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Existence and Multiplicity of Solutions for Nonlinear Elliptic Equations of p-Laplace Type in $\mathbb{R}^N$

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Abstract. In this paper, we discuss the following elliptic equation:

$$-\mathrm{div}(\varphi(x, \nabla u)) = \lambda f(x, u) \quad \text{in} \quad \mathbb{R}^N,$$

where the function $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is of type $|v|^{p-2} v$ with a real constant $p > 1$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition.

1. Introduction

The differential equations involving p-Laplacian have been interested since they arise in various contexts of physical phenomena, for instance, in the study of non-Newtonian fluids. The quantity p is a characteristic of the medium. Media with $p = 2$ (respectively, $p < 2$, $p > 2$) are called Newtonian (respectively, pseudoplastic, dilatant). The p-Laplacian also appears in the search of flow through porous media ($p = 3/2$), nonlinear elasticity ($p \geq 2$) and glaciology ($p \in (1, 4/3]$). Other applications of such problems are to obtaining soliton-like solutions of Lorentz invariant nonlinear field equations. We refer to [2, 5, 8, 9, 14, 15] for details and further references therein.

In this paper, we establish the existence and multiplicity results of nontrivial weak solutions to nonlinear elliptic equations of the p-Laplace type

$$-\mathrm{div}(\varphi(x, \nabla u)) = \lambda f(x, u) \quad \text{in} \quad \mathbb{R}^N, \tag{B}$$

where the function $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is of type $|v|^{p-2} v$ with a real constant $p > 1$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. The p-Laplace type operator $\mathrm{div}(\varphi(x, \nabla u))$ is the more generalized form of the p-Laplacian $\mathrm{div}(|\nabla u|^{p-2} \nabla u)$.

As considering an abstract critical point theory which is one of the crucial tools for finding solutions to elliptic equations of variational type, Ambrosetti and Rabinowitz [1] investigated the existence of solutions of the second order uniformly elliptic equations

$$-\mathrm{div}(a(x)\nabla u) + b(x)u = f(x, u) \ \text{ in } \ \Omega, \tag{1}$$

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subject to Dirichlet boundary condition where f is odd with respect to u and Ω is a smooth bounded domain in $\mathbb{R}^N$. Moreover, they proved that the equations above had infinitely many distinct pairs of solutions under suitable conditions. When Ω is the whole space $\mathbb{R}^N$, the existence of a positive solution, and under some conditions, infinitely many solutions of the semilinear elliptic equations (1) are observed by Bartsch and Wang [3]. Yu [18] showed the existence of solutions for the p-Laplacian problem

$$\begin{cases} -\operatorname{div}(a(x)\,|\nabla u|^{p-2}\,\nabla u) + b(x)\,|u|^{p-2}\,u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \lim_{|x|\to\infty} u = 0, \end{cases}$$

where $1 < p < N$ and Ω is an exterior domain. De Nápoli and Mariani [14] established the existence and multiplicity results of nontrivial weak solutions for problem of the p-Laplace type with Dirichlet boundary condition. In order to apply the Mountain pass theorem they assumed that the functional Φ which was induced by φ was uniform convex, i.e., there exists a positive constant k such that

$$\Phi\left(x, \frac{\xi+\eta}{2}\right) \le \frac{1}{2}\Phi(x,\xi) + \frac{1}{2}\Phi(x,\eta) - k\left|\xi-\eta\right|^p$$

for all $x \in \overline{\Omega}$ and $\xi, \eta \in \mathbb{R}^N$, where Ω is a bounded domain in $\mathbb{R}^N$. But, for $1 < p < 2$, it is well known that this condition cannot be applicable for the p-Laplacian problems. For instance, the functional $\Phi(x,t) = (1/p)t^p$ is not uniformly convex for $t > 0$ and $1 < p < 2$.

The first goal of this paper is to prove the existence of at least one nontrivial weak solution for problem (B) in the weighted Sobolev spaces, without the assumption about uniform convexity of the functional Φ. Moreover, we shall verify that there exist two different sequences of critical points using the variational method, namely, Fountain theorem (Theorem 3.6 in [17]). However, it is not easy to apply Fountain theorem to gain infinitely many solutions for nonlinear elliptic equations involving p-Laplace type on unbounded domain. In this view, we give a specific proof of the existence of the infinitely many solutions for problem (B) by using Fountain theorem.

The second purpose of this paper is to deduce the existence of two distinct nontrivial weak solutions for problem (B) without assuming Ambrosetti-Rabinowitz condition (see [1]) which was inspired by [7]. But, without this condition, we can not ensure the boundedness of the Palais-Smale sequence of the Euler-Lagrange functional which is crucial to apply critical point theory. To overcome this difficulty, we show the coercivity of the functional corresponding to our problem under appropriate hypotheses, and then we employ a result from Theorem 1.2 in Struwe [16] in order to obtain a critical point of the functional.

This paper is organized as follows. In Section 2, we introduce some definitions of the basic function space which is treated in this paper (see [6, 11, 18]). Under certain conditions on φ and f, we establish the existence of at least one nontrivial solution for problem (B) and infinitely many solutions by applying variational methods. In Section 3, we show there exist at least two distinct nontrivial weak solutions for problem (B) without Ambrosetti-Rabinowitz condition.

2. Existence of Solutions

In this section, we investigate the existence of at least one nontrivial weak solution and infinitely many solutions by employing the Mountain pass theorem and Fountain theorem. Before dealing with our main results, we recall some definitions and properties of the weighted Lebesgue and Sobolev spaces. For a profound treatments on these spaces, we refer to [6, 11, 18]. Let $1 < p < N$ and $p^* := Np/(N-p)$. Assume that

(A) $a \in L^\infty(\mathbb{R}^N)$ and there exists a positive constant a_0 such that

$$a(x) \ge a_0 \qquad \text{for almost all} \qquad x \in \mathbb{R}^N.$$

Recall that

$$w(x) = \frac{1}{(1+|x|)^p} \quad \text{for } x \in \mathbb{R}^N$$

is the weight function which appears in Hardy's inequality. Let $L^p(\mathbb{R}^N, w)$ be the weighted Lebesgue space that consists of all measurable real-valued functions u satisfying

$$\int_{\mathbb{R}^N} w(x)\,|u|^p\,dx < \infty,$$

endowed with the norm

$$\|u\|_{L^p(\mathbb{R}^N,w)} = \left(\int_{\mathbb{R}^N} w(x)\,|u|^p\,dx\right)^{\frac{1}{p}}.$$

Consider the weighted Sobolev space $X := W^{1,p}(\mathbb{R}^N, a, w)$ denoted as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} a(x)\,|\nabla u|^p\,dx + \int_{\mathbb{R}^N} w(x)\,|u|^p\,dx\right)^{\frac{1}{p}}.$$

It follows from Hardy's inequality and the assumption (A) that

$$\int_{\mathbb{R}^N} w(x)\,|u|^p\,dx \le \frac{1}{a_0}\left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} a(x)\,|\nabla u|^p\,dx,$$

which implies that on X, the norm $\|\cdot\|_X$ is equivalent to the other norm $\|\cdot\|_a$ given by

$$\|u\|_a = \left(\int_{\mathbb{R}^N} a(x)\,|\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

Note that there exist positive constants c_0 and c_1 such that

$$c_0\|u\|_X \le \|u\|_a \le c_1\|u\|_X \tag{2}$$

for all $u \in X$. The following *Sobolev inequality* will be used in the sequel:

$$\left(\int_{\mathbb{R}^N} |u|^{p^*}\,dx\right)^{\frac{1}{p^*}} \le d_0 \left(\int_{\mathbb{R}^N} a(x)\,|\nabla u|^p\,dx\right)^{\frac{1}{p}}$$

for some positive constant d_0 (see [6]).

Definition 2.1. *We say that $u \in X$ is a weak solution of problem* (B) *if*

$$\int_{\mathbb{R}^N} \varphi(x, \nabla u)\cdot\nabla v\,dx = \lambda \int_{\mathbb{R}^N} f(x,u)v\,dx$$

for all $v \in X$.

Let $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be the continuous derivative with respect to v of the mapping $\Phi_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $\Phi_0 = \Phi_0(x,v)$, that is, $\varphi(x,v) = \frac{d}{dv}\Phi_0(x,v)$. Assume that φ and Φ_0 satisfy the following conditions:

(J1) The following equalities

$$\Phi_0(x, \mathbf{0}) = 0 \quad \text{and} \quad \Phi_0(x,v) = \Phi_0(x,-v)$$

hold for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

(J2) There are a function $\sigma \in L^{p'}(\mathbb{R}^N)$ and a nonnegative constant b such that

$$\left|\varphi(x, v)\right| \le \sigma(x) + b\,|v|^{p-1}$$

for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

(J3) $\Phi_0(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \mathbb{R}^N$.

(J4) The following relation

$$c_* a(x)\,|v|^p \le \varphi(x, v) \cdot v \le p\Phi_0(x, v)$$

holds for all $x \in \mathbb{R}^N$ and $v \in \mathbb{R}^N$, where c_* is a positive constant.

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\,dx$$

for any $u \in X$. Under the assumptions (A), (J1)–(J2) and (J4), it is easy to check that the functional Φ is well defined on X, by the similar calculations as in [12]. And then we can modify the proof of Lemma 3.2 in [10] to get that $\Phi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is given by

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v\,dx \tag{3}$$

for any $u, v \in X$.

As a key tool in obtaining our main results, we give the operator Φ' is a mapping of type (S_+).

Lemma 2.2. *Assume that* (A) *and* (J1)–(J4) *hold. Then the functional $\Phi : X \to \mathbb{R}$ is convex and weakly lower semicontinuous on X. Moreover, the operator Φ' is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in X as $n \to \infty$ and* $\limsup_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \le 0$, *then $u_n \to u$ in X as $n \to \infty$.*

Proof. The analogous arguments as in Lemma 2.2 in [4] yield the assertion clearly. □

Until now, we have considered some properties for the integral operator corresponding to the divergence part in problem (B). Now we need the assumptions for f to establish our main results in this section. Let us put $F(x, t) = \int_0^t f(x, s)\,ds$. For $1 < p < q < p^*$, we assume that

(F1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition in the sense that $f(\cdot, t)$ is measurable for all $t \in \mathbb{R}$ and $f(x, \cdot)$ is continuous for almost all $x \in \mathbb{R}^N$.

(F2) f satisfies the following growth condition: for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$,

$$\left|f(x, t)\right| \le |m(x)|\,|t|^{q-1},$$

where $m \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\mathrm{meas}\{x \in \mathbb{R}^N : m(x) > 0\} > 0$.

(F3) There exists a positive constant θ such that $\theta > p$ and

$$0 < \theta F(x, t) \le f(x, t)t \ \text{ for all } t \in \mathbb{R} \setminus \{0\} \text{ and } x \in \mathbb{R}^N.$$

(F4) $\lim_{|t|\to 0} \frac{f(x,t)}{w(x)|t|^{p-1}} = 0$ uniformly for all $x \in \mathbb{R}^N$.

Define the functionals $\Psi, I_\lambda : X \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^N} F(x,u)\,dx \ \text{ and } \ I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for any $u \in X$. Then we obtain that $\Psi, I_\lambda \in C^1(X,\mathbb{R})$ and these Fréchet derivatives are

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v\,dx$$

and

$$\langle I_\lambda'(u), v\rangle = \int_{\mathbb{R}^N} \varphi(x,\nabla u)\cdot\nabla v\,dx - \lambda\int_{\mathbb{R}^N} f(x,u)v\,dx$$

for any $u, v \in X$, respectively.

Lemma 2.3. *Assume that* (A) *and* (F1)–(F2) *hold. Then* Ψ *and* Ψ' *are compact operators.*

Proof. The proof is absolutely the same as those of Lemma 4.4 in [6] and is omitted here. □

With the aid of Lemma 2.3, we investigate that the functional I_λ satisfies the Palais-Smale condition, which is denoted by (PS)-condition for short in the sequel. The functional I_λ satisfies the (PS)-condition if and only if each sequence $\{u_n\}$ in X satisfying $I_\lambda(u_n) \to c$ and $I_\lambda'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence for $c \in \mathbb{R}$. This fact plays an important role in finding at least one nontrivial weak solution for problem (B). The basic idea of the following assertions is derived by Lemmas 3.1 and 3.2 in [14]. But we give a specific proof because we deal with the case of unbounded domain and the basic function spaces which are treated in this paper are different from those in [14].

Lemma 2.4. *Assume that* (A), (J1)–(J4) *and* (F1)–(F3) *hold. Then* I_λ *satisfies the* (PS)*-condition for all* $\lambda > 0$.

Proof. Since the operator Ψ' is compact, Ψ' is a mapping of type (S_+). Let $\{u_n\}$ be a (PS)-sequence in X, i.e., $I_\lambda(u_n) \to c$ and $I_\lambda'(u_n) \to 0$ as $n \to \infty$. Because I_λ' is of type (S_+) and X is reflexive, it is enough to verify that the sequence $\{u_n\}$ is bounded in X. Suppose to the contrary that $\|u_n\|_X \to \infty$, in the subsequence sense. By the assumption (J4), we deduce that

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{\theta}\langle I_\lambda'(u_n), u_n\rangle &= \int_{\mathbb{R}^N}\Big(\Phi_0(x,\nabla u_n) - \frac{1}{\theta}\varphi(x,\nabla u_n)\cdot\nabla u_n\Big)dx \\ &\qquad + \lambda\int_{\mathbb{R}^N}\Big(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\Big)dx \\ &\geq \Big(1-\frac{p}{\theta}\Big)\int_{\mathbb{R}^N}\Phi_0(x,\nabla u_n)\,dx + \lambda\int_{\mathbb{R}^N}\Big(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\Big)dx, \end{aligned}$$

where θ is the positive constant from (F3). It follows that

$$\Big(1-\frac{p}{\theta}\Big)\int_{\mathbb{R}^N}\Phi_0(x,\nabla u_n)\,dx \leq I_\lambda(u_n) - \frac{1}{\theta}\langle I_\lambda'(u_n), u_n\rangle$$

for n large enough. Therefore, we get

$$c_0^p\Big(1-\frac{p}{\theta}\Big)\frac{c_*}{p}\|u_n\|_X^p \leq I_\lambda(u_n) + \frac{1}{\theta}\|I_\lambda'(u_n)\|_{X^*}\|u_n\|_X$$

for the positive constant c_0 which appears in the relation (2). Since $1 < p < \theta$, this is a contradiction. □

As our first main result, we establish the following consequence for the existence of a nontrivial weak solution for problem (B) by employing the Mountain pass theorem and Lemma 2.4.

Theorem 2.5. *Assume that* (A), (J1)–(J4) *and* (F1)–(F4) *hold. Then problem* (B) *has a nontrivial weak solution for all* $\lambda > 0$.

Proof. Note that $I_\lambda(0) = 0$. Since I_λ satisfies the (PS)-condition by Lemma 2.4, it suffices to show the geometric conditions in the Mountain pass theorem, i.e.,

(1) there is a positive constant R such that

$$\inf_{\|u\|_X = R} I_\lambda(u) > 0;$$

(2) there exists an element v in X satisfying

$$I_\lambda(tv) \to -\infty \quad \text{as} \quad t \to \infty.$$

First, we prove the condition (1). By Hardy's inequality, there exists a positive constant C^* such that $\|u\|_{L^p(\mathbb{R}^N,w)} \leq C^*\|u\|_X$. Let $\varepsilon > 0$ be small enough such that $\lambda\varepsilon C^* \leq c_0^p c_*/(2p)$ for the positive constant c_* from (J4). By the assumptions (F2) and (F4), for any $\varepsilon > 0$, there exists a positive constant denoted by $C(\varepsilon)$ such that

$$|F(x,t)| \leq \varepsilon w(x)|t|^p + C(\varepsilon)|m(x)||t|^q$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Then it follows from the assumptions (A), (J4) and the Sobolev and Hölder's inequalities that

$$\begin{aligned}
I_\lambda(u) &= \int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx - \lambda \int_{\mathbb{R}^N} F(x,u)\,dx \\
&\geq \int_{\mathbb{R}^N} \frac{c_*}{p} a(x)\,|\nabla u|^p\,dx - \lambda \int_{\mathbb{R}^N} (\varepsilon w(x)\,|u|^p + C(\varepsilon)\,|m(x)|\,|u|^q)\,dx \\
&\geq \frac{c_*}{p}\|u\|_a^p - \lambda\varepsilon C^*\|u\|_X^p - \lambda C(\varepsilon)\|m\|_{L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)}\|u\|_{L^{p^*}(\mathbb{R}^N)}^q \\
&\geq \frac{c_0^p c_*}{p}\|u\|_X^p - \lambda\varepsilon C^*\|u\|_X^p - \lambda C(\varepsilon)C_1\|u\|_X^q
\end{aligned}$$

for a positive constant C_1. Then we deduce that

$$I_\lambda(u) \geq \frac{c_0^p c_*}{2p}\|u\|_X^p - C(\lambda,\varepsilon)C_1\|u\|_X^q.$$

Since $q > p$, there exist $R > 0$ small enough and $\delta > 0$ such that $I_\lambda(u) \geq \delta > 0$ when $\|u\|_X = R$.

Let us show the condition (2). Observe that the assumption (J4) implies that

$$\Phi_0(x,s\xi) \leq s^p\Phi_0(x,\xi) \tag{4}$$

for all $s \geq 1$, $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^N$. Indeed, if we define $g(t) = \Phi_0(x,t\xi)$, then we have

$$g'(t) = \varphi(x,t\xi)\xi = \frac{1}{t}\varphi(x,t\xi)\cdot t\xi \leq \frac{p}{t}\Phi_0(x,t\xi) = \frac{p}{t}g(t).$$

It means that

$$\frac{g'(t)}{g(t)} \leq \frac{p}{t}.$$

By integrating the inequality above over $(1,s)$, we deduce

$$\ln g(s) - \ln g(1) \leq p \ln s$$

and thus we achieve

$$\frac{g(s)}{g(1)} \le s^p,$$

so that the inequality (4) holds. Similarly, we get from the assumption (F3) that

$$F(x, s\eta) \ge s^\theta F(x, \eta) \tag{5}$$

for all $\eta \in \mathbb{R} \setminus \{0\}$, $x \in \mathbb{R}^N$ and $s \ge 1$. Take $v \in X \setminus \{0\}$. Then by the inequalities (4) and (5), we obtain

$$\begin{aligned} I_\lambda(tv) &= \int_{\mathbb{R}^N} \Phi_0(x, t\nabla v)\,dx - \lambda \int_{\mathbb{R}^N} F(x, tv)\,dx \\ &\le t^p \int_{\mathbb{R}^N} \Phi_0(x, \nabla v)\,dx - \lambda t^\theta \int_{\mathbb{R}^N} F(x, v)\,dx, \end{aligned}$$

where $t \ge 1$. Since $1 < p < \theta$, we conclude that $I_\lambda(tv) \to -\infty$ as $t \to \infty$. This completes the proof. □

From now on, adding the oddity on f with respect to t and using Fountain theorem (Theorem 3.6 of [17]), we shall verify the existence of infinitely many pairs of weak solutions for problem (B). To do this, we consider the following Lemma which holds for a reflexive and separable Banach space.

Lemma 2.6. *Let W be a reflexive and separable Banach space. Then there are $\{e_n\} \subseteq W$ and $\{f_n^*\} \subseteq W^*$ such that*

$$W = \overline{span\{e_n : n = 1, 2, \cdots\}}, \quad W^* = \overline{span\{f_n^* : n = 1, 2, \cdots\}},$$

and

$$\langle f_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j. \end{cases}$$

Let us denote $W_n = \mathrm{span}\{e_n\}$, $Y_k = \bigoplus_{n=1}^k W_n$ and $Z_k = \overline{\bigoplus_{n=k}^\infty W_n}$.

Theorem 2.7. *Assume that* (A), (J1)–(J4) *and* (F1)–(F4) *hold. If $f(x, -t) = -f(x, t)$ holds for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, then I_λ has a sequence of critical points $\{\pm u_n\}$ in X such that $I_\lambda(\pm u_n) \to \infty$ as $n \to \infty$.*

Proof. Obviously, I_λ is an even functional and satisfies (PS)-condition. It is enough to show that there exist $\rho_k > \delta_k > 0$ such that

(1) $b_k := \inf\{I_\lambda(u) : u \in Z_k, \|u\|_X = \delta_k\} \to \infty \quad \text{as} \quad n \to \infty$;
(2) $a_k := \max\{I_\lambda(u) : u \in Y_k, \|u\|_X = \rho_k\} \le 0$

for k large enough.

Denote

$$\alpha_k := \sup_{u \in Z_k, \|u\|_X = 1} \Big(\int_{\mathbb{R}^N} \frac{1}{p^*} |u|^{p^*}\,dx \Big).$$

Then $\alpha_k \to 0$ as $k \to \infty$. In fact, suppose that it is false. Then there exist $\varepsilon_0 > 0$ and a sequence $\{u_k\} \in Z_k$ such that

$$\|u_k\|_X = 1, \int_{\mathbb{R}^N} \frac{1}{p^*} |u_k|^{p^*}\,dx \ge \varepsilon_0$$

for all $k \ge k_0$. Since the sequence $\{u_k\}$ is bounded in X, there exists $u \in X$ such that $u_k \rightharpoonup u$ in X as $k \to \infty$ and

$$\langle f_j^*, u \rangle = \lim_{k \to \infty} \langle f_j^*, u_k \rangle = 0$$

for $j = 1, 2, \cdots$. Hence we get $u = 0$. But, we have

$$\varepsilon_0 \le \lim_{k\to\infty} \int_{\mathbb{R}^N} \frac{1}{p^*} |u_k|^{p^*} dx = \int_{\mathbb{R}^N} \frac{1}{p^*} |u|^{p^*} dx = 0,$$

which implies a contradiction.

For any $u \in Z_k$, it follows from the assumptions (J4),(F2) and the Sobolev and Hölder's inequalities that

$$\begin{aligned} I_\lambda(u) &= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\, dx - \lambda \int_{\mathbb{R}^N} F(x, u)\, dx \\ &\ge \int_{\mathbb{R}^N} \frac{c_*}{p} a(x) |\nabla u|^p\, dx - \frac{\lambda}{q} \int_{\mathbb{R}^N} |m(x)|\, |u|^q\, dx \\ &\ge \frac{c_*}{p} \|u\|_a^p - \frac{\lambda}{q} \|m\|_{L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)} \|u\|^q_{L^{p^*}(\mathbb{R}^N)} \\ &\ge \frac{c_0^p c_*}{p} \|u\|_X^p - \frac{\lambda C_2}{q} \alpha_k^q \|u\|_X^q \end{aligned}$$

for a positive constant C_2. Choose $\delta_k = (\lambda \alpha_k^q C_2 / c_0^p c_*)^{1/(p-q)}$. Then $\delta_k \to \infty$ since $p < q$ and $\alpha_k \to 0$ as $k \to \infty$. Therefore, if $u \in Z_k$ and $\|u\|_X = \delta_k$, we obtain that

$$I_\lambda(u) \ge c_0^p c_* \left(\frac{1}{p} - \frac{1}{q}\right) \delta_k^p \to \infty \quad \text{as} \quad k \to \infty,$$

which means the condition (1).

Next we show the condition (2). Set $\varepsilon = 1$, then by the assumption (F4), there exists $\delta > 0$ such that

$$f(x, s) \le w(x) |s|^{p-1} \tag{6}$$

for almost all $x \in \mathbb{R}^N$ and for all $|s| < \delta$. Also we know that there exists $\ell \in L^\infty(\mathbb{R}^N)$ such that $\ell(x) > 0$ for almost all $x \in \mathbb{R}^N$ and

$$F(x, s) \ge \ell(x)\, |s|^\theta \tag{7}$$

for almost all $x \in \mathbb{R}^N$ and for all $|s| \ge \delta$. Indeed, by the assumption (F3), we have that for all $t \ge \delta$

$$\frac{\theta}{t} \le \frac{f(x, t)}{F(x, t)} = \frac{\frac{d}{dt} F(x, t)}{F(x, t)}.$$

Integrating this inequality above over (δ, s), it follows that

$$\int_\delta^s \frac{\theta}{t}\, dt \le \int_\delta^s \frac{\frac{d}{dt} F(x, t)}{F(x, t)}\, dt,$$

and so

$$\ln\left(\frac{s}{\delta}\right)^\theta \le \ln \frac{F(x, s)}{F(x, \delta)}.$$

Hence we get that

$$F(x, s) \ge \frac{s^\theta}{\delta^\theta} F(x, \delta).$$

In a similar way, we obtain that

$$F(x, s) \ge \frac{|s|^\theta}{\delta^\theta} F(x, -\delta)$$

for all $s \le -\delta$. Therefore, $F(x,s) \ge \ell(x)|s|^\theta$ for almost all $x \in \mathbb{R}^N$ and for all $|s| \ge \delta$, where $\ell(x) = \min\{F(x,\delta)/\delta^\theta, F(x,-\delta)/\delta^\theta\}$. Also the assumptions (F2) and (F3) imply that $\ell \in L^\infty(\mathbb{R}^N)$ and $\ell(x) > 0$ for almost all $x \in \mathbb{R}^N$.

For any $u \in Y_k$, by the assumptions (A), (J2), (F3) and inequalities (6) and (7), we deduce

$$\begin{aligned}
I_\lambda(u) &= \int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx - \lambda \int_{\mathbb{R}^N} F(x,u)\,dx \\
&\le \int_{\mathbb{R}^N} |\sigma(x)|\,|\nabla u|\,dx + \frac{b}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx \\
&\quad - \lambda\Big(\int_{\{x\in\mathbb{R}^N:|u(x)|\ge\delta\}} \ell(x)\,|u|^\theta\,dx - \int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}} \frac{1}{p}w(x)\,|u|^p\,dx\Big) \\
&\le \|\sigma\|_{L^{p'}(\mathbb{R}^N)}\|\nabla u\|_{L^p(\mathbb{R}^N)} + \frac{b}{p}\|\nabla u\|^p_{L^p(\mathbb{R}^N)} + \frac{\lambda}{p}\int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}} w(x)\,|u|^p\,dx \\
&\quad + \lambda\int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}} \theta\min\{\frac{F(x,\delta)}{\delta^\theta},\frac{F(x,-\delta)}{\delta^\theta}\}\frac{|u|^\theta}{\theta}\,dx - \lambda\int_{\mathbb{R}^N} \ell(x)w(x)\,|u|^\theta\,dx \\
&\le C\|\sigma\|_{L^{p'}(\mathbb{R}^N)}\|u\|_X + \Big(\frac{C^p b}{p} + \frac{C^*\lambda}{p}\Big)\|u\|_X^p \\
&\quad + \lambda\int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}} \frac{\min\{f(x,\delta)\delta, f(x,-\delta)(-\delta)\}}{\delta^\theta}\frac{|u|^{\theta-p}}{\theta}|u|^p\,dx - \lambda\int_{\mathbb{R}^N} \ell(x)w(x)\,|u|^\theta\,dx \\
&\le C\|\sigma\|_{L^{p'}(\mathbb{R}^N)}\|u\|_X + \Big(\frac{C^p b}{p} + \frac{C^*\lambda}{p}\Big)\|u\|_X^p \\
&\quad + \lambda\int_{\{x\in\mathbb{R}^N:|u(x)|<\delta\}} \frac{\delta^{p-1}}{\delta^{\theta-1}}\frac{\delta^{\theta-p}}{\theta}w(x)\,|u|^p\,dx - \lambda\int_{\mathbb{R}^N} \ell(x)w(x)\,|u|^\theta\,dx \\
&\le C\|\sigma\|_{L^{p'}(\mathbb{R}^N)}\|u\|_X + \Big(\frac{C^p b}{p} + \frac{C^*\lambda}{p} + \frac{C^*\lambda}{\theta}\Big)\|u\|_X^p - \lambda\int_{\mathbb{R}^N} \ell(x)w(x)\,|u|^\theta\,dx, \qquad (8)
\end{aligned}$$

for a positive constant $C = c_1/(a_0)^{1/p}$, where c_1 is the positive constant from the relation (2). By the Hölder's inequality, we see that

$$\begin{aligned}
\int_{\mathbb{R}^N} \ell(x)w(x)\,|u|^\theta\,dx &\le \|\ell\|_{L^\infty(\mathbb{R}^N)}\|w\,|u|^\theta\,\|_{L^1(\mathbb{R}^N)} \\
&\le \|\ell\|_{L^\infty(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N} w(x)\,|u|^p\,dx\right)^{\frac{\theta}{p}}\left(\int_{\mathbb{R}^N} w(x)\,dx\right)^{\frac{p-\theta}{p}}.
\end{aligned}$$

Since $w \in L^1(\mathbb{R}^N)$, we conclude that

$$\int_{\mathbb{R}^N} \ell(x)w(x)\,|u|^\theta\,dx \le C_3\|u\|_X^\theta$$

for a positive constant C_3. Notice that in the finite dimensional subspace X_1, the norm $\|\cdot\|_{L^p(\mathbb{R}^N,\ell w)}$ is equivalent to the norm $\|\cdot\|_X$ (see [17]). Therefore, it follows from (8) that

$$I_\lambda(u) \le C\|\sigma\|_{L^{p'}(\mathbb{R}^N)}\|u\|_X + \Big(\frac{C^p b}{p} + \frac{C^*\lambda}{p} + \frac{C^*\lambda}{\theta}\Big)\|u\|_X^p - \lambda C_4\|u\|_X^\theta$$

for a positive constant C_4. Since $\theta > p$, we get that

$$I_\lambda(u) \to -\infty \text{ as } \|u\|_X \to \infty$$

and thus we can choose $\rho_k > \delta_k > 0$ such that $\|u\|_X = \rho_k$. This completes the proof. □

3. Multiplicity of Solutions

In this section, we will prove that problem (B) has two distinct nontrivial weak solutions in X without Ambrosetti-Rabinowitz condition. However, without this condition, the boundedness of the (PS)-sequence of the Euler-Lagrange functional is not guaranteed, so we can not apply critical point theory to obtain a nontrivial weak solution for problem (B). To overcome this difficulty, we modify some assumptions for φ and f. Then we shall investigate that the functional corresponding to our problem is coercive, which plays a key role in establishing the main result of this section, using the analogous arguments as in Theorem 2.1 in Struwe [16]. The multiplicity result is motivated by the work of Fu and Zhang [7] which had been studied the existence of at least two distinct nontrivial weak solutions for the elliptic equations with variable exponents.

We assume the following condition instead of the assumption (J2):

(J5) There exists a positive constant c_2 such that

$$\left|\varphi(x,v)\right| \le c_2 \, |v|^{p-1} ,$$

for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

Even though the condition (J2) is replaced by (J5), the analogous statements about the functional Φ in Section 2 hold in the usual manner.

Next, we assume that f satisfies the following conditions:

(FM1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition.

(FM2) For all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$

$$\left|f(x,t)\right| \le |h(x)| \, |t|^{\gamma-1} ,$$

where $h \in L^{\frac{p^*}{p^*-\gamma}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and γ is a positive constant with $1 < \gamma < p$.

(FM3) There exists $\delta > 0$ such that

$$f(x,t) \ge s(x)t^{\gamma_0-1}$$

for almost all $x \in \mathbb{R}^N$ and $0 < t \le \delta$, where $s \ge 0, s \in C(\mathbb{R}^N, \mathbb{R})$ and $1 < \gamma_0 < \gamma$.

Under the assumptions (A) and (FM1)–(FM2), we deduce that the functional Ψ is compact operator on X. Furthermore, we obtain the fact that the functional I_λ is coercive for all $\lambda > 0$.

Lemma 3.1. *Assume that* (A), (J1), (J3)–(J5) *and* (FM1)–(FM2) *hold. Then I_λ is coercive for all $\lambda > 0$, i.e., $I_\lambda(u) \to +\infty$ as $\|u\|_X \to +\infty$.*

Proof. For any $u \in X$ and $\lambda > 0$, it follows from the assumptions (J4) and (FM2) that

$$\begin{aligned}
I_\lambda(u) &= \Phi(u) - \lambda\Psi(u) \\
&= \int_{\mathbb{R}^N} \Phi_0(x,\nabla u)\,dx - \lambda \int_{\mathbb{R}^N} F(x,u)\,dx \\
&\ge \int_{\mathbb{R}^N} \frac{c_*}{p} a(x)\,|\nabla u|^p\,dx - \frac{\lambda}{\gamma}\int_{\mathbb{R}^N} |h(x)|\,|u|^\gamma\,dx \\
&\ge \frac{c_*}{p}\|u\|_a^p - \frac{\lambda}{\gamma}\|h\|_{L^{\frac{p^*}{p^*-\gamma}}(\mathbb{R}^N)} \|u\|^\gamma_{L^{p^*}(\mathbb{R}^N)} \\
&\ge \frac{c_0^p c_*}{p}\|u\|_X^p - \frac{\lambda C_1}{\gamma}\|u\|_X^\gamma
\end{aligned} \tag{9}$$

for a positive constant C_1. Since $1 < \gamma < p$, we conclude that

$$I_\lambda(u) \to +\infty \;\text{ as }\; \|u\|_X \to +\infty \;\text{ for all }\; u \in X \;\text{ and }\; \lambda > 0.$$

□

Lemma 3.2. *Assume that* (A), (J1), (J3)–(J5) *and* (FM1)–(FM2) *hold. Then there exist positive constants* ρ *and* r *such that* $I_\lambda(u) \geq r > 0$ *for any* $u \in X$ *with* $\|u\|_X = \rho$ *and for any* $\lambda \in (0, \lambda^*)$*, where* λ^* *is a positive constant.*

Proof. Let $u \in X$ with $\|u\|_X = \rho > 0$. By the inequality (9), we yield

$$\begin{aligned} I_\lambda(u) &\geq \frac{c_0^p c_*}{p}\rho^p - \frac{\lambda C_1}{\gamma}\rho^\gamma \\ &= \rho^\gamma\left(\frac{c_0^p c_*}{p}\rho^{p-\gamma} - \frac{\lambda C_1}{\gamma}\right) \end{aligned} \tag{10}$$

for the same positive constant C_1 as in Lemma 3.1. If we define the quantity

$$\lambda^* = \frac{c_0^p c_* \gamma \rho^{p-\gamma}}{C_1 p},$$

then it follows from the inequality (10) that there exists $r = c_0^p c_* \rho^p / p > 0$ such that

$$I_\lambda(u) \geq r > 0$$

for any $u \in X$ with $\|u\|_X = \rho$ and for any $\lambda \in (0, \lambda^*)$. □

Lemma 3.3. *Assume that* (A), (J1), (J3)–(J5) *and* (FM1)–(FM3) *hold. Then there exists* $\phi \in X$ *such that* $\phi \geq 0, \phi \neq 0$ *and* $I_\lambda(\eta\phi) < 0$ *for* $\eta > 0$ *small enough.*

Proof. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ such that $\phi(x) \equiv 1, x \in B_R(x_0); 0 \leq \phi(x) \leq 1, \left|\nabla\phi(x)\right| \leq 1/R$, for all $x \in \mathbb{R}^N$, where $B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| \leq R\}$. Then for any $\eta \in (0,1)$ it follows from the assumptions (J5) and (FM3) that

$$\begin{aligned} I_\lambda(\eta\phi) &= \Phi(\eta\phi) - \lambda\Psi(\eta\phi) \\ &= \int_{B_{2R}(x_0)} \Phi_0(x, \eta\nabla\phi)\,dx - \lambda\int_{B_{2R}(x_0)} F(x, \eta\phi)\,dx \\ &\leq \int_{B_{2R}(x_0)} \frac{c_2}{p}\eta^p \left|\nabla\phi\right|^p\,dx - \lambda\int_{B_{2R}(x_0)} \frac{\eta^{\gamma_0}}{\gamma_0}|s(x)|\left|\phi\right|^{\gamma_0}\,dx \\ &\leq \frac{c_2\eta^p}{p}\int_{B_{2R}(x_0)} \left|\nabla\phi\right|^p\,dx - \frac{\lambda\eta^{\gamma_0}}{\gamma_0}\int_{B_{2R}(x_0)} |s(x)|\,dx. \end{aligned}$$

Choose a positive constant δ such that

$$0 < \delta < \min\left\{1, \frac{\frac{\lambda p}{c_2\gamma_0}\int_{B_{2R}(x_0)}|s(x)|\,dx}{\int_{B_{2R}(x_0)}\left|\nabla\phi\right|^p\,dx}\right\},$$

then $\eta < \delta^{1/(p-\gamma_0)}$ implies that

$$I_\lambda(\eta\phi) < 0.$$

□

By the coercivity of the functional I_λ, we get that there exists a global minimizer $u_1 \in X$ of I_λ (Theorem 1.2 in [16]). This together with Lemma 3.3 yields that

$$I_\lambda(u_1) = \inf_{u \in X\setminus\{0\}} I_\lambda(u) < 0.$$

Consequently, we deduce that u_1 is a global minimizer of the functional I_λ in X for any $\lambda > 0$.

In the rest of this section, we will show that there exists another nontrivial weak solution for problem (B). Set

$$g(x,t) = \begin{cases} f(x,t) & \text{if } t \le u_1(x) \\ f(x,u_1(x)) & \text{if } t > u_1(x), \end{cases}$$

where u_1 is a nontrivial weak solution of problem (B). Let us put $G(x,t) = \int_0^t g(x,s)\,ds$, then we define the functional $\tilde{\Psi} : X \to \mathbb{R}$ by

$$\tilde{\Psi}(u) = \int_{\mathbb{R}^N} G(x,u)\,dx.$$

It is not hard to check that $\tilde{\Psi} \in C^1(X,\mathbb{R})$ and its Fréchet derivative is

$$\langle \tilde{\Psi}'(u), v\rangle = \int_{\mathbb{R}^N} g(x,u)v\,dx$$

for all $u, v \in X$. Next we define the functional $\tilde{I}_\lambda : X \to \mathbb{R}$ by

$$\tilde{I}_\lambda(u) = \Phi(u) - \lambda\tilde{\Psi}(u).$$

Then the same arguments as those used for the functional I_λ imply that $\tilde{I}_\lambda \in C^1(X,\mathbb{R})$ and its Fréchet derivative is given by

$$\langle \tilde{I}'_\lambda(u), v\rangle = \int_{\mathbb{R}^N} \varphi(x,\nabla u)\cdot\nabla v\,dx - \lambda\int_{\mathbb{R}^N} g(x,u)v\,dx$$

for any $u, v \in X$. In addition, the fact that $\tilde{I}_\lambda$ is also coercive for all $\lambda > 0$ is obtained immediately; see Lemma 3.1.

Now we prove the existence of at least two distinct nontrivial weak solutions for problem (B) for any $\lambda \in (0,\lambda^*)$.

Theorem 3.4. *Assume that* (A), (J1), (J3)–(J5) *and* (FM1)–(FM3) *hold. If $f(x,t)t \ge 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, then problem* (B) *has at least two distinct nontrivial weak solutions u_1, u_2 in X with $u_2(x) \le u_1(x)$ for almost all $x \in \mathbb{R}^N$ and for any $\lambda \in (0,\lambda^*)$.*

Proof. Let u_1 be a nontrivial weak solution of problem (B). Then it follows from Lemma 3.2 that there exists a positive constant r_0 with $0 < r_0 < \|u_1\|_X$ such that

$$\inf_{\|u\|_X = r_0} \tilde{I}_\lambda(u) > 0 = \tilde{I}_\lambda(0).$$

And by the definition of $\tilde{I}_\lambda$, we get that $\tilde{I}_\lambda(u_1) = I_\lambda(u_1) < 0$ for any $\lambda \in (0,\lambda^*)$ where λ^* is the quantity defined in Lemma 3.3. Hence we can apply the Mountain pass theorem to find another critical point of the functional $\tilde{I}_\lambda$. Then we deduce that there exists a sequence $\{u_n\}$ in X such that $\tilde{I}_\lambda(u_n) \to c > 0$ and $\tilde{I}'_\lambda(u_n) \to 0$ as $n \to \infty$, where $c = \inf_{\rho\in P}\max_{t\in[0,1]} \tilde{I}_\lambda(\rho(t))$ and $P = \{\rho \in C([0,1],X) : \rho(0) = 0, \rho(1) = u_1\}$. Since the functional $\tilde{I}_\lambda$ is coercive, we obtain that the sequence $\{u_n\}$ is bounded in X and passing to a subsequence, still denoted by $\{u_n\}$. We may assume that there exists an element $u \in X$ such that $u_n \rightharpoonup u$ in X as $n \to \infty$. In order to show that $u_n \to u$ in X as $n \to \infty$, we need to consider the following equality:

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = \langle \tilde{I}'_\lambda(u_n) - \tilde{I}'_\lambda(u), u_n - u\rangle + \lambda\langle \tilde{\Psi}'(u_n) - \tilde{\Psi}'(u), u_n - u\rangle.$$

Since $u_n \rightharpoonup u$ in X and $\langle \tilde{I}'_\lambda(u_n) - \tilde{I}'_\lambda(u), u_n - u\rangle \to 0$ as $n \to \infty$, it suffices to show that

$$\langle \tilde{\Psi}'(u_n) - \tilde{\Psi}'(u), u_n - u\rangle \to 0$$

as $n \to \infty$. Observe that

$$\begin{aligned}|\langle \tilde{\Psi}'(u_n) - \tilde{\Psi}'(u), u_n - u\rangle| &= \left|\int_{\mathbb{R}^N} (g(x,u_n) - g(x,u))\,(u_n - u)\; dx\right| \\ &\le \int_{B_K(0)} \left|g(x,u_n) - g(x,u)\right| |u_n - u|\, dx \\ &\quad + \int_{\mathbb{R}^N \setminus B_K(0)} \left|g(x,u_n) - g(x,u)\right| |u_n - u|\, dx, \end{aligned} \tag{11}$$

where $B_K(0) = \{x \in \mathbb{R}^N : |x| \le K\}$. For the first term of the right side of the inequality (11), we have that

$$\begin{aligned}\int_{B_K(0)} \left|g(x,u_n) - g(x,u)\right| |u_n - u|\, dx &\le \int_{B_K(0)} |h(x)|\,(|u_n|^{\gamma-1} + |u|^{\gamma-1})\,|u_n - u|\, dx \\ &\le \|h\|_{L^\infty(\mathbb{R}^N)}\, \|\,|u_n|^{\gamma-1} + |u|^{\gamma-1}\,\|_{L^{\gamma'}(B_K(0))}\, \|u_n - u\|_{L^\gamma(B_K(0))}.\end{aligned}$$

Since $1 < \gamma < p^*$, the embedding $W^{1,p}(B_K(0)) \hookrightarrow\hookrightarrow L^\gamma(B_K(0))$ implies $u_n \to u$ in $L^\gamma\,(B_K(0))$ as $n \to \infty$. Hence, for any $\varepsilon > 0$, there exists $N(K) \in \mathbb{N}$ so that $n \ge N(K)$ implies $\|u_n - u\|_{L^\gamma(B_K(0))} < \varepsilon$. Thus

$$\int_{B_K(0)} \left|g(x,u_n) - g(x,u)\right| |u_n - u|\, dx \le C_2\varepsilon \tag{12}$$

for a positive constant C_2. The second term of the right side of the relation (11) is estimated by

$$\int_{\mathbb{R}^N \setminus B_K(0)} \left|g(x,u_n) - g(x,u)\right| |u_n - u|\, dx \le \|g(x,u_n) - g(x,u)\|_{L^{(p^*)'}(\mathbb{R}^N \setminus B_K(0))}\, \|u_n - u\|_{L^{p^*}(\mathbb{R}^N \setminus B_K(0))}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N \setminus B_K(0)} |g(x,u_n) - g(x,u)|^{(p^*)'}\, dx \\ &\le \int_{\mathbb{R}^N \setminus B_K(0)} \left\{ |h(x)| \left(|u_n|^{\gamma-1} + |u|^{\gamma-1}\right) \right\}^{(p^*)'} dx \\ &\le C_3 \|h^{(p^*)'}\|_{L^{\frac{q}{(p^*)'}}(\mathbb{R}^N \setminus B_K(0))} \|(|u_n|^{\gamma-1} + |u|^{\gamma-1})^{(p^*)'}\|_{L^{\left(\frac{q}{(p^*)'}\right)'}(\mathbb{R}^N \setminus B_K(0))} \\ &\le C_3 \|h^{(p^*)'}\|_{L^{\frac{q}{(p^*)'}}(\mathbb{R}^N \setminus B_K(0))} \|(|u_n|^{\gamma-1} + |u|^{\gamma-1})^{\frac{p^*}{p^*-1}}\|_{L^{\frac{p^*-1}{\gamma-1}}(\mathbb{R}^N \setminus B_K(0))} \\ &\le C_3 \left(\int_{\mathbb{R}^N \setminus B_K(0)} |h(x)|^q\, dx\right)^{\frac{(p^*)'}{q}} \left\{ \int_{\mathbb{R}^N \setminus B_K(0)} \left(|u_n|^{\gamma-1} + |u|^{\gamma-1}\right)^{\frac{p^*}{\gamma-1}}\, dx \right\}^{\frac{\gamma-1}{p^*-1}} \\ &\le C_4 \left(\int_{\mathbb{R}^N \setminus B_K(0)} |h(x)|^q\, dx\right)^{\frac{(p^*)'}{q}} \left\{ \int_{\mathbb{R}^N \setminus B_K(0)} (|u_n| + |u|)^{p^*}\, dx \right\}^{\frac{\gamma-1}{p^*-1}} \\ &\le C_4 \left(\int_{\mathbb{R}^N \setminus B_K(0)} |h(x)|^q\, dx\right)^{\frac{(p^*)'}{q}} \|u_n + u\|_{L^{p^*}(\mathbb{R}^N)}^{(p^*)'(\gamma-1)}\end{aligned}$$

for positive constants C_3 and C_4. As the sequence $\{u_n\}$ is bounded in X, by the Sobolev inequality, the sequence $\{u_n\}$ is also bounded in $L^{p^*}(\mathbb{R}^N)$. So the sequence $\{u_n + u\}$ is bounded in $L^{p^*}(\mathbb{R}^N)$ and it is immediate from $h \in L^{\frac{p^*}{p^*-\gamma}}$ that

$$\int_{\mathbb{R}^N \setminus B_K(0)} \left|g(x,u_n) - g(x,u)\right| |u_n - u|\, dx \le C_5\varepsilon \tag{13}$$

for a positive constant C_5. It follows from the inequalities (12) and (13) that

$$|\langle \tilde{\Psi}'(u_n) - \tilde{\Psi}'(u), u_n - u\rangle| \leq C_6 \varepsilon$$

for a positive constant C_6 when $n \geq N(K)$. This implies that $\langle \tilde{\Psi}'(u_n) - \tilde{\Psi}'(u), u_n - u\rangle \to 0$ as $n \to \infty$. Since Φ' is a mapping of type (S_+), we deduce $u_n \to u$ in X as $n \to \infty$. Because of $\tilde{I}_\lambda \in C^1(X, \mathbb{R})$, we have $\tilde{I}_\lambda(u) = c > 0$ and $\tilde{I}'_\lambda(u) = 0$, i.e., u is a nontrivial critical point of the functional $\tilde{I}_\lambda$.

Denote it by $u = u_2$. Then we claim that

$$u_2(x) \leq u_1(x)$$

for almost all $x \in \mathbb{R}^N$. Indeed, it is clear that

$$\begin{aligned}
0 &= \langle \tilde{I}'_\lambda(u_2) - I'_\lambda(u_1), (u_2 - u_1)^+\rangle \\
&= \int_{\mathbb{R}^N} (\varphi(x, \nabla u_2) - \varphi(x, \nabla u_1)) \cdot \nabla (u_2 - u_1)^+ \, dx \\
&\quad - \lambda \int_{\mathbb{R}^N} (g(x, u_2) - f(x, u_1)) (u_2 - u_1)^+ \, dx \\
&= \int_{\{x \in \mathbb{R}^N : u_2(x) \geq u_1(x)\}} (\varphi(x, \nabla u_2) - \varphi(x, \nabla u_1)) \cdot (\nabla u_2 - \nabla u_1) \, dx,
\end{aligned}$$

where $(u_2 - u_1)^+ = \max\{0, u_2 - u_1\}$. Since φ is monotone, we obtain $u(x) = u_1(x)$ for almost all $x \in \{x \in \mathbb{R}^N : u_2(x) \geq u_1(x)\}$. Hence it follows that

$$\int_{\{x \in \mathbb{R}^N : u_2(x) \geq u_1(x)\}} |\nabla(u_2 - u_1)|^p \, dx = 0,$$

and thus

$$\int_{\mathbb{R}^N} |\nabla(u_2 - u_1)^+|^p \, dx = 0,$$

i.e., $\|(u_2 - u_1)^+\|_X = 0$. Since $(u_2 - u_1)^+ \in X$, we see that $(u_2(x) - u_1(x))^+ = 0$ for almost all $x \in \mathbb{R}^N$. Consequently, we have that $u_2(x) \leq u_1(x)$ for almost all $x \in \mathbb{R}^N$.

Finally, we establish that u_2 is another weak solution. Indeed, since $u_2(x) \leq u_1(x)$ for almost all $x \in \mathbb{R}^N$, we deduce that

$$g(x, u_2) = f(x, u_2) \qquad \text{and} \qquad \tilde{\Psi}(u_2) = \Psi(u_2).$$

Then we get that $\tilde{I}_\lambda(u_2) = I_\lambda(u_2)$ and $\tilde{I}'_\lambda(u_2) = I'_\lambda(u_2)$. So, we conclude that $\tilde{I}_\lambda(u_2) = c > 0 > \tilde{I}_\lambda(u_1)$, i.e., u_2 is another nontrivial weak solution for problem (B) with $u_2 < u_1$. □

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