On Some Mean Value Results for the Zeta-Function and a Divisor Problem

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Abstract. Let $\Delta(x)$ denote the error term in the classical Dirichlet divisor problem, and let the modified error term in the divisor problem be $\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x)$. We show that

$$\int_T^{T+H} \Delta^*\left(\frac{t}{2\pi}\right)|\zeta(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t \ \ll \ HT^{1/6}\log^{7/2} T \quad (T^{2/3+\varepsilon} \leqslant H = H(T) \leqslant T),$$

$$\int_0^T \Delta(t)|\zeta(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t \ \ll \ T^{9/8}(\log T)^{5/2},$$

and obtain asymptotic formulae for

$$\int_0^T \left(\Delta^*\left(\frac{t}{2\pi}\right)\right)^2 |\zeta(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t, \qquad \int_0^T \left(\Delta^*\left(\frac{t}{2\pi}\right)\right)^3 |\zeta(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t.$$

The importance of the Δ^*-function comes from the fact that it is the analogue of $E(T)$, the error term in the mean square formula for $|\zeta(\tfrac{1}{2} + it)|^2$. We also show, if $E^*(T) = E(T) - 2\pi\Delta^*(T/(2\pi))$,

$$\int_0^T E^*(t)E^j(t)|\zeta(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t \ \ll_{j,\varepsilon} \ T^{7/6+j/4+\varepsilon} \quad (j = 1, 2, 3).$$

1. Introduction

As usual, let

$$\Delta(x) := \sum_{n \leqslant x} d(n) - x(\log x + 2\gamma - 1) \qquad (x \geqslant 2) \tag{1.1}$$

denote the error term in the classical Dirichlet divisor problem. Also let

$$E(T) := \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \,\mathrm{d}t - T\left(\log\left(\tfrac{T}{2\pi}\right) + 2\gamma - 1\right) \qquad (T \geqslant 2) \tag{1.2}$$

denote the error term in the mean square formula for $|\zeta(\frac{1}{2} + it)|$. Here $d(n)$ is the number of all positive divisors of n, $\zeta(s)$ is the Riemann zeta-function, and $\gamma = -\Gamma'(1) = 0.577215\ldots$ is Euler's constant. Long ago F.V. Atkinson [1] established a fundamental explicit formula for $E(T)$ (see also [5, Chapter 15] and [7, Chapter 2]), which indicated a certain analogy between $\Delta(x)$ and $E(T)$. However, in this context it seems that instead of the error-term function $\Delta(x)$ it is more exact to work with the modified function $\Delta^*(x)$ (see M. Jutila [12], [13] and T. Meurman [14]), where

$$\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \tfrac{1}{2}\Delta(4x) = \tfrac{1}{2}\sum_{n\leqslant 4x}(-1)^n d(n) - x(\log x + 2\gamma - 1), \tag{1.3}$$

since it turns out that $\Delta^*(x)$ is a better analogue of $E(T)$ than $\Delta(x)$. Namely, M. Jutila (op. cit.) investigated both the local and global behaviour of the difference

$$E^*(t) := E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right), \tag{1.4}$$

and in particular in [13] he proved that

$$\int_T^{T+H}(E^*(t))^2\,\mathrm{d}t \ll_\varepsilon HT^{1/3}\log^3 T + T^{1+\varepsilon} \qquad (1 \leqslant H \leqslant T). \tag{1.5}$$

Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a(x) \ll_\varepsilon b(x)$ (same as $a(x) = O_\varepsilon(b(x))$) means that the $|a(x)| \leqslant Cb(x)$ for some $C = C(\varepsilon) > 0, x \geqslant x_0$. The significance of (1.5) is that, in view of (see e.g., [5, Chapter 15])

$$\int_0^T(\Delta^*(t))^2\,\mathrm{d}t \sim AT^{3/2}, \qquad \int_0^T E^2(t)\,\mathrm{d}t \sim BT^{3/2} \quad (A, B > 0,\ T \to \infty), \tag{1.6}$$

it transpires that $E^*(t)$ is in the mean square sense of a lower order of magnitude than either $\Delta^*(t)$ or $E(t)$. We also refer the reader to the review paper [18] of K.-M. Tsang on this subject.

2. Statement of Results

Mean values (or moments) of $|\zeta(\frac{1}{2} + it)|$ represent one of the central themes in the theory of $\zeta(s)$, and they have been studied extensively. There are two monographs dedicated solely to them: the author's [7], and that of K. Ramachandra [17]. We are interested in obtaining mean value results for $\Delta^*(t)$ and $|\zeta(\frac{1}{2} + it)|^2$, namely how the quantities in question relate to one another. Our results are as follows.

Theorem 1. *For $T^{2/3+\varepsilon} \leqslant H = H(T) \leqslant T$ we have*

$$\int_T^{T+H}\Delta^*\left(\frac{t}{2\pi}\right)|\zeta(\tfrac{1}{2} + it)|^2\,\mathrm{d}t \ll HT^{1/6}\log^{7/2} T. \tag{2.1}$$

Remark 1. If one uses the first formula in (1.6), the classical bound (see e.g., [5, Chapter 4])

$$\int_0^T|\zeta(\tfrac{1}{2} + it)|^4\,\mathrm{d}t \ll T\log^4 T \tag{2.2}$$

and the Cauchy-Schwarz inequality for integrals, one obtains

$$\int_0^T\Delta^*\left(\frac{t}{2\pi}\right)|\zeta(\tfrac{1}{2} + it)|^2\,\mathrm{d}t \ll T^{5/4}\log^2 T,$$

which is considerably poorer than (2.1) for $H = T$, thus showing that this bound of Theorem 1 is non-trivial.

Theorem 2. *If γ is Euler's constant and*

$$C := \frac{2\zeta^4(3/2)}{3\sqrt{2\pi}\zeta(3)} = \frac{2}{3\sqrt{2\pi}} \sum_{n=1}^{\infty} d^2(n)n^{-3/2} = 10.3047\ldots, \tag{2.3}$$

then

$$\int_0^T \left(\Delta^*\!\left(\frac{t}{2\pi}\right)\right)^2 |\zeta(\tfrac{1}{2}+it)|^2 \, \mathrm{d}t = \frac{C}{4\pi^2} T^{3/2}\!\left(\log\frac{T}{2\pi} + 2\gamma - \frac{2}{3}\right) + O_\varepsilon(T^{17/12+\varepsilon}). \tag{2.4}$$

Remark 2. Note that (2.4) is a true asymptotic formula (17/12 = 3/2 - 1/12). It would be interesting to analyze the error term in (2.4) and see how small it can be, i.e., to obtain an omega-result (recall that $f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \to \infty$).

Theorem 3. *For some explicit constant $D > 0$ we have*

$$\int_0^T \left(\Delta^*\!\left(\frac{t}{2\pi}\right)\right)^3 |\zeta(\tfrac{1}{2}+it)|^2 \, \mathrm{d}t = DT^{7/4}\!\left(\log\frac{T}{2\pi} + 2\gamma - \frac{4}{7}\right) + O_\varepsilon(T^{27/16+\varepsilon}). \tag{2.5}$$

Remark 3. Like (2.4), the formula in (2.5) is also a true asymptotic formula (27/16 = 7/4 - 1/16). Moreover, the main term is positive, which shows that, in the mean, $\Delta^*\!\left(\frac{t}{2\pi}\right)$ is more biased towards positive values.

In the most interesting case when $H = T$, Theorem 1 can be improved. Indeed, we have

Theorem 4. *We have*

$$\int_0^T \Delta(t)|\zeta(\tfrac{1}{2}+it)|^2 \, \mathrm{d}t \ll T^{9/8}(\log T)^{5/2}, \tag{2.6}$$

and (2.6) remains true if $\Delta(t)$ is replaced by $\Delta^*(t)$, $\Delta(t/(2\pi))$ or $\Delta^*(t/(2\pi))$.

Remark 3. The presence of $\Delta^*\!\left(\frac{t}{2\pi}\right)$ instead of the more natural $\Delta^*(t)$ in (2.1), (2.4) and (2.5) comes from the defining relation (1.4). It would be interesting to see what could be proved if in the integrals in (2.4) and (2.5) one had $\Delta^*(t)$ (or $\Delta(t)$) instead of of $\Delta^*\!\left(\frac{t}{2\pi}\right)$.

Remark 4. In the case of (2.1) (when $H = T$), Theorem 4 answers this question. However, obtaining a short interval result for $\Delta(t)|\zeta(\tfrac{1}{2}+it)|^2$ is not easy. The method of proof of Theorem 4 cannot be easily adapted to yield the analogues of (2.4) and (2.5) for $\Delta(t)$ in place of $\Delta^*(t/(2\pi))$.

There are some other integrals which may be bounded by the method used to prove previous theorems. For example, one such result is

Theorem 5. *For $j = 1, 2, 3$ we have*

$$\int_0^T E^*(t)E^j(t)|\zeta(\tfrac{1}{2}+it)|^2 \, \mathrm{d}t \ll_{j,\varepsilon} T^{7/6+j/4+\varepsilon}. \tag{2.7}$$

3. The Necessary Lemmas

In this section we shall state some lemmas needed for the proof of our theorems. The proofs of the theorems themselves will be given in Section 4.

The first lemma embodies some bounds for the higher moments of $E^*(T)$.

Lemma 1. *We have*

$$\int_0^T |E^*(t)|^3 \, \mathrm{d}t \ll_\varepsilon T^{3/2+\varepsilon}, \tag{3.1}$$

$$\int_0^T |E^*(t)|^5 \, dt \ll_\varepsilon T^{2+\varepsilon}, \tag{3.2}$$

and also

$$\int_0^T (E^*(t))^4 \, dt \ll_\varepsilon T^{7/4+\varepsilon}. \tag{3.3}$$

The author proved (3.1) in [8, Part IV], and (3.2) in [8, Part II]. The bound (3.3) follows from (3.1) and (3.2) by the Cauchy-Schwarz inequality for integrals.

For the mean square of $E(t)$ we need a more precise formula than (1.6). This is

Lemma 2. *With C given by (2.3) we have*

$$\int_0^T E^2(t) \, dt = CT^{3/2} + R(T), \quad R(T) = O(T \log^4 T). \tag{3.4}$$

The first result on $R(T)$ is due to D.R. Heath-Brown [2], who obtained $R(T) = O(T^{5/4} \log^2 T)$. The sharpest known result at present is $R(T) = O(T \log^4 T)$, due independently to E. Preissmann [16] and the author [7, Chapter 2].

For the mean square of $E^*(t)$ we have a result which is different from (3.4). This is

Lemma 3. *We have*

$$\int_0^T (E^*(t))^2 \, dt = T^{4/3} P_3(\log T) + O_\varepsilon(T^{7/6+\varepsilon}), \tag{3.5}$$

where $P_3(y)$ is a polynomial of degree three in y with positive leading coefficient, and all its coefficients may be evaluated explicitly.

This formula was proved by the author in [9]. It sharpens (1.4) when $H = T$. It seems likely that the error term in (3.5) is $O_\varepsilon(T^{1+\varepsilon})$, but this seems difficult to prove.

Lemma 4. *We have*

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \, dt = T Q_4(\log T) + O(T^{2/3} \log^8 T), \tag{3.6}$$

where $Q_4(x)$ is an explicit polynomial of degree four in x with leading coefficient $1/(2\pi^2)$.

This result was proved first (with error term $O(T^{2/3} \log^C T)$) by Y. Motohashi and the author [10]. The value $C = 8$ was given by Y. Motohashi in his monograph [15].

Lemma 5. *For $1 \leqslant N \ll x$ we have*

$$\Delta^*(x) = \frac{1}{\pi \sqrt{2}} x^{\frac{1}{4}} \sum_{n \leqslant N} (-1)^n d(n) n^{-\frac{3}{4}} \cos(4\pi \sqrt{nx} - \tfrac{1}{4}\pi) + O_\varepsilon(x^{\frac{1}{2}+\varepsilon} N^{-\frac{1}{2}}). \tag{3.7}$$

The expression for $\Delta^*(x)$ (see [5, Chapter 15]) is the analogue of the classical truncated Voronoï formula for $\Delta(x)$ (ibid. Chapter 3), which is the expression in (3.7) without $(-1)^n$.

Lemma 6. *We have*

$$\int_0^T E(t)|\zeta(\tfrac{1}{2} + it)|^2 \, dt = \pi T\left(\log \tfrac{T}{2\pi} + 2\gamma - 1\right) + U(T), \tag{3.8}$$

where

$$U(T) = O(T^{3/4} \log T), \quad U(T) = \Omega_\pm(T^{3/4} \log T).$$

The asymptotic formula (3.8) is due to the author [6]. Here the symbol $f(x) = \Omega_{\pm}(g(x))$ has its standard meaning, namely that both $\limsup_{x\to\infty} f(x)/g(x) > 0$ and $\liminf_{x\to\infty} f(x)/g(x) < 0$ holds.

Lemma 7. *We have*

$$\int_1^T E^3(t)\,\mathrm{d}t = C_1 T^{7/4} + O_\varepsilon(T^{5/3+\varepsilon}), \tag{3.9}$$

$$\int_1^T E^4(t)\,\mathrm{d}t = C_2 T^2 + O_\varepsilon(T^{23/12+\varepsilon}), \tag{3.10}$$

where C_1, C_2 are certain explicit, positive constants.

These asymptotic formulae are due to P. Sargos and the author [11].

Lemma 8. *We have*

$$\sum_{n\leqslant x} d^2(n) = \frac{1}{\pi^2} x \log^3 x + O(x \log^2 x). \tag{3.11}$$

This is a well-known elementary formula; see e.g., page 141 of [5].

Lemma 9. *For real $k \in [0, 9]$ the limits*

$$E_k := \lim_{T\to\infty} T^{-1-k/4} \int_0^T |E(t)|^k\,\mathrm{d}t$$

exist.

This is a result of D.R. Heath-Brown [4]. The limits of moments without absolute values also exist when $k = 1, 3, 5, 7$ or 9.

Lemma 10. *For $4 \leqslant A \leqslant 12$ we have*

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^A\,\mathrm{d}t \ll_A T^{1+\frac{1}{8}(A-4)} \log^{C(A)} T \tag{3.12}$$

with some positive constant $C(A)$.

These are at present the strongest upper bounds for moments of $|\zeta(\tfrac{1}{2}+it)|$ for the range in question. They follow by convexity from the fourth moment bound (2.2) and the twelfth moment

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{12}\,\mathrm{d}t \ll T^2 \log^{17} T$$

of D.R. Heath-Brown [3] (see e.g., [5, Chapter 8] for more details).

4. Proofs of the Theorems

We begin with the proof of (2.1). We start from

$$\int_T^{T+H} E^*(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t \ll \left\{\int_T^{T+H} (E^*(t))^2\,\mathrm{d}t \int_T^{T+H} |\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t\right\}^{1/2}$$
$$\ll \left(HT^{1/3}\log^3 T \cdot H \log^4 T\right)^{1/2} = HT^{1/6}\log^{7/2} T. \tag{4.1}$$

Here we assumed that $T^{2/3+\varepsilon} \leqslant H = H(T) \leqslant T$ and used (3.6) of Lemma 4, (1.5) and the Cauchy-Schwarz inequality for integrals. On the other hand, by the defining relation (1.4) we have

$$
\begin{aligned}
\int_T^{T+H} E^*(t)|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t &= \int_T^{T+H} E(t)|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t \\
&\quad - 2\pi \int_T^{T+H} \Delta^*\!\left(\frac{t}{2\pi}\right)|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t.
\end{aligned}
\tag{4.2}
$$

Using (3.8) of Lemma 6 and (4.1), we obtain then from (4.2)

$$
\begin{aligned}
2\pi &\int_T^{T+H} \Delta^*\!\left(\frac{t}{2\pi}\right)|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t \\
&= \int_T^{T+H} E^*(t)|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t - \int_T^{T+H} E(t)|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t \\
&= O(HT^{1/6}\log^{7/2} T) + \pi t\left(\log \frac{t}{2\pi} + 2\gamma - 1\right)\Big|_T^{T+H} + O(T^{3/4}\log T) \\
&= O(HT^{1/6}\log^{7/2} T) + O(H\log T) + O(T^{3/4}\log T) \\
&\ll HT^{1/6}\log^{7/2} T,
\end{aligned}
$$

since $T^{2/3+\varepsilon} \leqslant H = H(T) \leqslant T$. This completes the proof of Theorem 1.

The proof of Theorem 2 is somewhat more involved. It suffices to consider the integral from T to $2T$, and then at the end of the proof to replace T by $T2^{-j}$ and sum the resulting expressions when $j = 1, 2, \ldots$. First, by squaring (1.4), we have

$$
\begin{aligned}
\int_T^{2T} (E^*(t))^2|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t &= \int_T^{2T} (E(t))^2|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t \\
&\quad - 2\int_T^{2T} E(t)2\pi\Delta^*\!\left(\frac{t}{2\pi}\right)|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t + 4\pi^2 \int_T^{2T} \left(\Delta^*\!\left(\frac{t}{2\pi}\right)\right)^2|\zeta(\tfrac{1}{2} + it)|^2 \, \mathrm{d}t.
\end{aligned}
\tag{4.3}
$$

The expression in the middle of the right-hand side of (4.3) equals, on differentiating (1.2),

$$
\begin{aligned}
&- 2\int_T^{2T} E(t)2\pi\Delta^*\!\left(\frac{t}{2\pi}\right)\left(\log \frac{t}{2\pi} + 2\gamma + E'(t)\right) \mathrm{d}t \\
&= -2\int_T^{2T} E(t)2\pi\Delta^*\!\left(\frac{t}{2\pi}\right)\left(\log \frac{t}{2\pi} + 2\gamma\right) \mathrm{d}t + J(T),
\end{aligned}
\tag{4.4}
$$

say, where

$$
J(T) := -2\int_T^{2T} E(t)2\pi\Delta^*\!\left(\frac{t}{2\pi}\right)E'(t) \, \mathrm{d}t.
\tag{4.5}
$$

To bound $J(T)$ we use Lemma 5 with $N = N(T), 1 \ll N \ll T$, where N will be determined a little later. The error term in (3.7) trivially makes a contribution which is

$$
\ll_\varepsilon \int_T^{2T} |E(t)|\left(|\zeta(\tfrac{1}{2} + it)|^2 + \log T\right)T^{1/2+\varepsilon}N^{-1/2} \ll_\varepsilon T^{7/4+\varepsilon}N^{-1/2}
\tag{4.6}
$$

on using the second formula in (1.6), (2.2) and the Cauchy-Schwarz inequality for integrals. There remains the contribution of a multiple of

$$
\int_T^{2T} \left(E^2(t)\right)' t^{1/4} \sum_{n \leqslant N} (-1)^n d(n)n^{-3/4}\cos(\sqrt{8\pi nt} - \pi/4) \, \mathrm{d}t.
$$

This is integrated by parts. The integrated terms are $\ll T^{11/12}N^{1/3}\log T$, by using the standard estimate $E(T) \ll T^{1/3}$ (see e.g., [5, Chapter 15]) and trivial estimation. The main contribution comes from the differentiation of the sum over n. Its contribution will be, with $n \sim K$ meaning that $K < n \leqslant K' \leqslant 2K$,

$$
\ll T^{-1/4} \int_T^{2T} E^2(t) \Big| \sum_{n \leqslant N} (-1)^n d(n) n^{-1/4} \exp(i\sqrt{8\pi n t}) \Big| \, dt
$$

$$
\leqslant T^{-1/4} \left\{ \int_T^{2T} E^4(t) \, dt \int_T^{2T} \Big| \sum_{n \leqslant N} (-1)^n d(n) n^{-1/4} \exp(i\sqrt{8\pi n t}) \Big|^2 \, dt \right\}^{1/2} \tag{4.7}
$$

$$
\ll T^{3/4} \left(\int_T^{2T} \sum_{n \leqslant N} d^2(n) n^{-1/2} \, dt + \log^2 T \max_{K \ll N} \sum_{m \neq n \sim K} K^{\varepsilon - 1/2} \frac{\sqrt{T}}{|\sqrt{m} - \sqrt{n}|} \right)^{1/2}
$$

$$
\ll T^{3/4} (TN^{1/2} \log^3 T + T^{1/2+\varepsilon} N)^{1/2} \ll T^{5/4} N^{1/4} \log^{3/2} T
$$

for $T^\varepsilon \leqslant N = N(T) \leqslant T^{1-\varepsilon}$. Here we used the standard first derivative test (see e.g., Lemma 2.1 of [6]) for exponential integrals, Lemma 7, (3.10) and

$$
\sum_{m \neq n \sim K} \frac{1}{|\sqrt{m} - \sqrt{n}|} \ll \sum_{n \sim K} \sum_{m \sim K, m \neq n} \frac{\sqrt{K}}{|m - n|} \ll K^{3/2} \log K.
$$

From (4.6) and (4.7) we see that the right choice for N should be if we have

$$
T^{7/4} N^{-1/2} = T^{5/4} N^{1/4}, \quad N = T^{2/3},
$$

and with this choice of N we obtain $T^{11/12} N^{1/3} = T^{41/36}$ $(41/36 < 17/12)$, and

$$
J(T) \ll_\varepsilon T^{17/12+\varepsilon}. \tag{4.8}
$$

In view of (1.4), the formula (4.3) and the bound (4.8) give

$$
4\pi^2 \int_T^{2T} \left(\Delta^* \Big(\frac{t}{2\pi} \Big) \right)^2 |\zeta(\tfrac{1}{2} + it)|^2 \, dt = O_\varepsilon(T^{17/12+\varepsilon})
$$

$$
+ 2 \int_T^{2T} E(t) 2\pi \Delta^* \Big(\frac{t}{2\pi} \Big) \Big(\log \frac{t}{2\pi} + 2\gamma \Big) \, dt - \int_T^{2T} E^2(t) |\zeta(\tfrac{1}{2} + it)|^2 \, dt \tag{4.9}
$$

$$
+ \int_T^{2T} (E^*(t))^2 |\zeta(\tfrac{1}{2} + it)|^2 \, dt = O_\varepsilon(T^{17/12+\varepsilon}) + 2I_1 - I_2 + I_3,
$$

say. On using (3.3) of Lemma 1, (2.2) and the Cauchy-Schwarz inequality we obtain

$$
I_3 \ll \left\{ \int_T^{2T} (E^*(t))^4 \, dt \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^4 \, dt \right\}^{1/2} \ll_\varepsilon T^{11/8+\varepsilon}.
$$

Further we have

$$
2I_1 - I_2 = 2 \int_T^{2T} E(t) \Big(E(t) - E^*(t) \Big) \Big(\log \frac{t}{2\pi} + 2\gamma \Big) \, dt - I_2
$$

$$
= \int_T^{2T} E^2(t) \left\{ 2 \Big(\log \frac{t}{2\pi} + 2\gamma \Big) - |\zeta(\tfrac{1}{2} + it)|^2 \right\} \, dt
$$

$$
- 2 \int_T^{2T} E(t) E^*(t) \Big(\log \frac{t}{2\pi} + 2\gamma \Big) \, dt
$$

$$
= \int_T^{2T} E^2(t) \Big(\log \frac{t}{2\pi} + 2\gamma - E'(t) \Big) \, dt - 2 \int_T^{2T} E(t) E^*(t) \Big(\log \frac{t}{2\pi} + 2\gamma \Big) \, dt.
$$

The last integral is, by Lemma 2, Lemma 3 and the Cauchy-Schwarz inequality for integrals,

$$\ll \log T \Big\{ \int_T^{2T} E^2(t)\,\mathrm{d}t \int_T^{2T} (E^*(t))^2\,\mathrm{d}t \Big\}^{1/2} \ll T^{17/12} \log^{5/2} T.$$

On the other hand,

$$\begin{aligned}
&\int_T^{2T} E^2(t)\Big(\log \frac{t}{2\pi} + 2\gamma - E'(t)\Big)\,\mathrm{d}t \\
&= \int_T^{2T} E^2(t)\Big(\log \frac{t}{2\pi} + 2\gamma\Big)\,\mathrm{d}t - \tfrac{1}{3} E^3(t)\Big|_T^{2T} \\
&= \int_T^{2T} E^2(t)\Big(\log \frac{t}{2\pi} + 2\gamma\Big)\,\mathrm{d}t + O(T).
\end{aligned} \tag{4.10}$$

To evaluate the last integral in (4.10) we use Lemma 6 and integration by parts. This shows that the integral in question is

$$\begin{aligned}
&\Big(Ct^{3/2} + R(t)\Big)\Big(\log \frac{t}{2\pi} + 2\gamma\Big)\Big|_T^{2T} - \int_T^{2T} \Big(Ct^{1/2} + \frac{R(t)}{t}\Big)\,\mathrm{d}t \\
&= Ct^{3/2}\Big(\log \frac{t}{2\pi} + 2\gamma\Big)\Big|_T^{2T} + O(T\log^5 T) - \tfrac{2}{3} Ct^{3/2}\Big|_T^{2T} \\
&= Ct^{3/2}\Big(\log \frac{t}{2\pi} + 2\gamma - \frac{2}{3}\Big)\Big|_T^{2T} + O(T\log^5 T).
\end{aligned}$$

It transpires from (4.9) and (4.10) that

$$4\pi^2 \int_T^{2T} \Big(\Delta^*\big(\frac{t}{2\pi}\big)\Big)^2 |\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t = Ct^{3/2}\Big(\log \frac{t}{2\pi} + 2\gamma - \frac{2}{3}\Big)\Big|_T^{2T} + O_\varepsilon(T^{17/12+\varepsilon}),$$

which gives at once (2.4) of Theorem 2.

We turn now to the proof of Theorem 3. The basic idea is analogous to the one used in the proof of Theorem 2, so that we shall be relatively brief. The integral in (2.5) equals $1/(8\pi^3)$ times

$$\int_0^T \Big\{ E^3(t) - 3E^*(t)E^2(t) + 3(E^*(t))^2 E(t) - (E^*(t))^3 \Big\} |\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t. \tag{4.11}$$

The main term in (2.5) comes from

$$\begin{aligned}
\int_0^T E^3(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t &= \int_0^T E^3(t)\Big(\log \frac{t}{2\pi} + 2\gamma - E'(t)\Big)\,\mathrm{d}t \\
&= C_1 T^{7/4}\Big(\log \frac{T}{2\pi} + 2\gamma\Big) - \int_1^T C_1 t^{3/4}\,\mathrm{d}t + O_\varepsilon(T^{5/3+\varepsilon}) \\
&= C_1 T^{7/4}\Big(\log \frac{T}{2\pi} + 2\gamma - \frac{4}{7}\Big) + O_\varepsilon(T^{5/3+\varepsilon}),
\end{aligned}$$

where (3.9) of Lemma 7 was used. By Hölder's inequality for integrals, (3.3) of Lemma 1 and (3.12) of Lemma 10 (with $A = 5$) we obtain

$$\begin{aligned}
\int_0^T (E^*(t))^3 |\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t &\ll \Big(\int_0^T |E^*(t)|^5\,\mathrm{d}t \Big)^{3/5} \Big(\int_0^T |\zeta(\tfrac{1}{2}+it)|^5\,\mathrm{d}t \Big)^{2/5} \\
&\ll_\varepsilon T^{6/5+9/20+\varepsilon} = T^{33/20+\varepsilon}.
\end{aligned}$$

Similarly we obtain

$$\int_0^T (E^*(t))^2 E(t) |\zeta(\tfrac{1}{2} + it)|^2 \, dt$$

$$\ll \left(\int_0^T |E^*(t)|^{16/3} \, dt \right)^{3/8} \left(\int_0^T E^8(t) \, dt \right)^{1/8} \left(\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \, dt \right)^{1/2}$$

$$\ll_\varepsilon T^{\frac{3}{8}(2 + \frac{1}{9}) + \frac{3}{8} + \frac{1}{2} + \varepsilon} = T^{\frac{5}{3} + \varepsilon},$$

where we used (3.2) and the fact that $E^*(T) \ll T^{1/3}$, which follows from the definition of E^* and the classical estimates $\Delta(x) \ll x^{1/3}, E(T) \ll T^{1/3}$. Finally, by using (3.3), Lemma 9 with $k = 8$ and (2.2), we obtain

$$\int_0^T E^*(t) E^2(t) |\zeta(\tfrac{1}{2} + it)|^2 \, dt$$

$$\ll \left(\int_0^T |E^*(t)|^4 \, dt \right)^{1/4} \left(\int_0^T E^8(t) \, dt \right)^{1/4} \left(\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \, dt \right)^{1/2}$$

$$\ll_\varepsilon T^{7/16 + 3/4 + 1/2 + \varepsilon} = T^{27/16 + \varepsilon}.$$

Since $27/16 = 1.6875 > 5/3 > 33/20 = 1.65$, we obtain easily the assertion of Theorem 3.

We shall prove now (2.6) of Theorem 4. We suppose $T \leqslant t \leqslant 2T$ and take $N = T$ in (3.7) of Lemma 5. This holds both for $\Delta^*(x)$ and $\Delta(x)$, and one can see easily that the proof remains valid if we have an additional factor of $1/(2\pi)$ in the argument of Δ^* or Δ (or any constant $c > 0$, for that matter). Thus we start from

$$\Delta(t) = \frac{t^{1/4}}{\pi \sqrt{2}} \sum_{n \leqslant T} d(n) n^{-3/4} \cos(4\pi \sqrt{nt} - \pi/4) + O_\varepsilon(T^\varepsilon)$$

$$= \frac{t^{1/4}}{\pi \sqrt{2}} \left(\sum_{n \leqslant G} \cdots + \sum_{G < n \leqslant T} \cdots \right) + O_\varepsilon(T^\varepsilon), \tag{4.12}$$

say, where $T^\varepsilon \leqslant G = G(T) \leqslant T^{1-\varepsilon}$, and G will be determined a little later. The error term in (4.12) makes a contribution of $O_\varepsilon(T^{1+\varepsilon})$ to (2.6). We have

$$\int_T^{2T} t^{1/4} \sum_{n \leqslant G} d(n) n^{-3/4} \cos(4\pi \sqrt{nt} - \pi/4) |\zeta(\tfrac{1}{2} + it)|^2 \, dt$$

$$= \int_T^{2T} t^{1/4} \left(\log \frac{t}{2\pi} + 2\gamma + E'(t) \right) \sum_{n \leqslant G} d(n) n^{-3/4} \cos(4\pi \sqrt{nt} - \pi/4) \, dt \tag{4.13}$$

$$= I_1 + I_2,$$

say. By the first derivative test

$$I_1 := \int_T^{2T} t^{1/4} \left(\log \frac{t}{2\pi} + 2\gamma \right) \sum_{n \leqslant G} d(n) n^{-3/4} \cos(4\pi \sqrt{nt} - \pi/4) \, dt$$

$$\ll T^{1/4} \log T \cdot \sum_{n \leqslant G} d(n) n^{-3/4} T^{1/2} n^{-1/2} \ll T^{3/4} \log T,$$

since $\sum_{n \geqslant 1} d(n) n^{-\alpha}$ converges for $\alpha > 1$. The integral I_2, namely

$$I_2 := \int_T^{2T} t^{1/4} E'(t) \sum_{n \leqslant G} d(n) n^{-3/4} \cos(4\pi \sqrt{nt} - \pi/4) \, dt$$

is integrated by parts. The integrated terms are trivially $O(T)$, and there remains

$$
- \int_T^{2T} \frac{1}{4} t^{-3/4} E(t) \sum_{n \leqslant G} d(n) n^{-3/4} \cos(4\pi \sqrt{nt} - \pi/4) \, dt
$$

$$
+ 2\pi \int_T^{2T} t^{-1/4} E(t) \sum_{n \leqslant G} d(n) n^{-1/4} \sin(4\pi \sqrt{nt} - \pi/4) \, dt.
$$
(4.14)

Both integrals in (4.14) are estimated analogously, and clearly it is the latter which is larger. By the Cauchy-Schwarz inequality for integrals it is

$$
\ll T^{-1/4} (J_1 J_2)^{1/2},
$$

where

$$
J_1 := \int_T^{2T} \left| \sum_{n \leqslant G} d(n) n^{-1/4} e^{4\pi i \sqrt{nt}} \right|^2 \, dt
$$

$$
J_2 := \int_T^{2T} E^2(t) \, dt \ll T^{3/2},
$$

on using Lemma 2 in bounding J_2. Using the first derivative test and (3.11) of Lemma 8, we find that

$$
J_1 = T \sum_{n \leqslant G} d^2(n) n^{-1/2} + \sum_{m \neq n \leqslant G} \frac{d(m)d(n)}{(mn)^{1/4}} \int_T^{2T} e^{4\pi i (\sqrt{m} - \sqrt{n}) \sqrt{t}} \, dt
$$

$$
\ll TG^{1/2} \log^3 T + T^{1/2} \sum_{m \neq n \leqslant G} \frac{d(m)d(n)}{(mn)^{1/4} | \sqrt{m} - \sqrt{n}|}.
$$

When $n/2 < m \leqslant 2n$ the contribution of the last double sum is

$$
\ll_\varepsilon T^{1/2} \sum_{n \leqslant G} n^{\varepsilon - 1/2} n^{1/2} \sum_{n/2 < m \leqslant 2n, m \neq n} \frac{1}{|m - n|} \ll_\varepsilon T^{1/2 + \varepsilon} G.
$$

If $m \leqslant n/2$ then $| \sqrt{m} - \sqrt{n}|^{-1} \ll n^{-1/2}$, and when $m > 2n$ it is $\ll m^{-1/2}$. Thus the total contribution of the double sum above is certainly

$$
\ll_\varepsilon T^{1/2 + \varepsilon} G \ll TG^{1/2} \log^3 T \qquad (T^\varepsilon \leqslant G = G(T) \leqslant T^{1-\varepsilon}).
$$

We infer that

$$
T^{-1/4} (J_1 J_2)^{1/2} \ll T^{-1/4} (TG^{1/2} \log^3 T \cdot T^{3/2})^{1/2} = TG^{1/4} (\log T)^{3/2}.
$$

In a similar vein it is found that

$$
\int_T^{2T} t^{1/4} \sum_{G < n \leqslant T} d(n) n^{-3/4} \cos(4\pi \sqrt{nt} - \pi/4) |\zeta(\tfrac{1}{2} + it)|^2 \, dt
$$

$$
\ll T^{1/4} \left\{ \int_T^{2T} \left| \sum_{G < n \leqslant T} d(n) n^{-3/4} e^{4\pi i \sqrt{nt}} \right|^2 \, dt \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^4 \, dt \right\}^{1/2}
$$

$$
\ll T^{3/4} \log^2 T \left\{ \int_T^{2T} \left(\sum_{n > G} \frac{d^2(n)}{n^{3/2}} + \sum_{G < m \neq n \leqslant T} \frac{d(m)d(n)}{(mn)^{3/4}} e^{4\pi i (\sqrt{m} - \sqrt{n}) \sqrt{t}} \right) dt \right\}^{1/2}
$$

$$
\ll_\varepsilon T^{3/4} \log^2 T \left\{ TG^{-1/2} \log^3 T + T^{1/2 + \varepsilon} \right\}^{1/2} \ll T^{5/4} G^{-1/4} (\log T)^{7/2}.
$$

We finally infer that, for $T^\varepsilon \leqslant G = G(T) \leqslant T^{1-\varepsilon}$,

$$\int\limits_T^{2T} \Delta(t)|\zeta(\tfrac{1}{2} + it)|^2 \, dt \ll TG^{1/4}(\log T)^{3/2} + T^{5/4}G^{-1/4}(\log T)^{7/2} \ll T^{9/8}(\log T)^{5/2}$$

with the choice $G = T^{1/2} \log^4 T$. This leads to (2.6) on replacing T by $T2^{-j}$ and adding the resulting estimates.

Corollary. We have

$$\int_0^T E^*(t)|\zeta(\tfrac{1}{2} + it)|^2 \, dt \ll T^{9/8}(\log T)^{5/2}. \tag{4.15}$$

Namely

$$\int_T^{2T} E^*(t)|\zeta(\tfrac{1}{2} + it)|^2 \, dt = \int_T^{2T} \left\{E(t) - 2\pi\Delta^*(t/(2\pi))\right\}|\zeta(\tfrac{1}{2} + it)|^2 \, dt.$$

The integral with Δ^* is $\ll T^{9/8}(\log T)^{5/2}$ by Theorem 4. There remains

$$\int_T^{2T} E(t)|\zeta(\tfrac{1}{2} + it)|^2 \, dt = \pi T\Big(\log \frac{2T}{\pi} + 2\gamma - 1\Big) + O(T^{3/4}\log T) = O(T\log T)$$

by (3.8) of Lemma 6. This gives

$$\int_T^{2T} E^*(t)|\zeta(\tfrac{1}{2} + it)|^2 \, dt \ll T^{9/8}(\log T)^{5/2}.$$

To complete the proof of (4.15), again one replaces T by $T2^{-j}$ and adds the resulting estimates.

It remains to prove (2.7) of Theorem 5 (the bound (4.15) gives a result when $j = 0$). The proof is analogous to the proofs given before, so we shall be brief. We have

$$\int_T^{2T} E^*(t)E^j(t)|\zeta(\tfrac{1}{2} + it)|^2 \, dt$$
$$= \int_T^{2T} E^*(t)E^j(t)\Big(\log \frac{t}{2\pi} + 2\gamma + E'(t)\Big) \, dt = I' + I'',$$

say. By the Cauchy-Schwarz inequality for integrals, Lemma 3 and Lemma 9 (with $k = 2j$), it follows that

$$I' := \int_T^{2T} E^*(t)E^j(t)\Big(\log \frac{t}{2\pi} + 2\gamma\Big) \, dt$$
$$\ll \log T\Big\{\int_T^{2T} (E^*(t))^2 \, dt \int_T^{2T} E^{2j}(t) \, dt\Big\}^{1/2}$$
$$\ll \log T(T^{4/3}\log^3 T \cdot T^{1+j/2})^{1/2} = T^{7/6+j/4}\log^{5/2} T.$$

On the other hand, by (1.4) we have

$$I'' := \int_T^{2T} E^*(t)E^j(t)E'(t) \, dt$$
$$= \int_T^{2T} E^{j+1}(t)E'(t) \, dt - 2\pi\int_T^{2T} \Delta^*\Big(\frac{t}{2\pi}\Big)E^j(t)E'(t) \, dt. \tag{4.16}$$

Note that

$$\int_T^{2T} E^{j+1}(t)E'(t)\,dt = \tfrac{1}{j+2}E^{j+2}(t)\Big|_T^{2T} = O(T^{(j+2)/3}),$$

and $(j+2)/3 < 7/6 + j/4$ for $0 < j \leqslant 6$. By using (3.7) of Lemma 4 it is seen that the last integral in (4.16) is a multiple of

$$\int_T^{2T} t^{1/4} \sum_{n \leqslant N}(-1)^n d(n)n^{-3/4}\cos(\sqrt{8\pi nt} - \pi/4)E^j(t)E'(t)\,dt + \mathcal{J}(T), \tag{4.17}$$

say, where $T^\varepsilon \leqslant N = N(T) \leqslant T^{1-\varepsilon}$. Using Lemma 9 we have

$$\mathcal{J}(T) \ll_\varepsilon T^{1/2+\varepsilon}N^{-1/2}\int_T^{2T}|E^j(t)||E'(t)|\,dt$$

$$\ll_\varepsilon T^{1/2+\varepsilon}N^{-1/2}\left\{\int_T^{2T}E^{2j}(t)\,dt\int_T^{2T}\left(\log^2 T + |\zeta(\tfrac{1}{2}+it)|^4\right)dt\right\}^{1/2}$$

$$\ll_\varepsilon T^{1/2+\varepsilon}N^{-1/2}\left(T^{1+j/2}\cdot T\log^4 T\right)^{1/2} = T^{3/2+j/4+\varepsilon}N^{-1/2}.$$

The remaining integral in (4.17) is again integrated by parts. The major contribution will come from a multiple of

$$\int_T^{2T}E^{j+1}(t)t^{-1/4}\sum_{n \leqslant N}(-1)^n d(n)n^{-1/4}\sin(\sqrt{8\pi nt} - \pi/4)\,dt$$

$$\ll T^{-1/4}\left\{\int_T^{2T}E^{2j+2}(t)\,dt\int_T^{2T}\left|\sum_{n \leqslant N}(-1)^n d(n)n^{-1/4}e^{i\sqrt{8\pi nt}}\right|^2 dt\right\}^{1/2}$$

$$\ll T^{-1/4}\left\{T^{1+(j+1)/2}\cdot TN^{1/2}\log^3 T\right\}^{1/2} = T^{3/4+(j+1)/4}N^{1/4}\log^{3/2}T,$$

where Lemma 9 was used with $k = 2j + 2 \leqslant 8$. The choice $N = T^{2/3}$ gives

$$T^{3/4+(j+1)/4}N^{1/4} = T^{3/2+j/4+\varepsilon}N^{-1/2} = T^{7/6+j/4},$$

as asserted by Theorem 5. The bound in (2.7) is an expected one, since (in the mean square sense) $E^*(t)$ is of the order $\ll t^{1/6}\log^{3/2}t$, $E(t)$ is of the order $\ll t^{1/4}$, and $|\zeta(\tfrac{1}{2}+it)|^2$ is of logarithmic order. However, by Hölder's inequality for integrals (2.7) does not follows directly, since it would require the yet unknown estimates for higher moments of $|\zeta(\tfrac{1}{2}+it)|$.

References

[1] F.V. Atkinson, *The mean value of the Riemann zeta-function*, Acta Math. **81**(1949), 353-376.
[2] D.R. Heath-Brown, *The distribution and moments of the error term in the Dirichlet divisor problems*, Acta Arith. **60**(1992), 389-415.
[3] D.R. Heath-Brown, *The mean value theorem for the Riemann zeta-function*, Mathematika **25**(1978), 177-184.
[4] D.R. Heath-Brown, *The twelfth power moment of the Riemann zeta-function*, Quart. J. Math. (Oxford) **29**(1978), 443-462.
[5] A. Ivić, *Mean values of the Riemann zeta-function*, LN's **82**, Tata Inst. of Fundamental Research, Bombay, 1991 (distr. by Springer Verlag, Berlin etc.).
[6] A. Ivić, *On some integrals involving the mean square formula for the Riemann zeta-function*, Publications Inst. Math. (Belgrade) **46(60)** (1989), 33-42.
[7] A. Ivić, *On the mean square of the zeta-function and the divisor problem*, Annales Acad. Scien. Fennicae Mathematica **23**(2007), 1-9.
[8] A. Ivić, *On the Riemann zeta-function and the divisor problem*, Central European J. Math. **(2)(4)** (2004), 1-15, II, ibid. **(3)(2)**(2005), 203-214, III, Annales Univ. Sci. Budapest, Sect. Comp. **29**(2008), 3-23, and IV, Uniform Distribution Theory **1**(2006), 125-135.
[9] A. Ivić, *The Riemann zeta-function*, John Wiley & Sons, New York, 1985 (2nd ed. Dover, Mineola, New York, 2003).
[10] A. Ivić and Y. Motohashi, *On the fourth power moment of the Riemann zeta-function*, J. Number Theory **51**(1995), 16-45.
[11] A. Ivić and P. Sargos, *On the higher power moments of the error term in the divisor problem*, Illinois J. Math. **81**(2007), 353-377.

[12] M. Jutila, *On a formula of Atkinson*, Topics in classical number theory, Colloq. Budapest 1981, Vol. I, Colloq. Math. Soc. János Bolyai **34**(1984), 807-823.
[13] M. Jutila, *Riemann's zeta-function and the divisor problem*, Arkiv Mat. **21**(1983), 75-96 and II, ibid. **31**(1993), 61-70.
[14] T. Meurman, *A generalization of Atkinson's formula to L-functions*, Acta Arith. **47**(1986), 351-370.
[15] Y. Motohashi, *Spectral theory of the Riemann zeta-function*, Cambridge University Press, Cambridge, 1997.
[16] E. Preissmann, *Sur la moyenne quadratique du terme de reste du probléme du cercle*, C. R. Acad. Sci. Paris Sér. I Math. **306**(1988), no. 4, 151-154.
[17] K. Ramachandra, *On the mean-value and omega-theorems for the Riemann zeta-function*, LN's **85**, Tata Inst. of Fundamental Research (distr. by Springer Verlag, Berlin etc.), Bombay, 1995.
[18] K.-M. Tsang, *Recent progress on the Dirichlet divisor problem and the mean square of the Riemann zeta-function*, Sci. China Math. **53**(2010), no. 9, 2561-2572.