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Remarks on Upper and Lower Bounds for Matching Sequencibility of Graphs

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Abstract. In 2008, Alspach [The Wonderful Walecki Construction, Bull. Inst. Combin. Appl. 52 (2008) 7–20] defined the matching sequencibility of a graph G to be the largest integer k such that there exists a linear ordering of its edges so that every k consecutive edges in the linear ordering form a matching of G, which is denoted by $\mathrm{ms}(G)$. In this paper, we show that every graph G of size q and maximum degree Δ satisfies $\frac{1}{2}\left\lfloor \frac{q}{\Delta+1} \right\rfloor \le \mathrm{ms}(G) \le \left\lfloor \frac{q-1}{\Delta-1} \right\rfloor$ by using the edge-coloring of G, and we also improve this lower bound for some particular graphs. We further discuss the relationship between the matching sequencibility and a conjecture of Seymour about the existence of the kth power of a Hamilton cycle.

1. Introduction

In this paper, we consider only finite graphs having at least one edge. For terminology and notation not defined in this paper, we refer the readers to [3]. Unless stated otherwise, "graph" means simple graph. A *multigraph* may contain multiple edges but no loops. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively, and $|V(G)|$ and $|E(G)|$ is called the *order* and the *size* of G, respectively. A *vertex-degree*, or simply a *degree*, of a vertex in G is the number of edges incident to it, and we denote by $\Delta(G)$ the maximum degree of G. For $X \subseteq V(G)$, we denote by $G[X]$ a subgraph of G induced by X. An *independent edge set*, also called a *matching*, of G is a subset of $E(G)$ such that no two edges in the set have a vertex in common. We call a matching of cardinality m an *m-matching*. A *maximum matching* of G is a matching of largest possible cardinality, and by $\mathrm{m}(G)$ we denote the cardinality of a maximum matching of G. Let X be a finite set. We denote by $X_{(2)}$ the set of unordered pairs of distinct elements of X. For a positive integer k, let $\mathbf{N}_k = \{1, 2, \dots, k\}$. For a map $f : \mathbf{N}_k \to X$, we often denote by $[\, l\,]$ the image of $l \in \mathbf{N}_k$ instead of $f(l)$ if there is no danger of confusion.

In 2008, Alspach [1] introduced a new graph invariant for matchings, which comes from the problem of how to schedule a round-robin tournament in which the minimum amount of time any participant has between games is maximized. Let G be a graph. For an integer k, we call a bijection $f : \mathbf{N}_{|E(G)|} \to E(G)$ a *map with sequential k-matching* of G if $\left\{[\, l\,], [\, l+1\,], \dots, [\, l+k-1\,]\right\}$ forms a k-matching of G for each l with $l \in \mathbf{N}_{|E(G)|-k+1}$. We define

$$\mathrm{ms}(G) = \max\{k : G \text{ has a map with sequential } k\text{-matching}\},$$

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which is called a *matching sequencibility* of G. In [1], Alspach determined the matching sequencibility of the complete graph by using the Walecki decomposition of it into Hamilton cycles (or Hamilton paths). He actually proved that the complete graph K_n ($n \geq 2$) satisfies $\mathrm{ms}(K_n) = \left\lfloor \frac{n-1}{2} \right\rfloor$. In 2012, Brualdi, Kiernan, Meyer and Schroeder [2] pointed out that it is not so difficult to determine the matching sequencibility of the complete bipartite graph $K_{m,n}$ ($m \geq n \geq 2$) by using the biadjacency matrix, and they remarked that if $m > n$, then $\mathrm{ms}(K_{m,n}) = n$; if $m = n$, then $\mathrm{ms}(K_{m,n}) = n - 1$.

In this note, we give the upper and lower bounds for matching sequencibility of graphs as follows. (We can actually obtain the better lower bound for some particulars class of graphs, see Remark 2.2 in Section 2 and also Section 3.)

Theorem 1.1. *Every graph G of size q and maximum degree Δ (≥ 2) satisfies*

$$\frac{1}{2}\left\lfloor \frac{q}{\Delta + 1} \right\rfloor \leq \mathrm{ms}(G) \leq \left\lfloor \frac{q-1}{\Delta - 1} \right\rfloor.$$

In particular, in order to show the left side inequality in Theorem 1.1, we use the edge-coloring of graphs (i.e., the decomposition of the graph into matchings), and the essential part of the proof is the following. Here for a multigraph G and $X \subseteq V(G)_{(2)}$, $G + X$ means the graph with the vertex set $V(G)$ and the edge set $E(G) \cup X$.

Theorem 1.2. *Let G be a multigraph and M be a subset of $V(G)_{(2)}$ such that M is a matching, and let k be an integer with $k \leq \left\lceil \frac{|M|}{2} \right\rceil$. If $\mathrm{ms}(G) \geq k$, then the multigraph $G + M$ satisfies $\mathrm{ms}(G + M) \geq k$.*

The paper is organized as follows. In Section 2, we give the proof of Theorem 1.1 after proving Theorem 1.2. In Section 3, we discuss the relation between the matching sequencibility and the matching number of regular graphs, and we improve the lower bound of the matching sequencibility in Theorem 1.1 for 2-regular graphs and 3-regular graphs with an additional condition. In the last section (Section 4), we discuss the relationship between the matching sequencibility and Seymour's conjecture [13] concerning the existence of the kth power of a Hamilton cycle, and we give conjectures for matching sequencibility.

2. Proof of Theorem 1.1

As mentioned in Section 1, we first prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a multigraph of order n and write $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let M be a subset of $V(G)_{(2)}$ such that M is a matching. Let $m = |M|$ for convenience, and let k be an integer with $k \leq \left\lceil \frac{m}{2} \right\rceil$. Suppose that $\mathrm{ms}(G) \geq k$, and we show that $G^* = G + M$ also satisfies $\mathrm{ms}(G^*) \geq k$. We may assume that $k \geq 2$. Since $\mathrm{ms}(G) \geq k$, it follows that G has a map g with sequential k-matching. Note that

$$\Big\{g(1), g(2), \ldots, g(k)\Big\} \text{ is a matching of } G. \tag{1}$$

So, without loss of generality, we may assume that

$$g(s) = v_{2s-1}v_{2s} \text{ for } 1 \leq s \leq k. \tag{2}$$

For each edge $e = v_iv_j \in M$, we further define $\mathrm{i}_{\min}(e) = \min\{i, j\}$.

We now define the map $f : \mathbf{N}_{|E(G^*)|} \to E(G^*)$ by the following procedure.

(I) Assign integers $1, 2, \ldots, m$ to edges of M so that $(1 \leq)\ \mathrm{i}_{\min}([\ 1\ ]) < \mathrm{i}_{\min}([\ 2\ ]) < \cdots < \mathrm{i}_{\min}([\ m-1\ ]) < \mathrm{i}_{\min}([\ m\ ])$ (note that we can actually assign integers to M in this way because M is a matching).

(II) Assign integers $m + 1, m + 2, \ldots, m + |E(G)|$ to edges of $G^* - M$ ($= G$) so that $[\ m + r\ ] = g(r)$ for $1 \leq r \leq |E(G)|$.

Since g is a bijection from $\mathbf{N}_{|E(G)|}$ to $E(G)$, it follows from (I) and (II) that f is also bijective. Thus, it suffices to show that the set $\{[\,l\,], [\,l+1\,], \ldots, [\,l+k-1\,]\}$ forms a matching of G^* for each l with $l \in \mathbf{N}_{|E(G^*)|-k+1}$. Since g is a map with sequential k-matching of G, the set $\{g(r), g(r+1), \ldots, g(r+k-1)\}$ is a matching of G for $r \in \mathbf{N}_{|E(G)|-k+1}$. Hence by (II), we see that $\{[\,l\,], [\,l+1\,], \ldots, [\,l+k-1\,]\}$ also forms a matching of G^* for $m+1 \le l \le |E(G^*)| - k + 1$.

We next show that $\{[\,l\,], [\,l+1\,], \ldots, [\,l+k-1\,]\}$ forms a matching of G^* for $1 \le l \le m$. Let l be an arbitrary integer with $1 \le l \le m$. Since $M = \{[\,1\,], [\,2\,], \ldots, [\,m\,]\}$ by (I), if $l \le m-k+1$, then $\{[\,l\,], [\,l+1\,], \ldots, [\,l+k-1\,]\}$ is clearly a matching of G^*. Thus we may assume that $l \ge m-k+2$. Since $l \le \mathrm{i}_{\min}([\,l\,]) < \cdots < \mathrm{i}_{\min}([\,m\,])$ by (I) (recall that $\mathrm{i}_{\min}(e) = \min\{i, j\}$ for $e = v_iv_j \in M$), it follows that

$$\{[\,l\,], [\,l+1\,], \ldots, [\,m\,]\} \text{ is a matching of } G^*\big[\{v_i : l \le i \le n\}\big].$$

On the other hand, by (1), (2) and (II), it follows that

$$\{[\,m+1\,], [\,m+2\,], \ldots, [\,m+\varepsilon\,]\} \text{ is a matching of } G^*\big[\{v_i : 1 \le i \le l-1\}\big],$$

where we let $\varepsilon = \min\left\{k, \left\lfloor \frac{l-1}{2} \right\rfloor\right\}$. So, if we can show that $(m-l+1)+\varepsilon \ge k$, our result follows. Hence it is sufficient to show that $\varepsilon + m - l + 1 - k \ge 0$. If $\varepsilon = k$, then obviously $\varepsilon + m - l + 1 - k \ge 0$ holds because $l \le m$. Thus we may assume that $\varepsilon = \left\lfloor \frac{l-1}{2} \right\rfloor$. If $l \le m-1$, then

$$\begin{aligned}\left\lfloor \frac{l-1}{2} \right\rfloor + m - l + 1 - k &\ge \left\lfloor \frac{l-1}{2} \right\rfloor + m - l + 1 - \left\lceil \frac{m}{2} \right\rceil \\ &\ge \frac{l-2}{2} + m - l + 1 - \frac{m+1}{2} = \frac{m-l-1}{2} \ge 0;\end{aligned}$$

if $l = m$ and l is even, then

$$\left\lfloor \frac{l-1}{2} \right\rfloor + m - l + 1 - k = \frac{l-2}{2} + 1 - k \ge \frac{l-2}{2} + 1 - \left\lceil \frac{l}{2} \right\rceil = \frac{l-2}{2} + 1 - \frac{l}{2} = 0;$$

if $l = m$ and l is odd, then

$$\left\lfloor \frac{l-1}{2} \right\rfloor + m - l + 1 - k = \frac{l-1}{2} + 1 - k \ge \frac{l-1}{2} + 1 - \left\lceil \frac{l}{2} \right\rceil = \frac{l-1}{2} + 1 - \frac{l+1}{2} = 0.$$

Thus the inequality $\varepsilon + m - l + 1 - k \ge 0$ holds.

This completes the proof of Theorem 1.2. □

To complete the proof of Theorem 1.1, we further use the lemma concerning the equitable edge-coloring. Werra [15] and independently, McDiarmid [11] proved that if a multigraph G has an l-edge-coloring, then G also has an *equitable* l-edge-coloring, i.e., l-edge-coloring such that each color class has size $\left\lfloor \frac{|E(G)|}{l} \right\rfloor$ or $\left\lceil \frac{|E(G)|}{l} \right\rceil$. (Note that "edge-coloring (vertex-)" always means "proper edge-coloring (proper vertex-)" in this paper.) On the other hand, Vizing [14] proved that every graph G has a $(\Delta(G)+1)$-edge-coloring, which is a well known theorem in Graph Theory. Combining this two results, we can get the following. (We actually use only the fact that every graph G has a $(\Delta(G)+1)$-edge-coloring in which each color class has size at least $\left\lfloor \frac{|E(G)|}{\Delta(G)+1} \right\rfloor$, see the proof of Theorem 1.1.)

Lemma 2.1. *Every graph G of maximum degree Δ has an equitable $(\Delta+1)$-edge-coloring.*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let G be a graph of size q and maximum degree $\Delta \ge 2$. We first show the right side inequality.

Let f be a map with sequential ms(G)-matching in G, and let $e_1, \ldots, e_\Delta$ be distinct Δ edges which are incident to the vertex with maximum degree of G. We may assume that $[\,e_1\,]^{-1} < [\,e_2\,]^{-1} < \cdots < [\,e_\Delta\,]^{-1}$, where $[\,e_i\,]^{-1} = f^{-1}(e_i)$ for $1 \le i \le \Delta$. Since $\left\{[\,l\,], [\,l+1\,], \ldots, [\,l+\mathrm{ms}(G)-1\,]\right\}$ forms a matching of G for $l \in \mathbf{N}_{q-\mathrm{ms}(G)+1}$, it follows that $[\,e_{i+1}\,]^{-1} - [\,e_i\,]^{-1} \ge \mathrm{ms}(G)$ for $1 \le i \le \Delta-1$. Thus $q \ge \sum_{1\le i\le \Delta-1}([\,e_{i+1}\,]^{-1} - [\,e_i\,]^{-1}) + 1 \ge \mathrm{ms}(G)(\Delta-1)+1$, that is, $\mathrm{ms}(G) \le \left\lfloor \frac{q-1}{\Delta-1} \right\rfloor$.

We next show the left side inequality. By Lemma 2.1, there exists a partition $\{M_1, \ldots, M_{\Delta+1}\}$ of $E(G)$ such that each M_i is a matching of cardinality at least $\left\lfloor \frac{q}{\Delta+1} \right\rfloor$. We may assume that $|M_1| \ge |M_2| \ge \cdots \ge |M_{\Delta+1}|$. Let G_1 be a graph such that $V(G_1) = V(G)$ and $E(G_1) = M_1$, and let $G_i = G_{i-1} + M_i$ for $2 \le i \le \Delta + 1$. Since $|M_1| \ge |M_2| \ge \cdots \ge |M_{\Delta+1}|$, we easily see that the following holds.

(I) If $\mathrm{ms}(G_{i-1}) \ge \frac{1}{2}|M_{i-1}|$, then $\mathrm{ms}(G_{i-1}) \ge \frac{1}{2}|M_i|$ $(2 \le i \le \Delta + 1)$.

Since $M_1, \ldots, M_{\Delta+1}$ are matchings of G, it follows from the definition of $G_1, \ldots, G_{\Delta+1}$ and Theorem 1.2 that the following hold.

(II) $\mathrm{ms}(G_1) = |M_1| \ge \frac{1}{2}|M_1|$.

(III) If $\mathrm{ms}(G_{i-1}) \ge \frac{1}{2}|M_i|$, then $\mathrm{ms}(G_i) = \mathrm{ms}(G_{i-1} + M_i) \ge \frac{1}{2}|M_i|$ $(2 \le i \le \Delta + 1)$.

Thus, by applying (II), and (I), (III) inductively, we can get $(\mathrm{ms}(G) =)\ \mathrm{ms}(G_{\Delta+1}) \ge \frac{1}{2}|M_{\Delta+1}|$. Since $|M_{\Delta+1}| \ge \left\lfloor \frac{q}{\Delta+1} \right\rfloor$, we have $\mathrm{ms}(G) \ge \frac{1}{2}\left\lfloor \frac{q}{\Delta+1} \right\rfloor$.

This completes the proof of Theorem 1.1. □

Remark 2.2. *Note that by a theorem of Vizing [14], the edge chromatic number of a graph G, denoted by $\chi'(G)$, is $\Delta(G)$ or $\Delta(G) + 1$. Hence by the same argument as in the proof of Theorem 1.1, we can actually show that if a graph G is Class 1 (i.e., $\chi'(G) = \Delta(G)$), then $\mathrm{ms}(G) \ge \frac{1}{2}\left\lfloor \frac{|E(G)|}{\Delta(G)} \right\rfloor$; if a graph G is Class 2 (i.e., $\chi'(G) = \Delta(G) + 1$), then $\mathrm{ms}(G) \ge \frac{1}{2}\left\lfloor \frac{|E(G)|}{\Delta(G)+1} \right\rfloor$. (However, determining whether a graph is Class 1 or Class 2 is* **NP***-complete [9].) For example, if G is a 1-factorizable graph, that is, G is a regular Class 1 graph, then $\mathrm{ms}(G) \ge \frac{1}{2}\left\lfloor \frac{|E(G)|}{\Delta(G)} \right\rfloor = \frac{1}{2} \times \frac{|V(G)|}{2} = \frac{1}{2}\mathrm{m}(G)$ holds (recall that $\mathrm{m}(G)$ denotes the matching number of G). In the next section, we focus on the relation between the matching sequencibility and the matching number for regular graphs.*

3. The Matching Sequencibility and the Matching Number of Regular Graphs

In this section, we improve Theorem 1.1 for 2-regular graphs and 3-regular graphs with an additional condition.

A *perfect matching* of a graph G is a matching whose edges cover all the vertices of G. A graph is *even* (resp., *odd*) if it has even order (resp., odd order). A graph is said to be *d-regular* if every vertex has degree d.

As a corollary of Theorem 1.1, we can easily obtain the relationship between $\mathrm{ms}(G)$ and $\mathrm{m}(G)$ for a regular graph G.

Corollary 3.1. *For $d \ge 2$, every d-regular graph G of order n satisfies*

$$\mathrm{ms}(G) \ge \frac{1}{2}\left\lfloor \frac{d}{d+1} \times \frac{n}{2} \right\rfloor \left(\ge \frac{1}{2}\left\lfloor \frac{d}{d+1} \times \mathrm{m}(G) \right\rfloor \right).$$

Note that, in 2007, Henning and Yeo gave a tight lower bound on the matching number for d-regular graphs on n vertices (see [8]).

In fact, we can completely determine the matching sequencibility of 2-regular graphs. Here, for a graph G, if G contains an even component of order at least 4, then we let $\gamma(G) = 1$; otherwise, let $\gamma(G) = 0$.

Proposition 3.2. *If G is a 2-regular multigraph, then G satisfies $\mathrm{ms}(G) = \mathrm{m}(G) - \gamma(G)$.*

Using Theorem 1.2 and Proposition 3.2, we can also obtain the better lower bound than the one of Corollary 3.1 for 3-regular graphs having perfect matchings (we can obtain the same relation as the case of 1-factorizable graphs, see Remark 2.2 in Section 2).

Corollary 3.3. *If G is a 3-regular multigraph which has a perfect matching, then the graph G satisfies* $\mathrm{ms}(G) \geq \frac{1}{2}\mathrm{m}(G)$.

By Corollary 3.3 and the well known theorem of Petersen [12] concerning the existence of a perfect matching in 3-regular graphs, every bridgeless 3-regular graph satisfies $\mathrm{ms}(G) \geq \frac{1}{2}\mathrm{m}(G)$.

In the rest of this section, we prove Proposition 3.2 and Corollary 3.3. To do that, we prepare the following notation. Let C be a cycle with a fixed orientation. For an edge e in C, we denote by e^+ and e^- the successor and the predecessor of e on C, and we let $e^{+2} = (e^+)^+$.

Proof of Proposition 3.2. Let G be a 2-regular multigraph. In this proof, we consider that every component (cycle) has a fixed orientation. Note that every maximum matching of G covers all the vertices of each even cycle of order at least 4 (if G contains it), and this implies that $\mathrm{ms}(G) \leq \mathrm{m}(G) - \gamma(G)$. Thus it suffices to show that $\mathrm{ms}(G) \geq \mathrm{m}(G) - \gamma(G)$.

Let M be a maximum matching of G, and let $m = |M|$. Let m^o (resp., m^e) be the number of edges which belong to M in odd cycles (resp., even cycles). Moreover, let $M_1, M_2, \ldots, M_{o(G)}$ be pairwise disjoint subsets of M such that each M_i is a maximum matching of some odd cycle (here for a graph H, $o(H)$ denotes the number of odd components of H) and we denote each edge of M_i by the following ordered pair: $M_i = \{(1, i), (2, i), (3, i), \ldots, (|M_i|, i)\}$, where $(j, i)^{+2} = (j + 1, i)$ for $1 \leq j \leq |M_i| - 1$. Now we define the bijection $f : \mathbf{N}_{|E(G)|} \to E(G)$ by the following procedure.

(I) Assign integers $1, 2, \ldots, m^o$ to M in odd cycles so that $\{[\,1\,], [\,2\,], \ldots, [\,m^o\,]\} = M_1 \cup \cdots \cup M_{o(G)}$ and $[\,1\,], [\,2\,], \ldots, [\,m^o\,]$ appear in lexicographic order.

(II) Take any edge which belongs to M in any even cycle, and go around the cycle assigning integers to edges of M, and repeat this until integers have been assigned to all edges of M in each even cycle. (Note that we assigned integers $1, 2, \ldots, m$ to all edges of M at this stage.)

(III) Assign integers $m + 1, m + 2, \ldots, m + m^o$ to edges of $E(G) \setminus M$ in odd cycles so that $[\,m + r\,] = \big([\,r\,]\big)^-$ for $1 \leq r \leq m^o$.

(IV) Assign integers $m + m^o + 1, m + m^o + 2, \ldots, 2m$ to edges of $E(G) \setminus M$ in even cycles so that $[\,m + m^o + r\,] = \big([\,m^o + r\,]\big)^+$ for $1 \leq r \leq m^e$.

(V) Assign integers $2m+1, 2m+2, \ldots, 2m+o(G)$ to edges of $E(G)\setminus M$ in odd cycles so that $[\,2m+r\,] = \big([\,m+r\,]\big)^-$ for $1 \leq r \leq o(G)$.

This is illustrated in Figure 1. Let $e \in E(G)$, and suppose first that e is an edge of M in some odd cycle. Then by (I), (III) and (V), we have

$$\left|[\,e^+\,]^{-1} - [\,e\,]^{-1}\right| > \left|[\,e^-\,]^{-1} - [\,e\,]^{-1}\right| = m \geq \mathrm{m}(G) - \gamma(G).$$

Suppose next that e is an edge of M in some even cycle. Then by (II) and (IV), if e is the first edge in the even cycle, then

$$\left|[\,e^-\,]^{-1} - [\,e\,]^{-1}\right| \geq \left|[\,e^+\,]^{-1} - [\,e\,]^{-1}\right| = m \geq \mathrm{m}(G) - \gamma(G);$$

otherwise (note that, in this case, $\gamma(G) = 1$),

$$\left|[\,e^+\,]^{-1} - [\,e\,]^{-1}\right| > \left|[\,e^-\,]^{-1} - [\,e\,]^{-1}\right| = m - 1 = \mathrm{m}(G) - \gamma(G).$$

Suppose finally that e is an edge in some odd cycle such that $e, e^+ \notin M$ (i.e., $[\,e\,]^{-1} = 2m + r$ for some r with $1 \leq r \leq o(G)$). Then by (I) and (V), we have

$$\left|[\,e^-\,]^{-1} - [\,e\,]^{-1}\right| > \left|[\,e^+\,]^{-1} - [\,e\,]^{-1}\right| = m \geq \mathrm{m}(G) - \gamma(G).$$

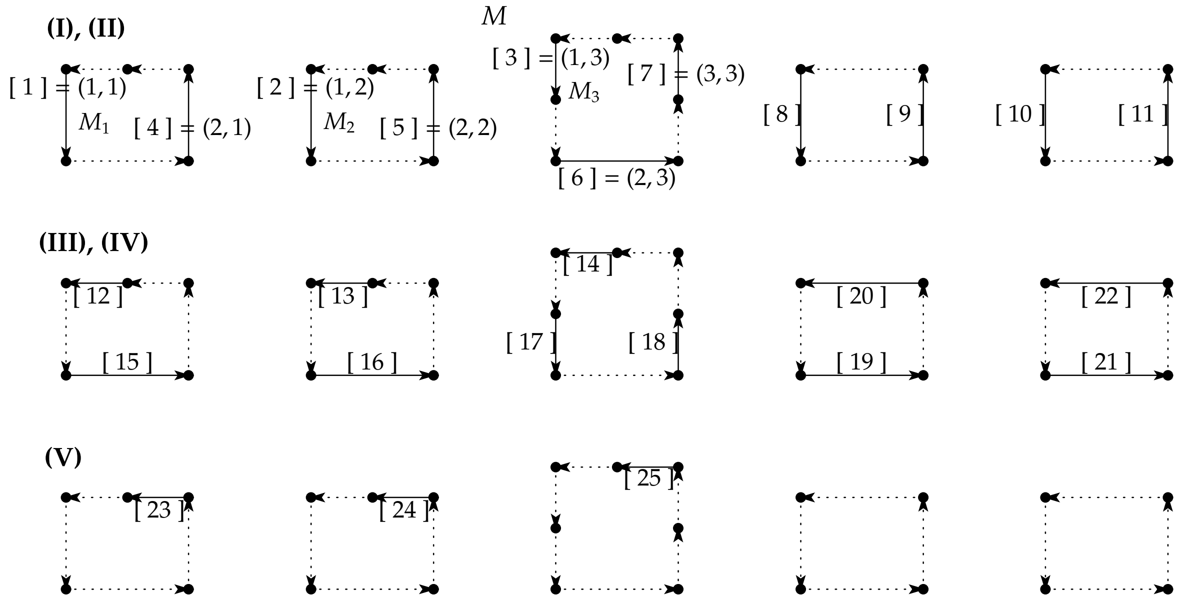

Figure 1: The definition of the map in the 2-regular graph G

This implies that for any two adjacent edges e_1 and e_2 in G, the inequality $\left|[\ e_1\ ]^{-1} - [\ e_2\ ]^{-1}\right| \geq \mathrm{m}(G) - \gamma(G)$ holds. Thus f is a map with sequential $(\mathrm{m}(G) - \gamma(G))$-matching of G. $\square$

Proof of Corollary 3.3. Let G be a 3-regular multigraph of order $2n$ which has a perfect matching M. Then, since $H = G - M$ is a spanning 2-regular subgraph of G, it follows from Proposition 3.2 that $\mathrm{ms}(H) = \mathrm{m}(H) - \gamma(H) = \frac{2n - o(H) - 2\gamma(H)}{2}$. Hence by Theorem 1.2, it suffices to show that $\frac{2n-o(H)-2\gamma(H)}{2} \geq \frac{n}{2}$ (because if it is true, then by Theorem 1.2, $\mathrm{ms}(G) = \mathrm{ms}(H + M) \geq \frac{n}{2} = \frac{1}{2}|M|$). Suppose first that $\gamma(H) = 0$ (i.e., H contains no even cycle of order at least 4). Then, since $o(H) \leq \frac{2n}{3}$, it follows that $\frac{2n-o(H)-2\gamma(H)}{2} - \frac{n}{2} = \frac{n-o(H)}{2} \geq \frac{n-(2n/3)}{2} > 0$. Suppose next that $\gamma(H) = 1$ (i.e., H contains an even cycle of order at least 4). Then, since $o(H) \leq \frac{2n-4}{3}$ and $n \geq 2$, it follows that $\frac{2n-o(H)-2\gamma(H)}{2} - \frac{n}{2} = \frac{n-o(H)-2}{2} \geq \frac{n-\left((2n-4)/3\right)-2}{2} = \frac{n-2}{6} \geq 0$. Thus the inequality $\frac{2n-o(H)-2\gamma(H)}{2} \geq \frac{n}{2}$ holds. $\square$

4. The Matching Sequencibility and the kth Power of Hamilton Paths in Graphs

In this section, we discuss the relationship between the matching sequencibility and Seymour's conjecture [13] concerning the existence of the kth power of a Hamilton cycle.

Let P be a path (or a cycle). The kth *power* of P is the graph obtained from P by joining every pair of vertices with distance at most k in P. We call the 2nd power of P a *square-path* (*-cycle*).

A well known theorem of Dirac [4] states that if H is a graph on n vertices with $\delta(H) \geq \frac{n}{2}$, then G contains a Hamilton cycle, where $\delta(H)$ denotes the minimum degree of H. In 1963, Posá conjectured that if H is a graph on n vertices with $\delta(H) \geq \frac{2}{3}n$, then H contains a Hamilton square-cycle (see Erdős [5]). In 1973, Seymour [13] proposed the more general conjecture as follows.

Conjecture 4.1 (Seymour [13]). *Let k, n be integers with $k \geq 2$ and $n \geq 3$. If H is a graph of order n such that $\delta(H) \geq \frac{(k-1)n}{k}$, then H contains the $(k-1)$th power of a Hamilton cycle.*

The motivation for Conjecture 4.1 comes from the conjecture of Erdős [5] stating that every graph G on n vertices with $\Delta(G) \leq r$ has an equitable $(r + 1)$-vertex-coloring. This Erdős' conjecture has been already settled by Hajnal and Szemerédi [7]. In fact, Hajnal and Szemerédi showed that the minimum degree condition of Conjecture 4.1 guarantees the existence of a spanning subgraph in which each component is isomorphic to the complete graph K_k for graphs of order divisible by k, and it follows from the result that Erdős' conjecture is true (see [7] for more details). However, Conjecture 4.1 is still open (it is known that Conjecture 4.1 is true for sufficiently large graphs, see [10] for more details).

On the other hand, by the definitions of the matching sequencibility and the k-th power of a Hamilton path, we can easily get the following relation. Here, for a graph G, we denote by $L(G)$ and $\overline{G}$ a line graph and a complement of G, respectively. A vertex subset of a graph G is called an *independent vertex set* of G if no two vertices in the set are adjacent.

Lemma 4.2. *Let G be a graph. Then $\mathrm{ms}(G) \geq k$ if and only if $\overline{L(G)}$ contains the $(k-1)$th power of a Hamilton path.*

Proof of Lemma 4.2. Since $E(G) = V(L(G))$, it follows from the definition of the matching sequencibility that $\mathrm{ms}(G) \geq k$ if and only if there exists a bijection $f' : \mathbf{N}_{|V(L(G))|} \to V(L(G))$ such that $\left\{f'(l), \ldots, f'(l+k-1)\right\}$ is an independent vertex set of $L(G)$ for $1 \leq l \leq |V(L(G))| - k + 1$, that is, $\overline{L(G)}\left[\{f'(l), \ldots, f'(l+k-1)\}\right]$ is isomorphic to a complete graph of order k for $1 \leq l \leq |V(L(G))| - k + 1$, which is true if and only if the $(k-1)$th power of $f'(1)f'(2)\ldots f'(|V(L(G))|)$ is contained in $\overline{L(G)}$. □

Now we propose the following conjecture, which is a sufficient condition to guarantee that the graph G satisfies $\mathrm{ms}(G) \geq k$. An *edge-degree* of an edge e in a graph G is defined as the number of edges adjacent with e in G. Hence the edge-degree of e in G corresponds to the vertex-degree of e in its line graph. We denote by $\Delta'(G)$ the maximum edge-degree of G.

Conjecture 4.3. *Let k, q be integers with $q \geq k \geq 2$. If G is a graph of size q such that $\Delta'(G) \leq \frac{q+1}{k} - 1$, then $\mathrm{ms}(G) \geq k$.*

By using Lemma 4.2, we show that the above conjecture is related to the following "Hamilton path version" of Conjecture 4.1 (Conjectures 4.4 and 4.5). Note that Conjecture 4.4 implies Conjecture 4.5. Note also that Conjecture 4.1 implies Conjecture 4.4 because if H is a graph of order n such that $\delta(H) \geq \frac{(k-1)n-1}{k}$, then the graph H^* obtained from H by adding a new vertex v and joining v to all vertices in H satisfies $|V(H^*)| = n+1$ and $\delta(H^*) \geq \frac{(k-1)(n+1)}{k}$, and further if H^* contains the $(k-1)$th power of a Hamilton cycle, then $H^* - v$ contains the $(k-1)$th power of a Hamilton path.

Conjecture 4.4. *Let k, n be integers with $n \geq k \geq 2$. If H is a graph of order n such that $\delta(H) \geq \frac{(k-1)n-1}{k}$, then H contains the $(k-1)$th power of a Hamilton path.*

Conjecture 4.5. *Let k, n be integers with $n \geq k \geq 2$. If H is a graph of order n such that $H = \overline{L(G)}$ for some graph G, and $\delta(H) \geq \frac{(k-1)n-1}{k}$, then H contains the $(k-1)$th power of a Hamilton path.*

Proposition 4.6. *Conjectures 4.3 and 4.5 are equivalent.*

Proof of Proposition 4.6. By the relationship between a graph and its line graph, $|E(G)| = |V(L(G))|$ $(= |V(\overline{L(G)})|)$ and $\Delta'(G) = \Delta(L(G))$ hold for a graph G. Moreover, by the relationship between a graph and its complement, $\delta(\overline{G}) + \Delta(G) + 1 = |V(G)|$ holds for a graph G. This in particular implies that for a graph G, $\Delta'(G) \leq \frac{|E(G)|+1}{k} - 1$ if and only if $\delta(\overline{L(G)}) \geq \frac{(k-1)|V(L(G))|-1}{k}$. Combining this with Lemma 4.2, we can easily see that Conjectures 4.3 and 4.5 are equivalent. □

We now see the following situation:

$$\text{Conjecture 4.1} \Longrightarrow \text{Conjecture 4.4} \Longrightarrow \boxed{\text{Conjecture 4.5} \overset{\text{Proposition 4.6}}{\Longleftrightarrow} \text{Conjecture 4.3}}$$

It is known that Conjecture 4.4 is true for $k = 3$ by Fan and Kierstead [6]. Combining this with Dirac's result [4] (every graph on n vertices with minimum degree at least $\frac{n-1}{2}$ has a Hamilton path), we see that Conjecture 4.3 is also true for $k = 2, 3$. Moreover, as mentioned in the paragraph following Conjecture 4.1, Komlós, Sárközy and Szemerédi [10] proved that for any integer $k \geq 2$ and a real number $\varepsilon > 0$, there is a constant $n_0 = n_0(k, \varepsilon)$ such that, if H is a graph of order $n \geq n_0$ with $\delta(H) \geq \left(\frac{k-1}{k} + \varepsilon\right)n$, then H contains the $(k-1)$th power of a Hamilton cycle. Hence by the same argument as in the proof of Proposition 4.6, we see that the following holds.

Corollary 4.7. *For any integer $k \geq 2$ and a real number $\varepsilon > 0$, there is a constant $q_0 = q_0(k, \varepsilon)$ such that, if G is a graph of size $q \geq q_0$ such that $\Delta'(G) \leq \left(\frac{1}{k} - \varepsilon\right)q - 1$, then $\mathrm{ms}(G) \geq k$.*

Proof of Corollary 4.7. Let k be an integer with $k \geq 2$ and ε be a real number with $\varepsilon > 0$. Then by the result of Komlós et al. [10], there is a constant $q_0 = q_0(k, \varepsilon)$ such that,

$$\begin{array}{c}\text{if } H \text{ is a graph of order } q \geq q_0 \text{ such that } \delta(H) \geq \left(\frac{k-1}{k} + \varepsilon\right)q, \\ \text{then } H \text{ contains the } (k-1)\text{th power of a Hamilton cycle.}\end{array} \tag{3}$$

Let G be a graph of size $q \geq q_0$ such that $\Delta'(G) \leq \left(\frac{1}{k} - \varepsilon\right)q - 1$. Then $H = \overline{L(G)}$ is a graph of order q such that $\delta(H) = \delta(\overline{L(G)}) = |V(L(G))| - (\Delta(L(G)) + 1) = q - (\Delta'(G) + 1) \geq q - \left(\frac{1}{k} - \varepsilon\right)q = \left(\frac{k-1}{k} + \varepsilon\right)q$. Therefore by (3), H contains the $(k-1)$th power of a Hamilton cycle, and hence by Lemma 4.2, we have $\mathrm{ms}(G) \geq k$. $\square$

Finally, we state Conjecture 4.3 in the version of analogue of Theorem 1.1.

Conjecture 4.8. *Every graph G of size q and maximum edge-degree Δ' (≥ 1) satisfies*

$$\mathrm{ms}(G) \geq \left\lfloor \frac{q+1}{\Delta' + 1} \right\rfloor.$$

Note that for any positive integers Δ' and q, $\Delta' = \left(\frac{\Delta'+1}{q+1}\right)(q+1) - 1 \leq \frac{q+1}{\lfloor \frac{q+1}{\Delta'+1} \rfloor} - 1$ holds, and this implies that any graph G of size q and maximum edge-degree Δ' satisfies $\mathrm{ms}(G) \geq \left\lfloor \frac{q+1}{\Delta'+1} \right\rfloor$ if Conjecture 4.3 is true. Therefore, Conjecture 4.3 implies Conjecture 4.8. On the other hand, if $\Delta' \leq \frac{q+1}{k} - 1$, then $q + 1 \geq k(\Delta' + 1)$ holds, and hence it is easy to see that Conjecture 4.8 implies Conjecture 4.3. Combining this with Proposition 4.6, we get the following relation.

Proposition 4.9. *Conjectures 4.3, 4.5 and 4.8 are all equivalent.*

In Conjecture 4.8, the lower bound is best possible in the following sense if it's true. Let k, t be integers with $k \geq 2$ and $t \geq 1$, and let $G = kK_{1,t+1}$, i.e., G is the union of k vertex-disjoint copies of the complete bipartite graph $K_{1,t+1}$. Then $\Delta'(G) = t$ and $|E(G)| = k(t+1)$, and it is easy to see that $\mathrm{ms}(G) = k = \left\lfloor \frac{k(t+1)+1}{t+1} \right\rfloor = \left\lfloor \frac{|E(G)|+1}{\Delta'(G)+1} \right\rfloor$.

Since $\Delta(G) - 1 \leq \Delta'(G) \leq 2\Delta(G) - 2$ for a graph G, the lower bound of Theorem 1.1 is close to that of Conjecture 4.8 as $\Delta'(G)$ approaches to $2\Delta(G) - 2$ (e.g., it is closest for regular graphs), and see also Table 1.

	$\Delta' = \Delta - 1$	$\Delta' = \Delta$	$\cdots$	$\Delta' = 2\Delta - 3$	$\Delta' = 2\Delta - 2$
Theorem 1.1	$\mathrm{ms}(G) \geq \frac{1}{2}\left\lfloor \frac{q}{\Delta+1} \right\rfloor$	$\mathrm{ms}(G) \geq \frac{1}{2}\left\lfloor \frac{q}{\Delta+1} \right\rfloor$	$\cdots$	$\mathrm{ms}(G) \geq \frac{1}{2}\left\lfloor \frac{q}{\Delta+1} \right\rfloor$	$\mathrm{ms}(G) \geq \frac{1}{2}\left\lfloor \frac{q}{\Delta+1} \right\rfloor$
Conjecture 4.8	$\mathrm{ms}(G) \geq \left\lfloor \frac{q+1}{\Delta} \right\rfloor$	$\mathrm{ms}(G) \geq \left\lfloor \frac{q+1}{\Delta+1} \right\rfloor$	$\cdots$	$\mathrm{ms}(G) \geq \left\lfloor \frac{q+1}{2\Delta-2} \right\rfloor$	$\mathrm{ms}(G) \geq \left\lfloor \frac{q+1}{2\Delta-1} \right\rfloor$

Table 1: Comparison of Theorem 1.1 and Conjecture 4.8

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