\right. \\ &\quad \left.\frac{1}{2}[d(x_0, x_n) + d(H(x_n, \lambda_n), H(x_n, \lambda)) + d(x_n, H(x_0, \lambda))]\right\}, \end{aligned}$$

and that $x_0 \in U$ is straightforward from (P1).

We now turn to prove that A is open $[0, 1]$. Suppose that $\lambda_0 \in A$ and let us show that $(\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1] \subset A$, for some $\delta > 0$. Since $\lambda_0 \in A$, there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Consider $r > 0$ with $\overline{B(x_0, r)} \subset U$, and use (P3) to obtain $\delta > 0$ such that

$$d(H(x_0, \lambda_0), H(x_0, \lambda)) < \frac{1}{3}\min\left\{\frac{r}{2}, r\left[1 - \theta\left(\frac{r}{2}, r\right)\right]\right\}$$

for all $\lambda \in (\lambda_0 - \lambda, \lambda_0 + \delta) \cap [0, 1]$. This implies that

$$\begin{aligned} d(x_0, H(x_0, \lambda)) &= d(H(x_0, \lambda_0), H(x_0, \lambda)) \\ &< \frac{1}{3}\min\left\{\frac{r}{2}, r\left[1 - \theta\left(\frac{r}{2}, r\right)\right]\right\}. \end{aligned}$$

for all $\lambda \in (\lambda_0 - \lambda, \lambda_0 + \delta) \cap [0, 1]$. By using Theorem 3.1, we get the mapping $H(\cdot, \lambda) : \overline{B(x_0, r)} \to X$ has a fixed point. Therefore, for any $\lambda \in (\lambda_0 - \lambda, \lambda_0 + \delta) \cap [0, 1]$ is also in A. This means that A is a open in $[0, 1]$.

Consequently, we have A is both closed and open in $[0, 1]$ and hence $A = [0, 1]$. This completes the proof. □

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