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# On Arithmetic Toroidal Groups

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**Abstract.** Andreotti, Gherardelli, and Abe constructed toroidal groups from algebraic number fields. We call these toroidal groups arithmetic toroidal groups. We study the algebraic structures of arithmetic toroidal groups and we show two examples of arithmetic toroidal groups where one is a quasi-Abelian variety and the other has no nonconstant meromorphic functions on it.

## 1. Toroidal Groups

**Definition 1.1.** *A connected complex Lie group $X$ is called a toroidal group, if $H^0(X, \mathcal{O}) = \mathbb{C}$.*

By the definition, $X$ is a complex abelian Lie group and there exists a discrete subgroup $\Gamma$ of $\mathbb{C}^n$ such that $X \cong \mathbb{C}^n/\Gamma$, where $n$ is a complex dimension of $X$. Put $r = \operatorname{rank}\Gamma$, then there exist $\mathbb{R}$-linearly independent vectors $\lambda_1, \ldots, \lambda_r \in \mathbb{C}^n$ satisfying $\Gamma = \mathbb{Z}\{\lambda_1, \ldots, \lambda_r\}$. A matrix $P = [\lambda_1, \ldots, \lambda_r]$ is called a period matrix for $\mathbb{C}^n/\Gamma$. We have the following (cf. [2])

**Proposition 1.2.** *A complex abelian Lie group $\mathbb{C}^n/\Gamma$ is a toroidal group if and only if the following condition is satisfied:*

$$\text{for any } \sigma \neq 0 \in \mathbb{C}^n, \ {}^t\sigma P \notin \mathbb{Z}^{1\times r}. \tag{1.1}$$

Hence if a complex abelian Lie group $\mathbb{C}^n/\Gamma$ is a toroidal group, we have $n+1 \leq r \leq 2n$. Put $\mathbb{R}_\Gamma = \mathbb{R}\{\lambda_1, \ldots, \lambda_r\}$, then $K := \mathbb{R}_\Gamma/\Gamma$ is a maximal compact subgroup of $\mathbb{C}^n/\Gamma$. Let $\mathbb{C}_\Gamma := \mathbb{R}_\Gamma \cap \sqrt{-1}\mathbb{R}_\Gamma$ be the maximal complex subspace of $\mathbb{R}_\Gamma$. By the result of Matsushima and Morimoto [5], a complex abelian Lie group $\mathbb{C}^n/\Gamma$ is a Stein group if and only if $\mathbb{C}_\Gamma = \{0\}$. Hence for a toroidal group $\mathbb{C}^n/\Gamma$, we have $\dim_C \mathbb{C}_\Gamma > 0$.

**Definition 1.3.** *A toroidal group $\mathbb{C}^n/\Gamma$ is said to be type $q$, if $\dim_C \mathbb{C}_\Gamma = q \, (q > 0)$.*

Then, a toroidal group $\mathbb{C}^n/\Gamma$ is type $q$ if and only if $\operatorname{rank}\Gamma = n + q$.

Let $\mathbb{C}^n/\Gamma$ be a complex abelian Lie group with $\operatorname{rank}\Gamma = n + q$. By a suitable linear change of $\mathbb{C}^n$, a period matrix $P = [\lambda_1, \ldots, \lambda_{n+q}]$ can be written as $P = [I_n, V]$, where $I_n = [e_1, \ldots, e_n]$ is an identity matrix and $V = [v_{ij}; 1 \leq i \leq n, 1 \leq j \leq q] = [v_1, \ldots, v_q]$ is an $n \times q$ matrix. Put $V_1 = [v_{ij}; 1 \leq i, j \leq q]$, and $V_2 = [v_{ij}; q+1 \leq i \leq n, 1 \leq j \leq q]$. We may assume $\det(\operatorname{Im} V_1) \neq 0$.

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**Definition 1.4.** *A toroidal group $\mathbb{C}^n/\Gamma$ is said to be a quasi-Abelian variety if there exists a Hermitian form $H$ on $\mathbb{C}^n$ satisfying*

(1) $H\,|\,\mathbb{C}_\Gamma \times \mathbb{C}_\Gamma > 0$ *and*

(2) $E := \operatorname{Im} H\,|\,\Gamma \times \Gamma$ *is a $\mathbb{Z}$-valued skew-symmetric form.*

$H$ is called an ample Riemann form. We denote by the same symbol $E$ the matix $[E_{ij} = E(\lambda_i, \lambda_j);\ 1 \le i,\ j \le n+q]$. Then we have the following ([6])

**Theorem 1.5.** *Let $\mathbb{C}^n/\Gamma$ be a toroidal group of type $q$, with a period matrix of the form $P = [\lambda_1, \ldots, \lambda_{n+q}] = [I_n, V]$.*

(1) *If $\mathbb{C}^n/\Gamma$ is a quasi-Abelian variety with an ample Riemann form $H$, then $E := \operatorname{Im} H|\Gamma \times \Gamma$ is a $\mathbb{Z}$-valued skew-symmetric form satisfies the following conditions:*

(PI) : $\quad {}^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3 = 0$ *and*

(PII) : $\quad \dfrac{\sqrt{-1}}{2}({}^t\overline{V}E_1V + {}^tE_2V - {}^t\overline{V}E_2 + E_3) > 0,$

*where* $E = \begin{bmatrix} E_1 & E_2 \\ -{}^tE_2 & E_3 \end{bmatrix}$, $E_1 \in \mathbb{Z}^{n\times n}$, *and* $E_3 \in \mathbb{Z}^{q\times q}$.

(2) *Conversely, if we have a $\mathbb{Z}$-valued skew-symmetric matrix $E = [E_{ij}; 1 \le i, j \le n+q] \in \mathbb{Z}^{(n+q)\times(n+q)}$, which satisfies* (PI) *and* (PII), *then $\mathbb{C}^n/\Gamma$ is a quasi-Abelian variety with an ample Riemann form $H$ satisfying* $\operatorname{Im} H|\Gamma\times\Gamma = E$.

Consider the exact sequence

$$\to H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \xrightarrow{\iota} H^2(X, \mathcal{O}) \to$$

**Definition 1.6.** $\operatorname{NS}(X) := c_1H^1(X, \mathcal{O}^*)$ *is called a Néron-Severi group.*

In the proof of Theorem 1.5, we have shown the following

**Theorem 1.7.** *Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group of type $q$ with a period matrix $P = [I_n, V]$. Then $E \in H^2(X, \mathbb{Z})$ belongs to* NS(X) *if and only if $E$ is a $\mathbb{Z}$-valued skew-symmetric $(n+q, n+q)$-matrix satisfying the condition* (PI) *in* Theorm 1.5.

Further we have the following theorem([4]).

**Theorem 1.8.** *Let $X = \mathbb{C}^n/\Gamma$ be a toroidal group.*
*If the Néron-Severi group* NS (X) = 0, *then there are no nonconstant meromorphic functions on X.*

## 2. Arithmetic Toroidal Groups

### *2.1. Arithmetic Toroidal Group*

Let $K$ be a non-totally real algebraic number field of degree $n+q$ with $2q$ complex embeddings $\varphi_i,\ \overline{\varphi_i} : K \longrightarrow \mathbb{C}\ \ (i = 1, \ldots, q)$ and $n-q$ real embeddings $\psi_j : K \longrightarrow \mathbb{R}\ \ (j = 1, \ldots, n-q)$.

For $x \in K$, put $\Psi(x) = (\varphi_1(x), \ldots, \varphi_q(x), \psi_1(x), \ldots, \psi_{n-q}(x))$. Then we get a canonical mapping $\Psi : K \longrightarrow \mathbb{C}^q \times \mathbb{R}^{n-q} \subset \mathbb{C}^n$.

Let $o_K$ be the ring of integers of $K$. Then $\Gamma = \Psi(o_K)$ is a discrete subgroup of $\mathbb{C}^n$ with $\operatorname{rank}\Gamma = n+q$ and $X = \mathbb{C}^n/\Gamma$ is a complex Lie Group. Andreotti and Gherardelli [3], Abe [1] proved the following

**Theorem 2.1.** *The complex Lie group $X = \mathbb{C}^n/\Gamma$ defined by an algebraic number field $K$ is a toroidal group.*

**Definition 2.2.** *We call the toroidal group defined by an algebraic number field an arithmetic toroidal group.*

We study whether an arithmetic toroidal group is a quasi-Abelian variety. We consider the case $K = \mathbb{Q}(\sqrt[3]{2})$ and the case $K = \mathbb{Q}(\sqrt[5]{2})$.

*2.2. The Toroidal Group Defined by* $\mathbb{Q}(\sqrt[3]{2})$

Put $\alpha = \sqrt[3]{2}, K = \mathbb{Q}(\alpha)$ and $\omega = \exp(\frac{2\pi i}{3})$. Then $o_K = \mathbb{Z}\{1,\ \alpha,\ \alpha^2\}$.
Let $\sigma$ and $\omega$ be conjugate mappings defined by $\sigma(\alpha) = \alpha\,\omega$ and $\tau(\alpha) = \alpha$.

For $K \ni x = x_1 + x_2\,\alpha + x_3\,\alpha^2$, we set

$\varphi(x) = x_1 + x_2\,\sigma(\alpha) + x_3\,\sigma(\alpha^2) = x_1 + x_2\,\alpha\,\omega + x_3\,\alpha^2\,\omega^2$ and

$\psi(x) = x_1 + x_2\,\tau(\alpha) + x_3\,\tau(\alpha^2) = x_1 + x_2\,\alpha + x_3\,\alpha^2$.

Then the embedding $\Psi(x) = \begin{pmatrix}\varphi(x)\\ \psi(x)\end{pmatrix} \in \mathbb{C}^2$ is defined and

$$\Psi(o_K) = \mathbb{Z}\{\Psi(1),\ \Psi(\alpha),\ \Psi(\alpha^2)\} = \mathbb{Z}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}\alpha\,\omega\\ \alpha\end{pmatrix}, \begin{pmatrix}\alpha^2\,\omega^2\\ \alpha^2\end{pmatrix}\right\}$$

Put $A = \begin{pmatrix}1 & \alpha\,\omega\\ 1 & \alpha\end{pmatrix}^{-1}$ and $\Gamma = A\Psi(o_K)$. Then $\Gamma = \mathbb{Z}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-\alpha^2\,\omega\\ -\alpha\,\omega^2\end{pmatrix}\right\}$.

**Proposition 2.3.** *The toroidal group* $X = \mathbb{C}^2/\Gamma$ *defined by* $\mathbb{Q}(\alpha)$ *is a quasi-Abelian variety.*

*Proof.* We shall check the conditions PI and PII in Theorem 1.5.

Let $E = \begin{pmatrix}E_1 & E_2\\ -{}^tE_2 & E_3\end{pmatrix} \in \mathbb{Z}^{3\times 3},\ E_1 = \begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}, E_2 = \begin{pmatrix}b\\ c\end{pmatrix}, E_3 = 0$, and $V = \begin{pmatrix}-\alpha^2\,\omega\\ -\alpha\,\omega^2\end{pmatrix}$.

Put $K1 := {}^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3$, then $K1 \in \mathbb{C}$ and $K1 = {}^tK1 = -K1$. Hence $K1 = 0$ and the condition PI holds.

Next put $K2 := \dfrac{\sqrt{-1}}{2}({}^t\overline{V}E_1V + {}^tE_2V - {}^t\overline{V}E_2 + E_3)$.

Then $K2 = \dfrac{\sqrt{3}}{2}(-2a + \alpha^2\,b - \alpha\,c)$. We find $a,\ b,\ c \in \mathbb{Z}$ satisfying K2> 0 and the condition PII holds. Hence $X$ is a quasi-Abelian variety. □

We note that since $X$ is a 2-$\dim_C$ toroidal group and a principal bundle over a 1-$\dim_C$ complex torus we see $X$ is a quasi-Abelian variety. Here we show the proof using the conditions PI and PII.

*2.3. The Toroidal Group Defined by* $\mathbb{Q}(\sqrt[5]{2})$

Put $\alpha = \sqrt[5]{2}, K = \mathbb{Q}(\alpha)$ and $\omega = \exp(\frac{2\pi i}{5})$. Then $o_K = \mathbb{Z}\{1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4\}$.
Let $\sigma_1,\ \sigma_2,\ \tau$ be conjugate mappings defined by $\sigma_1(\alpha) = \alpha\,\omega,\ \sigma_2(\alpha) = \alpha\,\omega^2$ and $\tau(\alpha) = \alpha$.

For $K \ni x = x_1 + x_2\,\alpha + x_3\,\alpha^2 + x_4\,\alpha^3 + x_5\,\alpha^4$, we set

$\varphi_1(x) = x_1 + x_2\,\sigma_1(\alpha) + x_3\,\sigma_1(\alpha^2) + x_4\,\sigma_1(\alpha^3) + x_5\,\sigma_1(\alpha^4) = x_1 + x_2\,\alpha\,\omega + x_3\,\alpha^2\,\omega^2 + x_4\,\alpha^3\,\omega^3 + x_5\,\alpha^4\,\omega^4$,

$\varphi_2(x) = x_1 + x_2\,\sigma_2(\alpha) + x_3\,\sigma_2(\alpha^2) + x_4\,\sigma_2(\alpha^3) + x_5\,\sigma_2(\alpha^4) = x_1 + x_2\,\alpha\,\omega^2 + x_3\,\alpha^2\,\omega^4 + x_4\,\alpha^3\,\omega^6 + x_5\,\alpha^4\,\omega^8$, and

$\psi(x) = x_1 + x_2\,\tau(\alpha) + x_3\,\tau(\alpha^2) + x_4\,\tau(\alpha^3) + x_5\,\tau(\alpha^4) = x_1 + x_2\,\alpha + x_3\,\alpha^2 + x_4\,\alpha^3 + x_5\,\alpha^4$.

Then the embedding $\Psi(x) = \begin{pmatrix}\varphi_1(x)\\ \varphi_2(x)\\ \psi(x)\end{pmatrix} \in \mathbb{C}^3$ is defined and

$$\Psi(o_K) = \mathbb{Z}\{\Psi(1),\ \Psi(\alpha),\ \Psi(\alpha^2),\ \Psi(\alpha^3),\ \Psi(\alpha^4)\} = \mathbb{Z}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}\alpha\,\omega\\ \alpha\,\omega^2\\ \alpha\end{pmatrix}, \begin{pmatrix}\alpha^2\,\omega^2\\ \alpha^2\,\omega^4\\ \alpha^2\end{pmatrix}, \begin{pmatrix}\alpha^3\,\omega^3\\ \alpha^3\,\omega^6\\ \alpha^3\end{pmatrix}, \begin{pmatrix}\alpha^4\,\omega^4\\ \alpha^4\,\omega^8\\ \alpha^4\end{pmatrix}\right\}.$$

Put $A = \begin{pmatrix} 1 & \alpha\,\omega & \alpha^2\,\omega^2 \\ 1 & \alpha\,\omega^2 & \alpha^2\,\omega^4 \\ 1 & \alpha & \alpha^2 \end{pmatrix}^{-1}$ and $\Gamma = A\Psi(o_K)$, then $\Gamma = \mathbb{Z}\{e_1, e_2, e_3, V1, V2\}$, where

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $(1-\omega^2)V1 = \begin{pmatrix} \alpha^3(\omega+\omega^2+2\omega^3+\omega^4) \\ -\alpha^2(1+\omega)(1+2\omega+\omega^2+\omega^3) \\ \alpha(2+2\omega+\omega^2) \end{pmatrix}$, and

$(1-\omega^2)V2 = \begin{pmatrix} \alpha^4(1+2\omega^3+2\omega^4) \\ -\alpha^3(1+\omega)(\omega+2\omega^2+\omega^3+\omega^4) \\ \alpha^2(1+\omega+2\omega^2+\omega^3) \end{pmatrix}$.

**Proposition 2.4.** *The toroidal group $X = \mathbb{C}^3/\Gamma$ defined by $\mathbb{Q}(\alpha)$ has no nonconstant meromorphic functions on it.*

*Proof.* We shall show that $NS(X) = 0$.

Let $E = \begin{pmatrix} E_1 & E_2 \\ -{}^tE_2 & E_3 \end{pmatrix} \in \mathbb{Z}^{5\times 5}$, $E_1 = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} d & e \\ f & g \\ h & k \end{pmatrix}$,

$E_3 = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$, and $V = (V1, V2) \in \mathbb{C}^{3\times 2}$.

Let $K := {}^tVE_1V + {}^tE_2V - {}^tVE_2 + E_3 = \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$.

Suppose $K = 0$, then $Y = 0$, and $(1-\omega^2)Y = A_1 + A_2\omega + A_3\omega^2 + A_4\omega^3 = 0$, where

$A_1 = -m + 2k\alpha - h\alpha^2 - (e+f)\alpha^3 + (c+d)\alpha^4$,

$A_2 = -2b - (2a-2k)\alpha - (2g+h)\alpha^2 - f\alpha^3 + (2c+2d)\alpha^4$,

$A_3 = m - 4b + k\alpha - (2g+2h)\alpha^2 + f\alpha^3 + (c+2d)\alpha^4$, and

$A_4 = -4b + 2a\alpha - (g+h)\alpha^2 + (e+f)\alpha^3 + c\alpha^4$.

Since 1, $\omega$, $\omega^2$ and $\omega^3$ are linearly independent over $\mathbb{R}$, we have $A_1 = A_2 = A_3 = A_4 = 0$.

Further $a, b, \ldots, m \in \mathbb{Z}$, we get $a, b, \ldots, m = 0$. Hence $NS(X) = 0$ and $X$ has no noncostant meromorphic functions on it. □

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