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Spectral Radius and Traceability of Connected Claw-Free Graphs

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Abstract. Let G be a connected claw-free graph on n vertices and $\overline{G}$ be its complement. Let $\mu(G)$ be the spectral radius of G. Denote by $N_{n-3,3}$ the graph consisting of K_{n-3} and three disjoint pendent edges. In this note we prove that:
(1) If $\mu(G) \geq n-4$, then G is traceable unless $G = N_{n-3,3}$.
(2) If $\mu(\overline{G}) \leq \mu(\overline{N_{n-3,3}})$ and $n \geq 24$, then G is traceable unless $G = N_{n-3,3}$.
Our works are counterparts on claw-free graphs of previous theorems due to Lu et al., and Fiedler and Nikiforov, respectively.

1. Introduction

Let G be a graph. The *eigenvalues* of G are the eigenvalues of the adjacency matrix of G. Since the adjacency matrix of G is real and symmetric, all its eigenvalues are real. The *spectral radius* of G, denoted by $\mu(G)$, is the spectral radius of its adjacency matrix, i.e., the maximum among the absolute values of its eigenvalues. By Perron-Frobenius' theorem (see Theorem 0.3 of [4]), $\mu(G)$ is equal to the largest eigenvalue of G.

Let G be a graph. We use $e(G)$ to denote *the number of edges* of G. Let $S \subset V(G)$. We use $G[S]$ to denote the subgraph of G induced by S and $G-S$ to denote the subgraph of G induced by $V(G)\backslash S$. For a subgraph H of G, we use $G-H$ instead of $G-V(H)$. For two subgraphs H, H' of G, we use $e_G(H,H')$ (or shortly, $e(H,H')$) to denote the number of edges with one vertex in H and the other one in H'.

By $\overline{G}$ we denote the *complement* of G. Let G_1 and G_2 be two graphs. We denote by $G_1 + G_2$ the *disjoint union* of G_1 and G_2, and by $G_1 \vee G_2$ the *join* of G_1 and G_2.

A graph G is *traceable* if it has a Hamilton path, i.e., a path containing all vertices of G; and G is *Hamiltonain* if it has a Hamilton cycle, i.e., a cycle containing all vertices of G. Note that every Hamiltonian graph is traceable. Hamiltonian properties of graphs have received much attention from graph theorists. A fundamental theorem due to Dirac [5] states that every graph on n vertices is traceable if the degree of every vertex is at least $(n-1)/2$. Up to now, there also has been some references on the spectral conditions for Hamilton paths or cycles. We refer the reader to [3, 8, 10, 15, 17, 19].

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In particular, Fiedler and Nikiforov [8] gave tight sufficient conditions for the existence of a Hamilton path in terms of the spectral radii of a graph and its complement.

Theorem 1 (Fiedler and Nikiforov [8]). *Let G be a graph on n vertices. If* $\mu(G) \geq n-2$*, then G is traceable unless* $G = K_{n-1} + K_1$.

Theorem 2 (Fiedler and Nikiforov [8]). *Let G be a graph on n vertices. If* $\mu(\overline{G}) \leq \sqrt{n-1}$*, then G is traceable unless* $G = K_{n-1} + K_1$.

Remark 1. *Note that* $\mu(K_{n-1} + K_1) = \mu(K_{n-1}) = n-2$ *and* $\mu(\overline{K_{n-1} + K_1}) = \mu(K_{1,n-1}) = \sqrt{n-1}$.

Since the connectedness is necessary for studying traceability of graphs. Lu, Liu and Tian [15] presented a sufficient condition for a connected graph to be traceable.

Theorem 3 (Lu, Liu and Tian [15]). *Let G be a connected graph of order* $n \geq 7$*. If* $\mu(G) \geq \sqrt{(n-3)^2+3}$*, then G is traceable.*

Lu et al.'s lower bound of spectral radius was sharpened in [17].

Theorem 4 (Ning and Ge [17]). *Let G be a connected graph on* $n \geq 7$ *vertices. If* $\mu(G) \geq n-3$*, then G is traceable unless* $G = K_1 \vee (K_{n-3} + 2K_1)$.

The bipartite graph $K_{1,3}$ is called a *claw*. A graph is called *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$. Claw-free graphs have been a very popular field of study, not only in the context of Hamiltonian properties. One reason is that the very natural class of line graphs turns out to be a subclass of the class of claw-free graphs. However, not every claw-free graph is Hamiltonian. There are examples of 3-connected non-Hamiltonian claw-free (even line) graphs, but it is a long-standing conjecture that all 4-connected claw-free graphs are Hamiltonian (and then, traceable). It is interesting to note that the lower bound on the degrees in Dirac's theorem for traceability was lowered to $(n-2)/3$ by Matthews and Sumner [16] for claw-free graphs. For a survey on claw-free graphs, we refer the reader to Faudree et al. [7].

Motivated by the relationship between Dirac's theorem and Matthews-Sumner's theorem, in this note we will improve the lower bound in Theorem 3 and give an analogue of Theorem 2 for connected claw-free graphs.

Our main results will be listed as follows. By $N_{n-3,3}$ we denote the graph consisting of a complete graph K_{n-3} with three disjoint pendent edges.

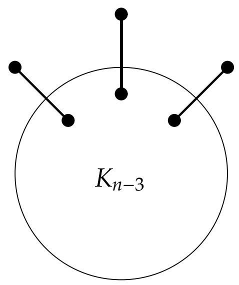

Fig. 1. Graph $N_{n-3,3}$.

Theorem 5. *Let G be a connected claw-free graph on n vertices. If* $\mu(G) \geq n-4$*, then G is traceable unless* $G = N_{n-3,3}$.

Theorem 6. *Let G be a connected claw-free graph on* $n \geq 24$ *vertices. If* $\mu(\overline{G}) \leq \mu(\overline{N_{n-3,3}})$*, then G is traceable unless* $G = N_{n-3,3}$.

2. Preliminaries

In this section, we first extend the concept of claw-free graphs to a general one. Let R be a given graph. The graph G is called *R-free* if G contains no induced subgraph isomorphic to R. We will also use three special graphs L, M and N (see Fig. 2). Note that $N = N_{3,3}$.

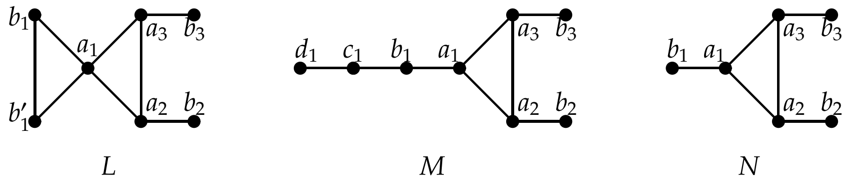

Fig. 2. Graphs L, M and N.

The following two theorems concerning traceability of claw-free graphs are used in our proofs.

Theorem 7 (Duffus, Gould and Jacobson [6]). *Every connected claw-free and N-free graph is traceable.*

Adopting the terminology of [9], we say that a graph is a *block-chain* if it is nonseparable or it has connectivity 1 and has exactly two end-blocks.

Theorem 8 (Li, Broersma and Zhang [13]). *Let G be a block-chain. If G is claw-free and M-free, then G is traceable.*

One important tool for studying Hamiltonian properties of claw-free graphs is the closure theory introduced by Ryjáček [18]. It is also useful for our proof. To ensure the completeness of our text, we include all the terminology and notations as follows. For other more information, see [18].

Let G be a graph. Following [18], for a vertex $x \in V(G)$, if the neighborhood of x induces a connected but non-complete subgraph of G, then we say that x is *eligible* in G. Set $B_G(x) = \{uv : u, v \in N(x), uv \notin E(G)\}$. The graph G'_x, constructed by $V(G'_x) = V(G)$ and $E(G'_x) = E(G) \cup B_G(x)$, is called the *local completion of G at x*.

As shown in [18], the *closure* of a claw-free graph G, denoted by $cl(G)$, is defined by a sequence of graphs $G_1, G_2, \ldots, G_t$, and vertices $x_1, x_2 \ldots, x_{t-1}$ such that
(1) $G_1 = G$, $G_t = cl(G)$;
(2) x_i is an eligible vertex of G_i, $G_{i+1} = (G_i)'_{x_i}$, $1 \le i \le t-1$; and
(3) $cl(G)$ has no eligible vertices.

Theorem 9 (Ryjáček [18]). *Let G be a claw-free graph. Then cl(G) is also claw-free.*

Theorem 10 (Brandt, Favaron and Ryjáček [1]). *Let G be a claw-free graph. Then G is traceable if and only if cl(G) is traceable.*

A claw-free graph G is said to be *closed* if $cl(G) = G$. It is not difficult to see that for every vertex x of a closed graph G, $N_G(x)$ is either a clique, or the disjoint union of two cliques in G (see [18]). In the following, we say a vertex x of a graph G is a *bad vertex* of G if $N_G(x)$ is neither a clique, nor the disjoint union of two cliques. So every closed graph has no bad vertices.

Lemma 1. *Let G be a closed claw-free graph. If there are two nonadjacent vertices of G have degree sum at least $n-1$, then G is traceable.*

Proof. Let x, y be two nonadjacent vertices of G with degree sum at least $n-1$. Note that a vertex is nonadjacent to itself. Hence x, y have at least one common neighbor.

Firstly we assume that x, y have at least three common neighbors, say z, z', z''. Since G is claw-free, either zz' or zz'' or $z'z''$ is in $E(G)$. Without loss of generality, we assume that $zz' \in E(G)$. Then z is a bad vertex, a contradiction.

Secondly we assume that x, y have two common neighbors, say z, z'. If $zz' \in E(G)$, then z will be a bad vertex. So we have that $zz' \notin E(G)$. Let C_x, C'_x, C_y, C'_y be the maximal cliques of G containing $\{x, z\}, \{x, z'\}, \{y, z\}, \{y, z'\}$, respectively. Clearly $H = G[C_x \cup C'_x \cup C_y \cup C'_y]$ has a Hamilton cycle. Note that there is at most one vertex in $V(G)\backslash V(H)$. Since G is connected, we have that G is traceable.

Finally we assume that x, y have only one common neighbor z. Then every vertex is adjacent either to x or to y. This implies that G consists of at most four maximal cliques and G is a block-chain. Clearly in this case G is traceable. □

The following two lemmas are crucial in the proofs of our two theorems. We guess that they are of interest in their own rights.

Lemma 2. *Let G be a connected claw-free graph on n vertices and m edges. If*

$$m \geq \binom{n-3}{2} + 2,$$

then G is traceable unless $G = N_{n-3,3}$ or L.

Proof. Let $G' = cl(G)$ be the closure of G. Then

$$e(G') \geq m \geq \binom{n-3}{2} + 2.$$

If G' is N-free, then by Theorems 7 and 9, G' is traceable, and so is G by Theorem 10. Now we assume that G' contains an induced subgraph $H \sim N$. We denote the vertices of H as in Fig. 2. In the following part of this proof, we set $N_H(x) = N_{G'}(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$.

For any $x \in V(G - H)$, note that the neighborhood of x in G' is either a clique or the disjoint union of two cliques. But any at least four vertices of H do not form a clique or a disjoint union of two cliques. This implies that $d_H(x) \leq 3$ for any $x \in V(G - H)$. Thus

$$e(G') = e(H) + e(G' - H) + e_{G'}(H, G' - H) \leq 6 + \binom{n-6}{2} + 3(n-6) = \binom{n-3}{2} + 3.$$

Recall that $e(G') \geq \binom{n-3}{2} + 2$. Thus we have $e(G') = \binom{n-3}{2} + 2$ or $\binom{n-3}{2} + 3$.

Case 1. $e(G') = \binom{n-3}{2} + 3$.

In this case, $G' - H$ is complete and every vertex in $G' - H$ has exactly three neighbors in H. Suppose first that there is a vertex x in $G' - H$ such that $N_H(x) = \{a_1, a_2, a_3\}$. We claim for every vertex x' in $G' - H$, $N_H(x') = \{a_1, a_2, a_3\}$. Since $N_H(x') \neq \{b_1, b_2, b_3\}$, we assume without loss of generality that $a_1 \in N_H(x')$. Note that $xx' \in E(G)$ and $G'[N_{G'}(x)]$ is a clique or disjoint union of two cliques. We can see that $a_2, a_3 \in N_H(x')$. Hence as we claimed $N_H(x') = \{a_1, a_2, a_3\}$. Thus $G' = N_{n-3,3}$.

Suppose that $E(G')\backslash E(G) \neq \emptyset$. Then $e(G) = \binom{n-3}{2} + 2$ and there is only one edge e in $E(G')\backslash E(G)$. If e is a pendant edge, then G is disconnected, a contradiction. So we assume that $e = uv$ is not a pendant edge. Suppose without loss of generality that a_1 is a vertex in $\{a_1, a_2, a_3\}\backslash\{u, v\}$. Then the subgraph induced by $\{a_1, b_1, u, v\}$ is a claw in G, a contradiction. This implies that $E(G')\backslash E(G) = \emptyset$. Hence $G = G' = N_{n-3,3}$.

Now we assume that for every vertex $x \in V(G - H)$, $N_H(x) \neq \{a_1, a_2, a_3\}$.

If $V(G' - H) = \emptyset$, then $G' = N = N_{3,3}$. By the analysis above, we can also see that $G = G' = N_{3,3}$. So we assume that $V(G' - H) \neq \emptyset$.

Let x be a vertex in $G' - H$. Thus $N_H(x)$, and then $N(x)$ induces two disjoint cliques. Note that $N_H(x) \neq \{b_1, b_2, b_3\}$. We assume without loss of generality that $a_1 \in N_H(x)$. If $a_2 \in N_H(x)$, then $a_3 \in N_H(x)$; otherwise a_1 will be a bad vertex of G'. But in this case $N_H(x) = \{a_1, a_2, a_3\}$, a contradiction. This implies that $a_2 \notin N_H(x)$ and similarly, $a_3 \notin N_H(x)$. Note that $N_H(x) \neq \{a_1, b_2, b_3\}$. We have $b_1 \in N_H(x)$. Without loss of generality, we assume that $N_H(x) = \{a_1, b_1, b_2\}$. If $G' - H$ has the only one vertex x, then $b_1a_1xb_2a_2a_3b_3$

is a Hamilton path of G'. By Theorem 10, G is traceable. Now we assume that there is a second vertex $x' \in V(G' - H)$.

Since both $\{x, x', b_1, b_2\}$ and $\{x, x', b_2, a_2\}$ induce no claws, it follows either $a_1, b_1 \in N_H(x')$ or $b_2 \in N_H(x')$. If $a_1, b_1 \in N_H(x')$, then $b_2 \notin N_H(x')$; otherwise x is a bad vertex of G'. Similarly as the case of x above, we can see that $a_2, a_3 \notin N_H(x')$. Thus $N_H(x') = \{a_1, b_1, b_3\}$. If $b_2 \in N_H(x')$, then $a_1, b_1 \notin N_H(x')$; otherwise x is a bad vertex of G'. If $a_2 \in N_H(x')$, then b_2 is a bad vertex of G', a contradiction. Thus we have $N_H(x') = \{b_2, a_3, b_3\}$. In conclusion, either $N_H(x') = \{a_1, b_1, b_3\}$ or $N_H(x') = \{b_2, a_3, b_3\}$.

Suppose that there is a third vertex x''. Then similarly as the case of x', $N_H(x'') = \{a_1, b_1, b_3\}$ or $N_H(x'') = \{b_2, a_3, b_3\}$. But if x' and x'' have the same neighborhood in H, then x' will be a bad vertex, a contradiction. So we assume without loss of generality that $N_H(x') = \{a_1, b_1, b_3\}$ and $N_H(x'') = \{b_2, a_3, b_3\}$. Then x' is also a bad vertex, a contradiction. Thus x, x' are the only two vertices in $G - H$, and $b_1 x x' b_3 a_3 a_1 a_2 b_2$ is a Hamilton path of G'. By Theorem 10, G is traceable.

Case 2. $e(G') = \binom{n-3}{2} + 2$.

In this case $G = G'$ and there is a vertex x in $G-H$ such that $d_H(x) = 2$ or $xx' \notin E(G)$ for some $x' \in V(G-H)$. Let $G_1 = G - x$. Since every vertex in $G - H - x$ is adjacent to three vertices in H, G_1 is connected. Note that

$$e(G_1) = e(G) - d(x) = \binom{n-3}{2} + 2 - (n-5) = \binom{n-4}{2} + 3.$$

Using the conclusion of Case 1, we can obtain that G_1 is traceable or $G_1 = N_{n-4,3}$.

Suppose first that $G_1 = N_{n-4,3}$. Let a_1b_1, a_2b_2, a_3b_3 be the three pendent edges of G_1, where a_1, a_2, a_3 are contained in a clique of G_1. Note that G is closed and $N(x)$ is either a clique or the disjoint union of two cliques. Also note that if x is adjacent to some two vertices of a maximal clique of G, then x will be adjacent to every vertex of the maximal clique of G. Since $d(x) = n - 5$, the neighborhood of x does not include $V(G_1)\backslash\{b_1, b_2, b_3\}$. If x is adjacent to two pendant vertices, say b_1, b_2, then let P be a Hamilton path of the complete graph $G_1 - \{b_1, b_2, b_3\}$ from a_2 to a_3. Then $b_1 x b_2 a_2 P a_3 b_3$ is a Hamilton path of G. Now we assume that x is adjacent to exactly one vertex of $\{b_1, b_2, b_3\}$. Suppose without loss of generality that $b_1 \in N(x)$. Since $d(x) = n - 5$, we can see that $n = 7$, $G_1 = N$ and $N(x) = \{a_1, b_1\}$. Hence $G = L$.

Now we assume that G_1 is traceable. Let $P = v_1 v_2 \ldots v_{n-1}$ be a Hamilton path of G_1. If $v_1 x \in E(G)$ or $v_{n-1} x \in E(G)$, then G is traceable. So we assume that $v_1 x, v_{n-1} x \notin E(G)$. If x is adjacent to two successive vertices on P, then G is traceable. So we assume that x is not adjacent to two successive vertices on P. This implies that $n - 1 - d(x) \geq d(x) + 1$. Since $d(x) = n - 5$, we have $n \leq 8$. Note that $n \geq 7$. We can see that either xv_2 or xv_{n-2} is in $E(G)$. We assume without loss of generality that $xv_2 \in E(G)$. Thus $v_1 v_3 \in E(G)$; otherwise the subgraph induced by $\{v_2, v_1, v_3, x\}$ is a claw. Hence $P' = x v_2 v_1 v_3 \ldots v_{n-1}$ is a Hamilton path of G. □

Lemma 3. *Let G be a connected claw-free graph on $n \geq 24$ vertices and m edges. If*

$$m > \binom{n}{2} - \left(1 + \sqrt{3n-8}\right)^2,$$

then G is traceable unless $G \subseteq N_{n-3,3}$.

Proof. We assume the opposite.

Claim 1. *G is a block-chain.*

Proof. Suppose that G is not a block-chain. Since G is claw-free, every cut-vertex of G is contained in exactly two blocks. This implies that G has a block B_0 which contains at least three cut-vertices of G. Let a_1, a_2, a_3 be three cut-vertices of G contained in B_0. Let B_i, $i = 1, 2, 3$, be the component of $G - B_0$ which has a neighbor of a_i. Let $H_0 = G - (\bigcup_{i=1}^{3} B_i)$ and $H_i = G[V(B_i) \cup \{a_i\}]$, $i = 1, 2, 3$. Note that $\nu(H_0) \geq 3$. If $\nu(H_1) = \nu(H_2) = \nu(H_3) = 2$, then $G \subseteq N_{n-3,3}$. Now we assume without loss of generality that $\nu(H_1) \geq 3$.

Note that $\sum_{i=0}^{3} \nu(H_i) = n + 3$. Thus

$$e(G) = \sum_{i=0}^{3} e(H_i) \leq \sum_{i=0}^{3} \binom{\nu(H_i)}{2} \leq \binom{n-4}{2} + 5 \leq \binom{n}{2} - \left(1 + \sqrt{3n-8}\right)^2$$

(noting that $n \geq 24$), a contradiction. □

Let $G' = cl(G)$. If G' is M-free, then by Theorems 8 and 10, G', and then G, is traceable. Now we assume that G' has an induced subgraph $H \sim M$. We denote the vertices of H as in Fig. 2.

Claim 2. *Every vertex in $G' - H$ has at most 5 neighbors in H; and there is at most one vertex in $G' - H$ having exactly 5 neighbors in H.*

Proof. Let x be a vertex in $G' - H$. Note that $N_H(x)$ is either a clique or the disjoint union of two cliques. This implies that $d_H(x) \leq 5$. Moreover, if $d_H(x) = 5$, then $N_H(x) = \{a_1, a_2, a_3, c_1, d_1\}$.

If there are two vertices, say x and x', such that each one has 5 neighbors in H, then $N_H(x) = N_H(x') = \{a_1, a_2, a_3, c_1, d_1\}$. But in this case x will be a bad vertex of G', a contradiction. □

By Claim 2, we have

$$e(G) \leq e(G') = e(H) + e(G' - H) + e_{G'}(H, G' - H) \leq 8 + \binom{n-8}{2} + 4(n-8) + 1.$$

Thus

$$8 + \binom{n-8}{2} + 4(n-8) + 1 > \binom{n}{2} - \left(1 + \sqrt{3n-8}\right)^2.$$

This implies that $n \leq 20$, a contradiction. □

The next theorem we need is a famous theorem due to Hong [12]. In fact, the spectral inequality also works for graphs without isolated vertices, see [12].

Theorem 11 (Hong [12]). *Let G be a connected graph on n vertices and m edges. Then*

$$\mu(G) \leq \sqrt{2m - n + 1}.$$

The equality holds if and only if $G = K_n$ or $K_{1,n-1}$.

Theorem 12 (Hofmeister [11]). *Let G be a graph. Then*

$$\mu(G) \geq \sqrt{\frac{\sum_{v \in V(G)} d^2(v)}{n}}.$$

3. Proofs of the Main Results

Proof of Theorem 5. By Theorem 11, $\mu(G) \leq \sqrt{2m - n + 1}$. Thus $n - 4 \leq \sqrt{2m - n + 1}$ and

$$m \geq \left\lceil \frac{(n-3)(n-4) + 3}{2} \right\rceil = \binom{n-3}{2} + 2.$$

Note that $\mu(M) = 2.6935\ldots < 3$. By Lemma 2, G is traceable or $G = N_{n-3,3}$. □

Proof of Theorem 6. We first give a bound on the value of $\mu(\overline{N_{n-3,3}})$. By using Theorem 2.8 in [4] and some computing, we know

$$\mu(K_k \vee (n-k)K_1) = \frac{k-1+\sqrt{4kn-(3k-1)(k+1)}}{2}.$$

Thus $\mu(K_3 \vee (n-3)K_1) = 1+\sqrt{3n-8}$. From the fact $\overline{N_{n-3,3}} \subset K_3 \vee (n-3)K_1$, we obtain

$$\mu(\overline{N_{n-3,3}}) < 1+\sqrt{3n-8}$$

for any $n \geq 6$.

Now we prove the theorem. The idea of our proof comes from [8]. We assume that G is not traceable. Let $G' = cl(G)$. By Theorem 10, G' is not traceable. By Lemma 1, for any pair of nonadjacent vertices u, v of G', $d_{G'}(u) + d_{G'}(v) \leq n-2$, and hence

$$d_{\overline{G'}}(u) + d_{\overline{G'}}(v) \geq 2(n-1)-(n-2) = n.$$

Furthermore, we have

$$\sum_{v\in V(G)} d^2_{\overline{G'}}(v) = \sum_{uv\in E(\overline{G'})} (d_{\overline{G'}}(u) + d_{\overline{G'}}(v)) \geq ne(\overline{G'}).$$

Note that $\overline{G'} \subseteq \overline{G}$. By Theorem 12,

$$\mu(\overline{G}) \geq \mu(\overline{G'}) \geq \sqrt{\frac{\sum_{v\in V(G)} d^2_{\overline{G'}}(v)}{n}} \geq \sqrt{e(\overline{G'})}.$$

Thus we have

$$e(G') = \binom{n}{2} - e(\overline{G'}) \geq \binom{n}{2} - \mu^2(\overline{G}) > \binom{n}{2} - \left(1+\sqrt{3n-8}\right)^2.$$

Recall that G' is claw-free and not traceable. By Lemma 3, $G' \subseteq N_{n-3,3}$. Thus $G \subseteq N_{n-3,3}$. But if $G \subset N_{n-3,3}$, then $\mu(\overline{G}) > \mu(\overline{N_{n-3,3}})$, a contradiction. This implies $G = N_{n-3,3}$. The proof is complete. □

4. Concluding Remarks

In this section, we give a brief discussion of the existence of Hamilton cycles in claw-free graphs under spectral condition.

Following the notations in [2], we use $\mathcal{P}$ to denote the class of graphs obtained by taking two vertex-disjoint triangles $a_1a_2a_3a_1$ and $b_1b_2b_3b_1$, and by joining every pair of vertices $\{a_i, b_i\}$ by a triangle or by a path of order at least 3. We use P_{x_i,x_2,x_3} to denote the graph from $\mathcal{P}$, where $x_i = T$ if $\{a_i, b_i\}$ is joined by a triangle; and $x_i = k_i$ if $\{a_i, b_i\}$ is joined by a path of order $k_i \geq 3$.

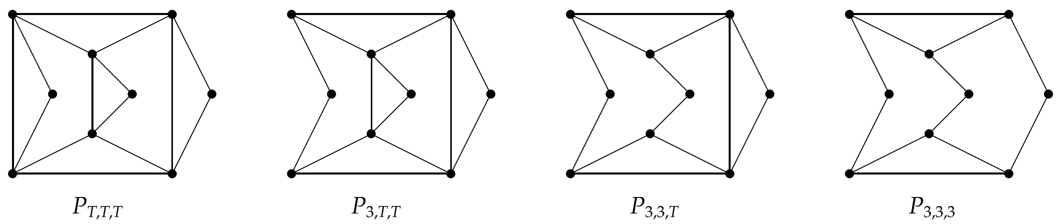

Fig. 3. 2-connected claw-free non-Hamiltonian graphs of order 9.

Brousek [2] showed that every 2-connected claw-free non-Hamiltonian graph contains a graph in $\mathcal{P}$ as an induced subgraph. By Brousek's result, we can see that the smallest 2-connected claw-free non-Hamiltonian graphs have order 9, and there are exactly four such graphs, namely, $P_{T,T,T}$, $P_{3,T,T}$, $P_{3,3,T}$ and $P_{3,3,3}$, see Fig. 3.

Let H be a graph from Fig. 3, and let G be a graph obtained from H by replacing one triangle by a complete graph K_{n-6}. Then G is not Hamiltonian and $\mu(G) > n-7$. Recently, we get the following result.

Theorem 13. *Suppose that G is a 2-connected claw-free graph of sufficiently large order n. If $\mu(G) \geq n-7$, then G is Hamiltonian or G is a subgraph of a graph which is obtained from $P_{T,T,T}$, $P_{3,T,T}$, $P_{3,3,T}$ or $P_{3,3,3}$ by replacing a triangle by K_{n-6}.*

For further works on this topic, we refer the reader to [14].

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