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Classes of Fuzzy Hyperideals

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Abstract. In this paper we study fuzzy hyperideals of a fuzzy hyperring, we define and analyze two particular kinds of fuzzy hyperideals, which extend similar notions of ring context, namely prime fuzzy hyperideals and maximal fuzzy hyperideals. Moreover, we study the hyperideal transfer through a fuzzy hyperring homomorphism, particularly for prime fuzzy hyperideals and for maximal fuzzy hyperideals.

1. Introduction

Most of the problems in biology, economics, ecology, engineering, environmental science, medical science, social science etc. have various uncertainties. Fuzzy set theory, rough set theory, vague set theory, interval mathematics, probability, soft set theory are different ways of expressing uncertainty.

There have been some developments in the study focusing on a fusion of algebra and theories modelling imprecision. The study of fuzzy algebraic structures, especially of fuzzy groups, dates back to the early 70-ies. Famous mathematicians were involved in it, such as A. Rosenfeld, J.N. Mordeson, D.S. Malik etc.

Fuzzy hyperstructures represent a connection between fuzzy sets [14] and algebraic hyperstructures [10]. This topic occurs in many up-to-date papers concerning fuzzy algebraic structures. There are several important applications of fuzzy algebra, such as in automata theory and coding theory. Concerning fuzzy sets and algebraic hyperstructures, there are three approaches in order to connect these topics. One approach is to consider a certain hyperoperation, defined through a fuzzy set, as in [1, 2]. Another approach is to consider fuzzy hyperstructures in a similar way as Rosenfeld did for fuzzy groups [11]. This study was initiated by Zahedi and his collaborations [15]. The third approach involves the definition and study of fuzzy hyperoperations. In a nonempty set H, fuzzy hyperstructures map a pair of elements of H to a fuzzy subset of H, see [3]. This idea was continued by Kehagias, Konstantinidou and Serafimidis ([5], [6], [12]). In 2007, Sen, Ameri and Chowdhury [13] used this idea for defining fuzzy hypersemigroups. Soon after, Leoreanu-Fotea and Davvaz introduced fuzzy hyperrings [7] and Leoreanu-Fotea extended this study to fuzzy hypermodules [8].

This topic is placed at the border between logic, computer science and universal algebra. It has been especially approached and developed in the past decade, especially due to its applicability in various

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domains: various subdomains of computer science and mathematics, biology, chemistry etc. To this end, a theoretical study of generalized algebraic structures and of their properties is required.

In this paper, we continue the study of fuzzy hyperideals of a fuzzy hyperring, initiated in [7]. We introduce and characterize prime fuzzy hyperideals and maximal fuzzy hyperideals. Moreover, we study the hyperideal transfer through a fuzzy hyperring homomorphism, particularly for prime fuzzy hyperideals and for maximal fuzzy hyperideals. In this paper, we present a study in detail of fuzzy hyperideals of a fuzzy hyperring, continuing [7]. We introduce and characterize prime fuzzy hyperideals and maximal fuzzy hyperideals. Moreover, we study the hyperideal transfer through a fuzzy hyperring homomorphism, particularly for prime fuzzy hyperideals and for maximal fuzzy hyperideals.

2. Preliminaries

Let us present some definitions that we need for the rest of our paper.

A mapping $\circ : H \times H \longrightarrow P^*(H)$ is called a *hyperoperation* (or a join operation), where $P^*(H)$ is the set of all non-empty subsets of H and the couple $(H, \circ)$ is called a *hypergroupoid*.
The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B\}.$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$, respectively. Generally, the singleton $\{a\}$ is identified by its element a.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H

$$(x \circ y) \circ z = x \circ (y \circ z),$$

which means that $\bigcup\limits_{u \in x \circ y} u \circ z = \bigcup\limits_{v \in y \circ z} x \circ v$.

We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, $x \circ H = H \circ x = H$. A subhypergroup $(S, \circ)$ of $(H, \circ)$ is a nonempty set S, such that for all $s \in S$, we have $s \circ S = S = S \circ s$.
Many examples of hypergroups can be found in [2]. We present here some of them:
Example 2.1. Let R be an equivalence relation on H. We define a hypergroupoid $(H, \circ_R)$, as follows:

$$\forall a, b \in H, \ \ a \circ_R a = \{y \mid (a, y) \in R\} \ \text{ and } \ a \circ_R b = a \circ_R a \cup b \circ_R b.$$

Then $(H, \circ_R)$ is a hypergroup.
Example 2.2. Let $(S, \cdot)$ be a semigroup and let P be a non-empty subset of S. For all x, y of S, we define $x \circ y = xPy$. Then $(S, \circ)$ is a semihypergroup. If $(S, \cdot)$ is a group, then $(S, \circ)$ is a hypergroup, called a *P-hypergroup*.
Example 2.3. If G is a group and for all x, y of G, $< x, y >$ denotes the subgroup generated by x and y, then we define $x \circ y =< x, y >$. Then $(G, \circ)$ is a hypergroup.

Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be two semihypergroups. A map $f : H_1 \longrightarrow H_2$ is called a *semihypergroup homomorphism* if for all $x, y \in H_1$, we have $f(x \circ_1 y) \subseteq f(x) \circ_2 f(y)$.

Hypergroups were introduced in 1934 by a French mathematician Marty at the VIIIth Congress of Scandinavian Mathematicians [10]. Till now, a lot of applications of hypergroups have been established in several fields, such as: combinatorics, cryptography, artificial intelligence, automata etc, see [2]. There are also other kinds of algebraic hyperstructures, such as hyperrings, hypermodules, hypervector spaces.

Several kinds of hyperrings on a nonempty set R can be defined, depending on which operation is replaced by a hyperoperation. In what follows, we shall consider one of the most general types of hyperrings [4]:

The triple $(R, +, \cdot)$ is a *hyperring* if:
(i) $(R, +)$ is a commutative hypergroup;
(ii) $(R, \cdot)$ is a semihypergroup;
(iii) the hyperoperation " $\cdot$ " is distributive over the hyperoperation " $+$ ".

Example 2.4. (M. Krasner) Let $(P, +, \cdot)$ be a ring and let G be a normal subgroup of its multiplicative semigroup (i.e., $xG = Gx$ for all $x \in P$). Set$\bar{P} = \{xG \mid x \in P\}$. For any two elements $x, y \in \bar{P}$, define $xG \oplus yG = \{(xp + yq)G \mid p, q \in G\}$ and $xG \odot yG = xyG$. Then $(\bar{P}, \oplus, \odot)$ is a hyperring.

Let us recall now some fuzzy hyperstructure definitions, [13].

Let S be a nonempty set. $F^*(S)$ denotes the set of all fuzzy subsets of S. A *fuzzy hyperoperation* on S is a mapping $\circ : S \times S \longmapsto F^*(S)$ written as $(a, b) \longmapsto a \circ b$. In other words the fuzzy hyperoperation "$\circ$", assigns to every pair (a, b) in H^2 , a nonempty fuzzy subset of H. The set S together with a fuzzy hyperoperation " $\circ$ " is called a *fuzzy hypergroupoid*.

(1) A fuzzy hypergroupoid $(S, \circ)$ is called a *fuzzy hypersemigroup* if for all $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$, where for any fuzzy subset μ of S and for all $r \in S$:

$(a \circ \mu)(r) = \bigvee_{t \in S}((a \circ t)(r) \wedge \mu(t))$, $(\mu \circ a)(r) = \bigvee_{t \in S}((t \circ a)(r) \wedge \mu(t))$.

(2) If A is a nonempty subset of S and $x \in S$, then for all $t \in S$ we have

$(x \circ A)(t) = \bigvee_{a \in A}(x \circ a)(t)$ and $(A \circ x)(t) = \bigvee_{a \in A}(a \circ x)(t)$.

(3) Let μ, ν be two fuzzy subsets of a fuzzy hypergroupoid $(S, \circ)$ then for all $t \in S$,

$$(\mu \circ \nu)(t) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ q)(t) \wedge \nu(q)).$$

If A is a nonempty subset of S, then we denote the characteristic function of A by χ_A. If $A = S$, then for all $t \in S$ we have $\chi_S(t) = 1$. A *fuzzy hypergroup* $(H, \cdot)$ is a fuzzy semihypergroup, such that for all $x \in H$, $x \cdot H = H \cdot x = \chi_H$.

Example 2.5.[8] Consider the set N^* of all nonzero natural numbers and for all $a, b \in N^*$ we define the fuzzy set $a \circ b : N^* \to [0, 1]$ by $(a \circ b)(t) = min\{1/a,\ 1/b,\ 1/t\}$. It follows that $(N^*, \circ)$ is a fuzzy semihypergroup.

Example 2.6. [8] Let $(H, \cdot)$ be a fuzzy semihypergroup. Let x_0 be an external element of H and denote $H_0 = H \cup \{x_0\}$. For all $x \in H_0$ define $x \circ x_0 = x_0 \circ X = \chi_{H_0}$ and for all $a, b \in H$ define $(a \circ b)(x_0) = 1$. If $x \in H$, then $(a \circ b)(x) = (a \cdot b)(x)$. Then $(H_0, \circ)$ is a fuzzy hypergroup.

Continuing this idea Leoreanu-Fotea and Davvaz [7] introduced the notion of fuzzy hyperrings as follows:

Let R be a nonempty set and " $\oplus$ "," $\odot$ " be two fuzzy hyperoperations on R. The triple $(R, \oplus, \odot)$ is called a *fuzzy hyperring* if

(1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in R$;
(2) $x \oplus R = R \oplus x = \chi_S$ for all $x \in R$;
(3) $a \oplus b = b \oplus a$ for all $a, b \in R$;
(4) $a \odot (b \odot c) = (a \odot b) \odot c$ for all $a, b, c \in R$;
(5) $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ and $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ for all $x, y, z \in R$.

The element $1 \in R$ is called *unity* (or *identity*) if for all $r \in R$, we have $(r \odot 1)(r) > 0$.

Example 2.7. [7] Let $(R, +, \cdot)$ be a ring and $\mu \in (0, 1]$. If we define the following fuzzy hyperoperations on R: $\forall a, b \in R$, $a \oplus b = \chi_{\{a,b\}}$ and $(a \odot b)(t) = \mu$ iff $t = ab$ and $(a \odot b)(t) = 0$, otherwise, then $(R, \oplus, \odot)$ is a fuzzy hyperring.

Example 2.8. Let us consider the ring $\mathbf{Z}$ of integer numbers and $k \in (0, 1]$. We define the following fuzzy hyperoperation on $\mathbf{Z}$:

$\forall a, b \in \mathbf{Z}$, $a \uplus_1 b = \chi_{a+b}$ and

$$(a \circ_1 b)(t) = \begin{cases} k & \text{if } t = ab, \\ 0 & \text{otherwise.} \end{cases}$$

According to Theorem 3.1, [7], $(\mathbf{Z}, \uplus_1, \circ_1)$ is a fuzzy hyperring.
All fuzzy hyperrings we consider in sequel are commutative and have unity.

3. Prime Fuzzy Hyperideals and Maximal Fuzzy Hyperideals

In this section we shall introduce and analyze the notions of a prime fuzzy hyperideal and a maximal fuzzy hyperideal.

First we recall the fuzzy hyperideal notion as it was defined in [7]:

Definition 3.1. If $(R, \oplus, \odot)$ is a fuzzy hyperring and I is a nonempty set of R, then I is called a *fuzzy hyperideal* if the followings hold:
(i) if $s_1, s_2 \in I$ and $(s_1 \oplus s_2)(x) > 0$, then $x \in I$;
(ii) for all $s \in I$, we have $s \oplus I = \chi_I$;
(iii) if $r \in R$, $s \in I$ and $(r \odot s)(x) > 0$ or $(s \odot s)(x) > 0$ then $x \in I$.

Example 3.2. Let us consider the fuzzy hyperring $(\mathbf{Z}, \uplus_1, \circ_1)$, given in Example 2.8.
Consider $S = n\mathbf{Z}$, where $n \in \mathbf{N}$. We check that S is a fuzzy hyperideal of $(\mathbf{Z}, \uplus_1, \circ_1)$.
If $s_1, s_2 \in n\mathbf{Z}$ and $x \in \mathbf{Z}$, then

$$(s_1 \uplus_1 s_2)(x) = \begin{cases} 1 & \text{if } x = s_1 + s_2, \\ 0 & \text{otherwise,} \end{cases}$$

hence, $(s_1 \uplus_1 s_2)(x) > 0$ implies that $x = s_1 + s_2 \in n\mathbf{Z}$.
Now, $\forall s \in n\mathbf{Z}$, and $\forall x \in n\mathbf{Z}$,

$$(s \uplus_1 n\mathbf{Z})(x) = \bigvee_{s_1 \in n\mathbf{Z}} (s \uplus_1 s_1)(x)$$

$$= \bigvee_{s_1 \in n\mathbf{Z}} \begin{cases} 1 & \text{if } x = s + s_1, \\ 0 & \text{otherwise,} \end{cases}$$

so $(s \uplus_1 n\mathbf{Z})(x) = 1$, since we can consider $s_1 = x - s \in n\mathbf{Z}$.
For all $x \in \mathbf{Z} - n\mathbf{Z}$, $(s \uplus_1 n\mathbf{Z})(x) = 0$, because x cannot be equal to $s + s_1 \in n\mathbf{Z}$. Hence, $s \uplus_1 n\mathbf{Z} = \chi_S$.
Finally, if $s \in n\mathbf{Z}$, $r \in n\mathbf{Z}$ and $(s \circ_1 r)(x) > 0$, then $(s \circ_1 r)(x) = k$, and so $x = sr \in n\mathbf{Z}$. In other words, if $s \in n\mathbf{Z}$, $r \in \mathbf{Z}$ and $(s \circ_1 r)(x) > 0$ then $x \in n\mathbf{Z}$.
Therefore, $(n\mathbf{Z}, \uplus_1, \circ_1)$ is a fuzzy hyperideal of $(\mathbf{Z}, \uplus_1, \circ_1)$.

Moreover, we shall prove that for all fuzzy hyperideal S of $(\mathbf{Z}, \uplus_1, \circ_1)$, there exists $n \in \mathbf{N}$ such that $S = n\mathbf{Z}$. Suppose that $S \neq \{0\}$, which means that $\exists s \in S$, $s \neq 0$. We have $(s \circ_1 0)(0) = k$, which means that $0 \in S$. Moreover, $s \uplus_1 S = \chi_S$, so $(s \uplus_1 S)(0) = 1$, whence $\bigvee_{\acute{s} \in S} (s \uplus_1 \acute{s})(0) = 1$. Hence, $\exists \acute{s} \in S$, for which $(s \uplus_1 \acute{s})(0) > 0$, whence $0 = s + \acute{s}$. In other words, $-s \in S$. So, S contains a positive integer.
Let $n = min\{s \in S | s > 0\}$. We show that $S = n\mathbf{Z}$. Since $(n \uplus_1 n)(2n) = 1 > 0$, it follows that $2n \in S$. Similarly, from $(n \uplus_1 2n)(3n) > 0$ it follows that $3n \in S$ and so on. We obtain that for all $x \in \mathbf{N}$, $nx \in S$. In a similar way as above, from $nx \in S$, it follows that $-nx \in S$, whence $n\mathbf{Z} \subseteq S$.
Let now $s \in S \subseteq \mathbf{Z}$. Hence, there are $q, r \in \mathbf{Z}$, $r \geq 0$ such that $s = nq + r$. From $(s \uplus_1 -nq)(r) > 0$, it follows that $r \in S$ and by the minimality of n, it follows that $r = 0$. Then $s = nq \in n\mathbf{Z}$. Therefore, $S = n\mathbf{Z}$, which means that the fuzzy hyperideals of $\mathbf{Z}$ are $n\mathbf{Z}$, where n is a natural number.

We shall endow quotients of a fuzzy hyperring through its fuzzy hyperideals with hyperring structures, as follows:

Let $(R, \oplus, \odot)$ be a fuzzy hyperring and I be a fuzzy hyperideal. We define the next relation on R:

$$x \rho_I y \iff [(x \oplus I)(t) > 0 \iff (y \oplus I)(t) > 0]$$

where $(x \oplus I)(t) = \bigvee_{i \in I} (x \oplus i)(t)$.

It is easy to see that ρ_I is an equivalence relation. We obtain the equivalence classes as follows:

$$\hat{x} = \{y \in R|\ x\rho_I y\} = \{y \in R|\ (x \oplus I)(t) > 0 \Longleftrightarrow (y \oplus I)(t) > 0\}$$

and the quotient $R/I = \{\hat{x} | x \in R\}$. Finally, we define the following hyperoperations on R/I:

$$\hat{x} \boxplus \hat{y} = \{\hat{z}|\ (x \oplus y)(z) > 0\},$$

$$\hat{x} \boxdot \hat{y} = \{\hat{t}|\ (x \odot y)(t) > 0\}.$$

Theorem 3.3. *If R is a fuzzy hyperring and I is a fuzzy hyperideal, then R/I endowed with the hyperoperations "$\boxplus$", "$\boxdot$" is a hyperring.*

Proof. Let us first verify the associativity of "$\boxplus$". We have

$\hat{y} \boxplus \hat{z} = \{\hat{t} | (y \oplus z)(t) > 0\}$,

$\hat{x} \boxplus \hat{t} = \{\hat{s} | (x \oplus t)(s) > 0\}$,

whence, $(x \oplus (y \oplus z))(s) = \bigvee_{t \in R} ((x \oplus t)(s) \wedge (y \oplus z)(t) > 0$.

Since $(x \oplus t)(s) > 0$ and $(y \oplus z)(t) > 0$, then $(x \oplus (y \oplus z))(s) > 0$. Conversely, if $(x \oplus (y \oplus z))(s) > 0$ then there exists $t \in R$ such that $(x \oplus t)(s) > 0$ and $(y \oplus z)(t) > 0$. So $\hat{x} \boxplus (\hat{y} \boxplus \hat{z}) = \{\hat{s} | (x \oplus (y \oplus z))(s) > 0\}$.

On the other hand, we have: $(\hat{x} \boxplus \hat{y}) \boxplus \hat{z} = \{\hat{s} | ((x \oplus y) \oplus z)(s) > 0\}$. Since R is a fuzzy hyperring, the associativity law holds. Similarly, we can show the associativity of "$\boxdot$".

Now we have to check the reproduction axiom. If $x, y \in R$ then $\hat{x}, \hat{y} \in R/I$. Since $x \oplus R = \chi_R$, it follows that for all $y \in R$ we have $(x \oplus R)(y) = 1$. So there exists $r \in R$ such that $(x \oplus r)(y) > 0$ then $\hat{y} \in \hat{x} \boxplus \hat{r}$. Therefore for all $\hat{x} \in R/I$ we have $\hat{x} \boxplus R/I = R/I$.

Hence, $(R/I, \boxplus)$ is a hypergroup.

And by a similar way we show the distributivity. Hence $(R/I, \boxplus, \boxdot)$ is a hyperring. □

Definition 3.4. (see [9]) Let $(R, \oplus, \odot)$ be a fuzzy hyperring. An element x is called *zero* and it is denoted by "0" if $[(s \oplus x)(u) > 0 \Longleftrightarrow s = u]$.

From now on, we consider that all fuzzy hyperrings have zero and all fuzzy hyperideals of them contain zero, too.

Definition 3.5. A hyperring $(H, +, \cdot)$ with zero is called an *integral hyperring* if the following implication holds: $x \cdot y = 0 \Longrightarrow x = 0$ or $y = 0$.

Lemma 3.6. *In the quotient hyperring $(R/I, \boxplus, \boxdot)$, the next equivalence holds:*

$$x \in I \Longleftrightarrow \hat{x} = \hat{0}.$$

Proof. If $x \in I$ then $x \oplus I = \chi_I$. Also, we have $(x \oplus I)(t) > 0 \Longleftrightarrow \chi_I(t) > 0 \Longleftrightarrow t \in I$.

On the other hand, by the definition of "0", $(0 \oplus I)(t) > 0$ if and only if $t \in I$.

Hence, we will obtain $[(x \oplus I)(t) > 0 \Longleftrightarrow (0 \oplus I)(t) > 0]$ and according to the definition of ρ_I we can say $x\rho_I 0$, whence $\hat{x} = \hat{0}$.

Conversely, suppose $\hat{x} = \hat{0}$. This means that $x\rho_I 0$ and so the the following implication holds:

$$(x \oplus I)(k) > 0 \Longleftrightarrow (0 \oplus I)(k) > 0.$$

But $(0 \oplus I)(k) > 0$ means that $k \in I$. Hence,

$$(x \oplus I)(k) > 0 \Longleftrightarrow k \in I. \quad (1)$$

From $0 \in I$ and $(x \oplus 0)(x) > 0$, it follows that $(x \oplus I)(x) > 0$ and according to (1) we obtain that $x \in I$. □

Lemma 3.7. *If $\hat{x} \boxdot \hat{y} = \hat{0}$, then* $[(x \odot y)(u) > 0 \Longrightarrow u \in I]$.

Proof. Let $(x \odot y)(u) > 0$. Then $\hat{u} \in \hat{x} \boxdot \hat{y} = \hat{0}$, whence $\hat{u} = \hat{0}$, and by Lemma 3.6, it follows that $u \in I$. □

We can introduce now prime fuzzy hyperideals, as follows:

Definition 3.8. A fuzzy hyperideal P of a fuzzy hyperring $(R, \oplus, \odot)$ is called a *prime fuzzy hyperideal*, if $P \neq R$ and whenever the following condition holds:

$$(x \odot y)(u) > 0 \Longrightarrow u \in P,$$

then $x \in P$ or $y \in P$.

Example 3.9. We determine the prime fuzzy hyperideals of $(\mathbf{Z}, \uplus_1, \circ_1)$, given in Example 2.8.
By Corollary 3.20, all fuzzy hyperideals $p\mathbf{Z}$, where p is a prime natural number are prime fuzzy hyperideals, since they are maximal. We show that if $(n\mathbf{Z}, \uplus_1, \circ_1)$ is a prime fuzzy hyperideal, then $(n\mathbf{Z}, +, \cdot)$ is a prime ideal of $(\mathbf{Z}, +, \cdot)$, whence we obtain that n is a prime natural number or $n = 0$. Indeed, $(n\mathbf{Z}, \uplus_1, \circ_1)$ is prime if $[(a \circ_1 b)(x) > 0 \Longrightarrow x \in n\mathbf{Z}] \Longrightarrow a \in n\mathbf{Z}$ or $b \in n\mathbf{Z}$. But $(a \circ_1 b)(x) > 0$ if and only if $x = a \cdot b$. So the above condition becomes:

$$a \cdot b \in n\mathbf{Z} \Longrightarrow a \in n\mathbf{Z} \text{ or } b \in n\mathbf{Z}$$

which means that $n\mathbf{Z}$ is a prime ideal of the ring $\mathbf{Z}$.
Therefore, the prime fuzzy hyperideals of $(\mathbf{Z}, \uplus_1, \circ_1)$ are $\{0\}$ and $(p\mathbf{Z}, \uplus_1, \circ_1)$ where p is a prime natural number.

Theorem 3.10. *If P is a prime fuzzy hyperideal of a fuzzy hyperring $(R, \oplus, \odot)$, then R/P is an integral hyperring.*

Proof. Let $\hat{x} \boxdot \hat{y} = \hat{0}$. By Lemma 3.7, it follows that $[(x \odot y)(u) > 0 \Longrightarrow u \in P]$. Since P is prime, we obtain $x \in P$ or $y \in P$, and by Lemma 3.6, we obtain $\hat{x} = \hat{0}$ or $\hat{y} = \hat{0}$. Hence R/P is integral hyperring. □

Also the reverse of the above theorem holds:
Theorem 3.11. *If R/P is integral hyperring, then P is prime.*

Proof. Suppose that $(x \odot y)(u) > 0 \Longrightarrow u \in P$. We show that $x \in P$ or $y \in P$.
We have $(x \odot y)(u) > 0 \Longleftrightarrow \hat{u} \in \hat{x} \boxdot \hat{y}$ and by Lema 3.6, $u \in P \Longleftrightarrow \hat{u} = \hat{0}$.
Hence, the following implication hold:

$$\hat{u} \in \hat{x} \boxdot \hat{y} \Longrightarrow \hat{u} = \hat{0}, \text{ which means that } \hat{x} \boxdot \hat{y} = \hat{0}.$$

Since R/P is an integral hyperring, it follows that $\hat{x} = \hat{0}$ or $\hat{y} = \hat{0}$, which means that $x \in P$ or $y \in P$. □

Hence we have just proved the following:
Corollary 3.12. *If $(R, \oplus, \odot)$ is a fuzzy hyperring, then R/P is an integral hyperring if and only if P is a fuzzy prime hyperideal.*

Now, let R be a fuzzy hyperring with "0" and let I be a fuzzy hyperideal of R. We denote $(R/I)^* = R/I - \{\hat{0}\}$.
We recall that the definition of a hyperfield [4]:
A hyperring $(R, \boxplus, \boxdot)$ with "0" is called *hyperfield* if for all $x \in R^* = R - \{0\}$, we have $x \boxdot R = R$.
In other words, a hyperring $(R, \boxplus, \boxdot)$ is a hyperfield if $(R^*, \boxdot)$ is a hypergroup. Hence, in a hyperfield $(R, \boxplus, \boxdot)$, $(R, \boxplus)$ and $(R^*, \boxdot)$ are hypergroups and " $\boxdot$ " is distributive over " $\boxplus$ ".

Proposition 3.13. *Let R be a fuzzy hyperring with zero and I be a fuzzy hyperideal. Then the following equivalence holds:*

$$\hat{x} \boxdot (R/I)^* = (R/I)^*, \ \textit{where } \hat{x} \in (R/I)* \Longleftrightarrow \forall t \notin I, \exists y \notin I : (x \odot y)(t) > 0.$$

Proof. According to Lemma 3.6, $\hat{x} = \hat{0} \Longleftrightarrow x \in I$, hence $\hat{x} \in (R/I)^*$ means that $x \notin I$.
We have $\hat{x} \boxdot (R/I)^* = (R/I)^* \Longleftrightarrow \forall \hat{t} \in (R/I)^*, \exists \hat{y} \in (R/I)^*$ such that $\hat{t} \in \hat{x} \boxdot \hat{y}$
$\Longleftrightarrow \forall t \notin I, \exists y \notin I$ such that $(x \odot y)(t) > 0$. □

Definition 3.14. If S is a nonempty subset of R, then the smallest fuzzy hyperideal of R containing S is called the *fuzzy hyperideal generated by S* and it is denoted by $< S >$. In other words $< S > = \bigcap\limits_{I \supseteq S} I$, where I is a fuzzy hyperideal of R.

In what follows, by $\sum_{i=1}^{n} \mu_i$ we intend $\mu_1 \oplus \cdots \oplus \mu_n$, where μ_i are fuzzy subsets on a same set.

Proposition 3.15. *Let $(R, \oplus, \odot)$ be a fuzzy hyperring with unity and S be a nonempty subset of R. Then*

$$< S >= \{t \in R \mid \exists n \in \mathbf{N}^*, \ \sum_{i=1}^{n} (a_i \odot s_i)(t) > 0, \textit{ where } \forall i \in \{1, ..., n\}, a_i \in R, s_i \in S\}.$$

Proof. Let $A = \{t \in R \mid \exists n \in \mathbf{N}^*, \ \sum_{i=1}^{n} (a_i \odot s_i)(t) > 0$, where $\forall i \in \{1, ..., n\}, a_i \in R, s_i \in S\}$. We show that

(i) A is a fuzzy hyperideal;

(ii) A is the smallest fuzzy hyperideal of R containing S.

First, let $t_1, t_2 \in A$ and $(t_1 \oplus t_2)(x) > 0$. We check that $x \in A$. We have

$$(\sum_{i=1}^{n} a_i \odot s_i)(t_1) > 0, (\sum_{j=1}^{m} \acute{a_j} \odot \acute{s_j})(t_2) > 0, \text{ where for all } i, j;\ a_i \in R, s_i \in S, \acute{a_j} \in R, \acute{s_j} \in S.$$

We can check easily that the next equivalence holds:

$$(a \oplus b)(x) > 0 \Longleftrightarrow \exists k \in (0, 1] \text{ such that } k\chi_x \leq \chi_a \oplus \chi_b.$$

Indeed, we can take $k = (a \oplus b)(x)$, and similarly

$$(a \odot b)(y) > 0 \Longleftrightarrow \exists k \in (0, 1] \text{ such that } k\chi_y \leq \chi_a \odot \chi_b.$$

Hence, from $(\sum_{i=1}^{n} a_i \odot s_i)(t_1) > 0$ it follows that $\exists k_1 \in (0, 1] : k_1\chi_{t_1} \leq \sum_{i=1}^{n} \chi_{a_i} \odot \chi_{s_i}$. Similarly, $\exists k_2 \in (0, 1] : k_2\chi_{t_2} \leq \sum_{j=1}^{m} \chi_{\acute{a}_j} \odot \chi_{\acute{s}_j}$. Moreover, $\exists k \in (0, 1] : k\chi_x \leq \chi_{t_1} \oplus \chi_{t_2}$.

Set $\acute{k} = min\{k_1, k_2\}$. We obtain

$$k\acute{k}\chi_x \leq \acute{k}\chi_{t_1} \oplus \acute{k}\chi_{t_2} \leq (\sum_{i=1}^{n} \chi_{a_i} \odot \chi_{s_i}) \oplus (\sum_{j=1}^{m} \chi_{\acute{a}_j} \odot \chi_{\acute{s}_j}).$$

Since $k\acute{k} \in (0, 1]$, it follows that $((\sum_{i=1}^{n} a_i \odot s_i) \oplus (\sum_{j=1}^{m} \acute{a}_j \odot \acute{s}_j))(x) > 0$, which means that $x \in A$.

Now, let $t \in A, r \in R$ and $(r \odot t)(x) > 0$. We check that $x \in A$.

There are $n \in \mathbf{N}^*$ such that $\forall i \in \{1, ..., n\}, \exists a_i \in R, \exists s_i \in S$ and $(\sum_{i=1}^{n} a_i \odot s_i)(t) > 0$. Hence, $\exists k_1, k_2 \in (0, 1]$ such that $k_1\chi_x \leq \chi_r \odot \chi_t$ and $k_2\chi_t \leq \sum_{i=1}^{n} \chi_{a_i} \odot \chi_{s_i}$. By distributivity, we obtain $k_1k_2\chi_x \leq \sum_{i=1}^{n} (\chi_r \odot \chi_{a_i}) \odot \chi_{s_i}$, whence $(\sum_{i=1}^{n} (r \odot a_i) \odot s_i)(x) > 0$. Hence, there exist $u_1, \cdots, u_n \in R$ such that $(\sum_{i=1}^{n} u_i)(x) > 0$ and $((r \odot a_i) \odot s_i)(u_i) > 0$, for all i. Moreover, there are $y_i \in R$ such that $(r \odot a_i)(y_i) > 0$ and $(y_i \odot s_i)(u_i) > 0$. For all $i \in \{1, , ..., n\}$ we obtain

$$\acute{k}_i\chi_{u_i} \leq \chi_{y_i} \odot \chi_{s_i} \ \text{ and } \ k_3\chi_x \leq \sum_{i=1}^{m} \chi_{u_i}, \text{ for some } \ k_3, \acute{k}_i \in (0, 1].$$

Set $\acute{k}_i = min\{\acute{k}_i \mid i \in \{1, ..., n\}\}$. We obtain

$$k_3\acute{k}\chi_x \leq \sum_{i=1}^{n} \acute{k}\chi_{u_i} \leq \sum_{i=1}^{n} \chi_{y_i} \odot \chi_{s_i}.$$

So, $(\sum_{i=1}^{n} y_i \odot s_i)(x) > 0$. This means that $x \in A$. Therefore, A is a fuzzy hyperideal of R.
Moreover, since R has unity, we have $(1 \odot s)(s) > 0$, for all $s \in S$, and so $S \subseteq A$, as desired.

(ii) Next, we show that A is the smallest fuzzy hyperideal containg S. Let B be a fuzzy hyperideal, which contains S and let $t \in A$. Then $(\sum_{i=1}^{n} r_i \odot s_i)(t) > 0$, where for all $i \in \{1, ..., n\}$, $r_i \in R, s_i \in S$. Since

$$(\sum_{i=1}^{n} r_i \odot s_i)(t) = \bigvee_{u_1,...,u_n \in R} ((\sum_{i=1}^{n} u_i)(t) \wedge (r_1 \odot s_1)(u_1) \wedge ... \wedge (r_n \odot s_n)(u_n))$$

it follows that $\exists u_1, ..., u_n \in R$ such that $(\sum_{i=1}^{n} u_i)(t) > 0$, $\forall i$, $(r_i \odot s_i)(u_i) > 0$.
Since $s_i \in S \subseteq B$, it follows that $u_i \in B$, for all $i \in \{1, ..., n\}$, whence $t \in B$. Hence, $A \leq B$. Therefore, $A =< S >$. □

Definition 3.16. A fuzzy hyperideal M of a fuzzy hyperring R is called *maximal* if for all fuzzy hyperideal N, if $M \leq N \leq R$ then $M = N$ or $N = R$.

Example 3.17. Let us find the maximal fuzzy hyperideals of the fuzzy hyperring, given in Example 2.8. Notice that $n\mathbf{Z}$ is maximal if and only if n is a prime integer. Indeed, if $n = p$ is prime and $S = m\mathbf{Z}$ is a fuzzy hyperideal of $(\mathbf{Z}, \uplus_1, \circ_1)$ such that $p\mathbf{Z} \leq m\mathbf{Z} \leq \mathbf{Z}$, then $m|p$, whence $m \in \{1, p\}$. Hence, $m = 1$ and so $m\mathbf{Z} = \mathbf{Z}$ or $m = p$ and so $m\mathbf{Z} = p\mathbf{Z}$. This means that $p\mathbf{Z}$ is a maximal fuzzy hyperideal of $(\mathbf{Z}, \uplus_1, \circ_1)$.
On the other hand, if we suppose there is $n \in \mathbf{N}$ such that $S = n\mathbf{Z}$ is maximal and n is not prime, then $\exists m|x$, $m \in \mathbf{N}$, $m \notin \{1, n\}$, whence $S = n\mathbf{Z} \lneq m\mathbf{Z} \lneq \mathbf{Z}$ which is a contradiction. Hence, the only maximal fuzzy hyperideals of $(\mathbf{Z}, \uplus_1, \circ_1)$ are $n\mathbf{Z}$, where n is a prime natural number.

Now, we characterize maximal fuzzy hyperideals in fuzzy hyperrings.
Theorem 3.18. *If* $(R, \oplus, \odot)$ *is a fuzzy hyperring, M is a fuzzy hyperideal,* $M \neq R$, *then the quotient* $(R/M, \boxplus, \boxdot)$ *is a hyperfield if and only if M is a maximal fuzzy hyperideal.*

Proof. Let x be an whichever element of $R - M$. We denote the fuzzy hyperideals of R generated by M and x by $< M, x >$. In order to show that M is maximal we check that $< M, x >= R$. Suppose $(R/M, \boxplus, \boxdot)$ is a hyperfield, so according to Proposition 3.13,

$$\forall x \notin M,\ \forall t \notin M, \exists y \notin M\, ; (x \odot y)(t) > 0$$

On the other hand, $< M, x >$ is a fuzzy hyperideal and $x \in< M, x >$, $y \in R$. Hence, $t \in< M, x >$, for all $t \notin M$. Therefore, $R - M \subset< M, x >$, whence $< M, x >= R$. This means that M is a maximal fuzzy hyperideal.
Conversely, let $x \notin M$. It follows that $M \subseteq< M, x >$ and since M is maximal we obtain that $< M, x >= R$.
On the other hand,

$$< M, x >= \{u \in R | \exists n \in \mathbf{N}^* : ((\sum_{i=1}^{n} r_i \odot m_i) \oplus (r \odot x))(u) > 0, \text{ where } r \in R,\ \forall i,\ m_i \in M, r_i \in R\}.$$

Let $t \in R - M$ be arbitrary. Since $t \in R =< M, x >$, it follows that $((\sum_{i=1}^{n} r_i \odot m_i) \oplus (r \odot x))(t) > 0$. We have

$$((\sum_{i=1}^{n} r_i \odot m_i) \oplus (r \odot x))(t) = \bigvee_{\forall i \in \{1,...,n+1\}, u_i \in R} ((\sum_{i=1}^{n+1} u_i)(t) \wedge (r_1 \odot m_1)(u_1) \wedge$$
$$\wedge ... \wedge (r_n \odot m_n)(u_n) \wedge (r \odot x)(u_{n+1})),$$

whence there exist $u_1, ..., u_{n+1} \in R$, such that

$$(\sum_{i=1}^{n+1} u_i)(t) > 0, (r_1 \odot m_1)(u_1) > 0, ..., (r_n \odot m_n)(u_n) > 0, (r \odot x)(u_{n+1}) > 0.$$

This means that in R/M we have $\hat{t} \in \sum_{i=1}^{n+1} \hat{u_i}$, $\hat{u_1} \in \hat{r_1} \boxdot \hat{m_1} = \hat{0}$, since $m_1 \in M, \dots, \hat{u_n} \in \hat{r_n} \boxdot \hat{m_n} = \hat{0}$, and $\widehat{u_{n+1}} \in \hat{r} \boxdot \hat{x}$. Hence, $\hat{t} = \widehat{u_{n+1}} \in \hat{r} \boxdot \hat{x}$. So, $\forall \hat{t}, \hat{x} \in R/M$, $\hat{t} \neq \hat{0} \neq \hat{x}$, $\exists \hat{r} \in R/M$ such that $\hat{t} \in \hat{r} \boxdot \hat{x}$. Notice that $\hat{r} \neq \hat{0}$, otherwise $\hat{t} = \hat{0}$.
Therefore, $((R/M)^*, \boxdot)$ is a hypergroup and so $(R/M, \boxplus, \boxdot)$ is a hyperfield. □.

Theorem 3.19. *Every hyperfield is an integral hyperring.*

Proof. Let $(R, \boxplus, \boxdot)$ be a hyperfield and $a, b \in R$ such that $a \boxdot b = 0$. We show that $a = 0$ or $b = 0$. Suppose $a \neq 0$. Since $(R^*, \boxdot)$ is a hypergroup it follows that $a \boxdot R^* = R^*$. Hence, $a \boxdot b \notin a \boxdot R^*$, but $a \boxdot b \in a \boxdot R$, so b must be 0. □

Corollary 3.20. *Every maximal fuzzy hyperideal is a prime fuzzy hyperideal.*
It follows by the characterizations of prime fuzzy hyperideals, maximal fuzzy hyperideals and the above theorem. □

4. Fuzzy Ideal Transfer

Let $(R_1, \oplus_1, \odot_1)$ and $(R_2, \oplus_2, \odot_2)$ be two fuzzy hyperrings. In [7], a homomorphism of fuzzy hyperrings is defined as follows:

A map $f : R_1 \longrightarrow R_2$ is called *homomorphism* of fuzzy hyperrings if the following conditions hold:
$\forall a, b \in R_1, \quad f(a \oplus_1 b) \leq f(a_1) \oplus_2 f(b)$ and $f(a \odot_1 b) \leq f(a) \odot_2 f(b)$.

Definition 4.1. If in the above conditions we have " $=$ " instead of " $\leq$ ", then f is called a *good homomorphism* of fuzzy hyperrings.

Recall that if $R_1 \longrightarrow R_2$ is a map and μ is a fuzzy set on R_1, then $f(\mu) : R_2 \longrightarrow [0, 1]$ is defined as follows:

$$(f(\mu))(t) = \bigvee_{r \in f^{-1}(t)} \mu(r) \ \ if \ \ f^{-1}(t) \neq \emptyset,$$

otherwise we consider $(f(\mu))(t) = 0$.

Theorem 4.2. *Let $f : R_1 \longrightarrow R_2$ be a surjective good homomorphism of fuzzy hyperrings and I_1, I_2 be two fuzzy hyperideals of R_1 and R_2 respectively. The following statements hold:*
(i) if the next implication holds:

$$f(\mu) = \chi_{I_2} \ \ and \ \ \mu = \bigvee_{a \in A \subseteq R_1, b \in B \subseteq R_2} a \oplus b \Longrightarrow \mu = \chi_{f^{-1}(I_2)} \qquad (\tau)$$

then $f^{-1}(I_2)$ is a fuzzy hyperideal of R_1.
(ii) $f(I_1)$ is a fuzzy hyperideal of R_2.

Proof. (i) Let $s_1, s_2 \in f^{-1}(I_2)$ and $x \in R_1$ such that $(s_1 \oplus_1 s_2)(x) > 0$. We show that $x \in f^{-1}(I_2)$.
Since $(s_1 \oplus_1 s_2)(x) > 0$, it follows that

$$0 < \bigvee_{r \in f^{-1}(f(x))} (s_1 \oplus_1 s_2)(r) = f(s_1 \oplus_1 s_2)(f(x)) \leq (f(s_1) \oplus_2 f(s_2))(f(x)).$$

Since I_2 is a fuzzy hyperideal of R_2 and $f(s_1), f(s_2) \in I_2$, we obtain that $f(x) \in I_2$, which means that $x \in f^{-1}(I_2)$.

Here, we check that $s_1 \oplus_1 f^{-1}(I_2) = \chi_{f^{-1}(I_2)}$, using (τ). Indeed,

$$f(s_1 \oplus_1 f^{-1}(I_2)) = f(s_1) \oplus_2 f(f^{-1}(I_2)) = f(s_1) \oplus_2 I_2 = \chi_{I_2}$$

whence, $s_1 \oplus_1 f^{-1}(I_2) = \chi_{f^{-1}(I_2)}$, (we put $A = \{s_1\}$ and $B = f^{-1}(I_2)$, so $\mu = s_1 \oplus_1 f^{-1}(I_2)$).
Notice that for all $r \notin f^{-1}(I_2)$, we will obtain $(s_1 \oplus_1 f^{-1}(I_2))(r) = 0$ without using condition (τ) as follows:
Let $r \in R_1$, we have $(s_1 \oplus_1 f^{-1}(I_2)(r) = \bigvee\limits_{u \in f^{-1}(I_2)} (s_1 \oplus_1 u)(r)$. On the other hand, $f(s_1 \oplus_1 u) \leq f(s_1) \oplus_2 f(u) \in f(s_1) \oplus_2 I_2 = \chi_{I_2}$, hence for all $x \notin I_2$, $f(s_1 \oplus_1 u)(x) = 0$, which means that $\bigvee\limits_{r \in f^{-1}(x)} (s_1 \oplus_1 u)(r) = 0$, if $x \in Imf$.
Hence, $(s_1 \oplus_1 u)(r) = 0$ for all r such that $f(r) \notin I_2$, whence $(s_1 \oplus_1 f^{-1}(I_2))(r) = 0$, for all $r \notin f^{-1}(I_2)$.
Finally, if $s \in f^{-1}(I_2)$, $r \in R_1$ and $(r \odot_1 s)(x) > 0$, then we show that $x \in f^{-1}(I_2)$.
We have $0 < \bigvee\limits_{t \in f^{-1}(f(x))} (r \odot_1 s)(t) = f(r \odot_1 s)(f(x)) \leq (f(r) \odot_2 f(s))(f(x))$. Since I_2 is a fuzzy hyperideal of R_2 and $f(s) \in I_2$, it follows that $f(x) \in I_2$ which means that $x \in f^{-1}(I_2)$.

(ii) Let $s_1, s_2 \in I_1$ and $x \in R_2$ such that $(f(s_1) \oplus_2 f(s_2))(x) > 0$. We have $(f(s_1) \oplus_2 f(s_2))(x) = f(s_1 \oplus_1 s_2)(x) = \bigvee\limits_{r \in f^{-1}(x)} (s_1 \oplus_1 s_2)(r)$, since f is surjective. Hence there exists $r \in f^{-1}(x)$ such that $(s_1 \oplus_1 s_2)(r) > 0$ and since I_1 is a fuzzy hyperideal it follows that $r \in I_1$ and so $f(r) = x \in f(I_1)$.
Similarly, we check that if $s_1 \in I_1$; $r, x \in R_2$ and $(f(s_1) \odot_2 r)(x) > 0$, then we obtain that $x \in f(I_1)$, too.
Finally, for all $s \in I_1$, $f(s) \oplus_2 f(I_1) = f(s \oplus_1 I_1) = f(\chi_{I_1}) = \chi_{f(I_1)}$, since $f(\chi_{I_1})(x) = \bigvee\limits_{r \in f^{-1}(x)} \chi_{I_1}(r) = \chi_{f(I_1)}(x)$.
Therefore, $f(I_1)$ is a fuzzy hyperideal of R_2. □

Notice that since any fuzzy hyperideal contains zero, it follows that if I_2 is a fuzzy hyperideal of R_2, then $f^{-1}(I_2)$ contains $f^{-1}(0)$, which we denote by $Kerf$.

Now, let $f : R_1 \longrightarrow R_2$ be a map and let $x \in R_1$. Denote by $\bar{x} = \{t | f(x) = f(t)\}$. $\bar{x}$ is the euivalence class determined by the equivalence relation induced by f:

$$x \sim_f y \Longleftrightarrow f(x) = f(y)$$

Theorem 4.3. *Let $f : R_1 \longrightarrow R_2$ be a surjective good fuzzy homomorphism. If P is a prime fuzzy hyperideal of R_1 and the following condition holds:*

$$x \in P \Longrightarrow \bar{x} \subset P \qquad (*)$$

then $f(P)$ is a prime fuzzy hyperideal of R_2.

Proof. First, notice that $f(P) \neq R_2$. We have that $P \neq R_1$. Suppose that $f(P) = R_2$. Since $f(R_1) = R_2$, it follows that $f(P) = f(R_1)$, which means that for all $x \in R_1$, there exists $p \in P$ such that $f(x) = f(p)$, $x \in \bar{P}$. By $(*)$, it follows that $x \in P$, that is $R_1 = P$, which is a contradiction.
Moreover, By Theorem 4.2, $f(P)$ is a fuzzy hyperideal of R_2.
Now, suppose that $(\acute{a} \odot \acute{b})(y) > 0 \Longrightarrow y \in f(P)$ holds, where $\acute{a}, \acute{b}, y \in R_2$. We check that $\acute{a} \in f(P)$ or $\acute{b} \in f(P)$.
If $(a \odot b)(x) > 0$, then $f(a \odot b)(f(x)) = \bigvee\limits_{f(u)=f(x)} (a \odot b)(u) > 0$, whence $(f(a)) \odot f(b))(f(x)) > 0$.
Denote $f(a) = \acute{a}$, $f(b) = \acute{b}$, $f(x) = y$. We obtain $f(x) = y \in f(P)$ and by $(*)$ it follows that $x \in P$. Hence, $(a \odot b)(x) > 0$ implies that $x \in P$. Since P is prime, we obtain that $a \in P$ or $b \in P$, and so $f(a) = \acute{a} \in f(P)$ or

$f(b) = \acute{b} \in f(P)$. Therefore $f(P)$ is a prime fuzzy hyperideal of R_2. □

Theorem 4.4. *Let $f : R_1 \longrightarrow R_2$ be a surjective good fuzzy homomorphism. If M is a maximal fuzzy hyperideal of R_1 and the following condition holds:*

$$x \in M \Longrightarrow \bar{x} \subset M \qquad (*)$$

then $f(M)$ is a maximal fuzzy hyperideal of R_2.

Proof. The proof is the same as the proof in the classical context of maximal ideals of rings. □

Theorem 4.5. *Let $f : R_1 \longrightarrow R_2$ be a surjective good homomorphism, such that condition (τ) holds. If $\acute{P}$ is a prime (maximal) fuzzy hyperideal of R_2, then $f^{-1}(\acute{P})$ is a prime (maximal) fuzzy hyperideal of R_1.*

Proof. First, notice that $f^{-1}(\acute{P}) \neq R_1$, otherwise, if $f^{-1}(\acute{P}) = R_1$, then $f(f^{-1}(\acute{P})) = f(R_1) = R_2$. Since $f(f^{-1}(\acute{P})) \subseteq \acute{P}$, it follows that $R_2 \subseteq \acute{P} \subseteq R_2$, that is $\acute{P} = R_2$, which is a contradiction.
Moreover, according to Theorem 4.2, $f(P)$ is a fuzzy hyperideal of R_2.
Finally, suppose that the next implication holds:

$$(a \odot b)(x) > 0 \Longrightarrow x \in f^{-1}(\acute{P}) \qquad (**)$$

where $a, b, x \in R_1$
We check that $a \in f^{-1}(\acute{P})$ or $b \in f^{-1}(\acute{P})$.
If $(\acute{a} \odot \acute{b})(x) > 0$,where $\acute{a}, \acute{b}, y \in R_2$, then $\acute{a} = f(a)$, $\acute{b} = f(b)$, $y = f(x)$, where $a, b, x \in R_1$, and $f(a \odot b)(f(x)) > 0$. Hence, $\bigvee_{f(u)=f(x)} (a \odot b)(u) > 0$, whence there exists $u \in R_1 :\ f(u) = f(x)$, such that $(a \odot b)(u) > 0$. By $(**)$, it follows that $u \in f^{-1}(\acute{P})$, that is $f(u) = f(x) = y \in \acute{P}$.
Since $\acute{P}$ is prime, we obtain that $\acute{a} = f(a) \in \acute{P}$ or $\acute{b} = f(b) \in \acute{P}$. Hence, $a \in f^{-1}(\acute{P})$ or $b \in f^{-1}(\acute{P})$. Therefore $f^{-1}(\acute{P})$ is a prime fuzzy hyperideal of R_2.
The proof for maximal fuzzy hyperideal is the same as the proof in the classical context of maximal ideals of rings. □

Corollary 4.6. *If $R_1 \longrightarrow R_2$ is a surjective good homomorphism such that condition (τ) holds, then there is a bijection correspondence between the set of prime (maximal) fuzzy hyperideals of R_1, that satisfy condition $(*)$ and the set of prime (maximal) fuzzy hyperideals of R_2.*

Example 4.7. We endow the ring $(\mathbf{Z}_n, +, \cdot)$ with two fuzzy hyperoperatins:
$\forall \hat{a}, \hat{b} \in \mathbf{Z}_n,\ \ \hat{a} \uplus_2 \hat{b} = \chi_{\hat{a}+\hat{b}}$,
$\hat{a} \circ_2 \hat{b}(t) = k \in (0, 1]$ if $t = \hat{a} \cdot \hat{b}$ and otherwise, $\hat{a} \circ_2 \hat{b}(t) = 0$.
In other words, $\hat{a} \circ_2 \hat{b} = k\chi_{\hat{a}.\hat{b}}$. According to Theorem 3.1 [7], it follows that $(\mathbf{Z}_n, \uplus_2, \circ_2)$ is a fuzzy hyperring. Consider now the canonical projection $\pi : \mathbf{Z} \longrightarrow \mathbf{Z}_n$, where $(\mathbf{Z}, \uplus_1, \circ_1)$ and $(\mathbf{Z}_n, \uplus_2, \circ_2)$ are the above fuzzy hyperrings. For all $\hat{t} \in \mathbf{Z}_n$, we have

$$\pi(a \uplus_1 b)(\hat{t}) = \pi(\chi_{a+b})(\hat{t}) = \bigvee_{s \in \pi^{-1}(\hat{t})} \chi_{a+b}(s),$$

so $\pi(a \uplus_1 b)(\hat{t}) = 1$ if and only if $a + b \in \pi^{-1}(\hat{t})$, which means that $\pi(a + b) = \hat{t}$, or equivalently $\hat{t} = \widehat{a+b}$. Otherwise, $\pi(a \uplus_1 b)(\hat{t}) = 0$.
On the other hand, $(\pi(a) \uplus_2 \pi(b))(\hat{t}) = (\hat{a} \uplus_2 \hat{b})(\hat{t}) = \chi_{\hat{a}+\hat{b}}(t)$. Therefore, $\pi(a \uplus_1 b) = \pi(a) \uplus_2 \pi(b)$.
Similarly, for all $\hat{t} \in \mathbf{Z}_n$, we have

$$\pi(a \circ_1 b)(\hat{t}) = \pi(k\chi_{a \cdot b})(\hat{t}) = \bigvee_{s \in \pi^{-1}(\hat{t})} k\chi_{a \cdot b}(s) = k\chi_{\widehat{ab}}(\hat{t}) = (\pi(a) \circ_2 \pi(b))(\hat{t}).$$

Therefore, π is a surjective good homomorphism of fuzzy hyperrings.
Let us check condition (τ). Consider that $\mu = A \uplus_1 B$ is a fuzzy subset on $\mathbf{Z}$. Then $\mu = \chi_{A+B}$. If $\pi(\mu) = \chi_S$, where S is a fuzzy hyperideal of $\mathbf{Z}_n$, then

$$\chi_S(\hat{t}) = \pi(\mu)(\hat{t}) = \bigvee_{s\in\hat{t}} \mu(s) = \bigvee_{s\in\hat{t}} \chi_{A+B}(s)$$
$$= \begin{cases} 1 & \text{if } \hat{t} \cap (A+B) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} 1 & \text{if } \hat{t} \in \widehat{A+B}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\hat{t} \in S \Longleftrightarrow \hat{t} \in \widehat{A+B}$, whence $S = \widehat{A+B}$. Therefore, $\mu = \chi_{\pi^{-1}(S)}$.

On the other hand, the fuzzy hyperring $(\mathbf{Z}_n, \uplus_2, \circ_2)$ has zero, which is $\hat{0} = \pi(n\mathbf{Z})$, so all fuzzy hyperideals of $\mathbf{Z}_n$ contain it. This means that $n\mathbf{Z} = \pi^{-1}(\hat{0}) \subseteq \pi^{-1}(I_2)$, for all fuzzy hyperideal I_2 of $\mathbf{Z}_n$. According to Theorem 4.2, the set of fuzzy hyperideals of $(\mathbf{Z}_n, \uplus_2, \circ_2)$ is $\{\pi(m\mathbf{Z}) | \text{where } m|n\}$.

Recall that the prime fuzzy hyperideals of $(\mathbf{Z}, \uplus_1, \circ_1)$ are $\{0\}$ and $p\mathbf{Z}$, where p is a prime natural number, while the maximal fuzzy hyperideals of $(\mathbf{Z}, \uplus_1, \circ_1)$ are $p\mathbf{Z}$, where p is a prime natural number.
We look for the fuzzy hyperideals I which satisfy $(*)$. Condition $(*)$ is:

$$x \in I \Longrightarrow \hat{x} \subset I.$$

Clearly, $\{0\}$ does not satisfy $(*)$. We prove that a fuzzy hyperideal $I = m\mathbf{Z}$ of $(\mathbf{Z}, \uplus_1, \circ_1)$ satisfies $(*)$ if and only if $m|n$, where $\pi : \mathbf{Z} \longrightarrow \mathbf{Z}_n$.

Let $x \in m\mathbf{Z}$ and $y \in \bar{x}$. This means that $n|(y-x)$. We have $y \in m\mathbf{Z} \Longleftrightarrow m|y$ if and only if $m|(y-x)$. Hence, if $n|(y-x)$, then we must have $m|(y-x)$. This happens if and only if $m|n$.
For instance, if $n = 5$ and $m = 6$, then $x = 6 \in 6\mathbf{Z}$, but $1 \notin 6\mathbf{Z}$ and $1 \in \bar{6}$. If $x = 6$ and $m = 3$, then $x = 9 \in 3\mathbf{Z}$ and $\bar{x} = 6\mathbf{Z} + 3 \subseteq 3\mathbf{Z}$.
We can conclude that for the surjective good homomorphism $\pi : \mathbf{Z} \longrightarrow \mathbf{Z}_n$, the set of prime (maximal) fuzzy hyperideals of $\mathbf{Z}$ which satisfy $(*)$ is $\{p\mathbf{Z} | p \text{ is prime}, p|n\}$. According to Corollary 4.6, this set is in a bijective correspondence with the set of prime (maximal) fuzzy hyperideals of $\mathbf{Z}_n$.
Therefore, the set of prime (maximal) fuzzy hyperideals of $\mathbf{Z}_n$ is $\{\pi(p\mathbf{Z}) | p \text{ is prime}, \ p|n\}$. □

In [7] homomorphisms of fuzzy hyperrings are analysed; in particular, they are considered quotients of fuzzy hyperrings with respect to fuzzy regular relations, notions that are necessary for isomorphism theorems. It is proved that there exists a bijective map between fuzzy regular relations on a hyzzy hyperring and regular relations on the associated hyperring.Hence the study of quotient fuzzy hyperrings is reduced to the study of quotient hyperrings. Isomorphism theorems for hyperrings are presented in [4].

5. Conclusion.

We extend the study initiated in [13] about fuzzy semihypergroups and in [7] about fuzzy hyperrings. We introduce and characterize prime fuzzy hyperideals and maximal fuzzy hyperideals and study the hyperideal transfer through a fuzzy hyperring homomorphism. This study can be continued in several directions, such as: to examine the spectrum of fuzzy hyperrings, to analyse similar notions in the context of fuzzy hypermodules.

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