Filomat 31:1 (2017), 113–123
DOI 10.2298/FIL1701113L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Convolution Properties for Certain Meromorphically Multivalent Functions

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Abstract. Let Σ_p denote the class of functions of the form $f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in N)$. Two new subclasses $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$ of meromorphically multivalent functions starlike with respect to k-symmetric points in Σ_p are investigated. Certain convolution properties for these subclasses are obtained.

1. Introduction and Preliminaries

Throughout this paper we assume that

$$N = \{1, 2, 3, \cdots\}, k \in N \setminus \{1\}, -1 \leq B < 0, B < A \leq -B \text{ and } 1 \leq \lambda < +\infty. \tag{1.1}$$

For functions f and g analytic in the open unit disk $U = \{z : |z| < 1\}$, the function f is said to be subordinate to g, written $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ in U.

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in N), \tag{1.2}$$

which are analytic in the punctured open unit disk $U_0 = U \setminus \{0\}$.

Let

$$f_j(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,j} z^n \in \Sigma_p \quad (j = 1, 2).$$

The Hadamard product (or convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,1} a_{n,2} z^n.$$

2010 *Mathematics Subject Classification*. Primary 30C45, 30C80.
Keywords. Meromorphically multivalent function; Hadamard product (or convolution); subordination; k-symmetric point.
Received: 09 January 2017; Accepted: 13 January 2017
Communicated by Hari M. Srivastava
This work is supported by National Natural Science Foundation of China(Grant No.11571299) and Natural Science Foundation of Jiangsu Province(Grant No. BK20151304).
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In [3] (see also [4]) Dziok obtained the following result.
Lemma. Let $f \in \Sigma_p$ defined by (1.2) satisfy

$$\sum_{n=p}^{\infty}[p(1-A)\delta_{n,p,k} + (1-B)(n\lambda + p(\lambda-1))]|a_n| \leq p(A-B), \tag{1.3}$$

where

$$\delta_{n,p,k} = \begin{cases} 0 & \left(\frac{n+p}{k} \notin N\right), \\ 1 & \left(\frac{n+p}{k} \in N\right). \end{cases} \tag{1.4}$$

Then

$$\frac{p(1-\lambda)f(z) - \lambda z f'(z)}{p f_{p,k}(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{1.5}$$

where

$$f_{p,k}(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{jp} f(\varepsilon_k^j z) \quad and \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right). \tag{1.6}$$

Recently, Liu and Srivastava [9] introduced and investigated two new subclasses $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$ of Σ_p as follows.

A function $f \in \Sigma_p$ is said to be in the class $H_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (1.3).

Also, a function $f \in \Sigma_p$ is said to be in the class $Q_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality

$$\sum_{n=p}^{\infty} n[p(1-A)\delta_{n,p,k} + (1-B)(n\lambda + p(\lambda-1))]|a_n| \leq p^2(A-B). \tag{1.7}$$

For $f \in \Sigma_p$, one can see that

$$f \in Q_{p,k}(\lambda, A, B) \quad \text{if and only if} \quad 2z^{-p} + \frac{zf'(z)}{p} \in H_{p,k}(\lambda, A, B). \tag{1.8}$$

In [9] the authors pointed out that each function in the classes $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$ is meromorphically starlike with respect to k-symmetric points.

The subject of meromorphically univalent and multivalent functions continue to receive a great deal of attention. Very recently, Srivastava et al. (see, e.g., [14, 15, 16, 17, 18, 19, 20, 22]) investigated various subclasses of meromorphic functions. Certain properties such as distortion bounds, inclusion relations and integral transforms for these subclasses are studied. Motivated essentially by these works and some other works (see also, e.g., [1], [2], [5-13], [21] and [23-26]), we aim at investigating here convolution properties of the subclasses $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$.

2. Convolution Properties for the Classes $H_{p,k}(\lambda, A, B)$ and $Q_{p,k}(\lambda, A, B)$

In this section we assume that

$$-1 \le B_j < 0 \quad \text{and} \quad B_j < A_j \le -B_j \quad (j = 1, 2). \tag{2.1}$$

Furthermore, we denote by λ_1 the root in $(1, +\infty)$ of the equation:

$$h(\lambda) = a\lambda^2 + b\lambda + c = 0,$$

where

$$\begin{cases} a = -(1 - B_1)(1 - B_2), \\ b = p[(1 - B_1)(1 - B_2) - (1 - B_1)(A_2 - B_2) - (1 - B_2)(A_1 - B_1)], \\ c = p(A_1 - B_1)(A_2 - B_2). \end{cases} \tag{2.2}$$

We also denote

$$A(B) = B + \frac{1 - B}{2\lambda \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} - \sum_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda}} \tag{2.3}$$

and

$$\widetilde{A}(B) = B + \frac{p(1 - B)}{(2p + 1)\lambda - p} \prod_{j=1}^{2} \frac{A_j - B_j}{1 - B_j}. \tag{2.4}$$

Theorem 1. Let

$$f_j \in H_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2)$$

with

$$\frac{2p}{k} \in N \quad \textit{and} \quad -1 \le B \le \max\{B_1, B_2\}.$$

Then we have the following:

(i) If $p(1 - A_1)(1 - A_2) \le (1 - B_1)(1 - B_2)$ and $\lambda \ge 1$, then

$$f_1 * f_2 \in H_{p,k}(\lambda, A(B), B).$$

(ii) If $p(1 - A_1)(1 - A_2) > (1 - B_1)(1 - B_2)$ and $\lambda \ge \lambda_1$, then

$$f_1 * f_2 \in H_{p,k}(\lambda, A(B), B).$$

(iii) If $p(1 - A_1)(1 - A_2) > (1 - B_1)(1 - B_2)$ and $1 \le \lambda < \lambda_1$, then

$$f_1 * f_2 \in H_{p,k}(\lambda, \widetilde{A}(B), B).$$

In all cases (i)-(iii) the numbers $A(B)$ and $\widetilde{A}(B)$ are optimal in the sense that they cannot be decreased for each B.

Proof. Suppose that $-1 \le B \le \max\{B_1, B_2\} = B_j$ ($j = 1$ or 2). It follows from (2.1) and (2.3) that

$$\begin{aligned} \frac{1 - B}{A(B) - B} &= 2\lambda \prod_{j=1}^{2} \frac{1 - B_j}{A_j - B_j} - \sum_{j=1}^{2} \frac{1 - B_j}{A_j - B_j} + \frac{1}{\lambda} \\ &= 2\lambda \prod_{j=1}^{2} \frac{1 - B_j}{A_j - B_j} - \left(\prod_{j=1}^{2} \frac{1 - B_j}{A_j - B_j} - \prod_{j=1}^{2} \frac{1 - A_j}{A_j - B_j} + 1 \right) + \frac{1}{\lambda} \\ &\ge (2\lambda - 1) \prod_{j=1}^{2} \frac{1 - B_j}{A_j - B_j} - \frac{\lambda - 1}{\lambda} \\ &\ge \prod_{j=1}^{2} \frac{1 - B_j}{A_j - B_j} \ge -\frac{1 - B}{2B} > 0, \end{aligned}$$

which implies that $B < A(B) \leq -B$. Also, (2.1) and (2.4) give that

$$\frac{1-B}{\widetilde{A}(B)-B} = \frac{(2p+1)\lambda - p}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} \geq \frac{1-B_j}{A_j-B_j} \geq -\frac{1-B}{2B} > 0,$$

which implies that $B < \widetilde{A}(B) \leq -B$.

Let $\frac{2p}{k} \in N$ and

$$f_j(z) = z^{-p} + \sum_{n=p}^{\infty} a_{n,j}z^n \in H_{p,k}(\lambda, A_j, B_j) \quad (z \in U_0;\ j = 1,2).$$

Then

$$\sum_{n=p}^{\infty}\left\{\prod_{j=1}^{2}\frac{p(1-A_j)\delta_{n,p,k} + (1-B_j)(n\lambda + p(\lambda-1))}{p(A_j-B_j)}\right\}|a_{n,1}a_{n,2}|$$
$$\leq \prod_{j=1}^{2}\left\{\sum_{n=p}^{\infty}\frac{p(1-A_j)\delta_{n,p,k} + (1-B_j)(n\lambda + p(\lambda-1))}{p(A_j-B_j)}|a_{n,j}|\right\} \leq 1. \tag{2.5}$$

Also, $f_1 * f_2 \in H_{p,k}(\lambda, A, B)$ if and only if

$$\sum_{n=p}^{\infty}\frac{p(1-A)\delta_{n,p,k} + (1-B)(n\lambda + p(\lambda-1))}{p(A-B)}|a_{n,1}a_{n,2}| \leq 1. \tag{2.6}$$

In order to prove Theorem 1, it follows from (2.5) and (2.6) that we only need to find the smallest A such that

$$\frac{p(1-A)\delta_{n,p,k} + (1-B)(n\lambda + p(\lambda-1))}{p(A-B)} \leq \prod_{j=1}^{2}\frac{p(1-A_j)\delta_{n,p,k} + (1-B_j)(n\lambda + p(\lambda-1))}{p(A_j-B_j)} \quad (n \geq p). \tag{2.7}$$

For $n \geq p$ and $\frac{n+p}{k} \in N$, (2.7) is equivalent to

$$A \geq B + \frac{1-B}{\frac{\lambda(n+p)}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{2p}{\lambda(n+p)}} = \varphi_1(n). \tag{2.8}$$

It can be verified that $\varphi_1(n)$ $(n \geq p, \lambda \geq 1)$ is decreasing in n and so, in view of $\frac{2p}{k} \in N$,

$$\varphi_1(n) \leq \varphi_1(p) = B + \frac{1-B}{2\lambda\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda}}. \tag{2.9}$$

For $n \geq p$ and $\frac{n+p}{k} \notin N$, (2.7) becomes

$$A \geq B + \frac{1-B}{\frac{n\lambda + p(\lambda-1)}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = \psi_1(n) \tag{2.10}$$

and we have

$$\psi_1(n) \leq \psi_1(p+1) = B + \frac{1-B}{\frac{(2p+1)\lambda - p}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}}. \tag{2.11}$$

Now

$$2\lambda \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} - \sum_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda} - \frac{(2p+1)\lambda - p}{p} \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j}$$
$$= \frac{h(\lambda)}{p\lambda(A_1-B_1)(A_2-B_2)}, \tag{2.12}$$

where

$$\begin{aligned} h(\lambda) &= \lambda(p-\lambda)(1-B_1)(1-B_2) - p\lambda[(1-B_1)(A_2-B_2) \\ &\quad + (1-B_2)(A_1-B_1)] + p(A_1-B_1)(A_2-B_2) \\ &= a\lambda^2 + b\lambda + c \end{aligned} \tag{2.13}$$

with

$$\begin{aligned} &a = -(1-B_1)(1-B_2), \\ &b = p[(1-B_1)(1-B_2) - (1-B_1)(A_2-B_2) - (1-B_2)(A_1-B_1)], \\ &c = p(A_1-B_1)(A_2-B_2). \end{aligned}$$

Note that $a < 0$, $h(0) = c > 0$ and

$$\begin{aligned} h(1) &= (p-1)(1-B_1)(1-B_2) - p[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)] \\ &\quad + p(A_1-B_1)(A_2-B_2) \\ &= p(1-A_1)(1-A_2) - (1-B_1)(1-B_2). \end{aligned} \tag{2.14}$$

Therefore, if (i) or (ii) is satisfied, then it follows from (2.7) to (2.14) that $h(\lambda) \le 0$, $\psi_1(p+1) \le \varphi_1(p) = A(B)$, and $f_1 * f_2 \in H_{p,k}(\lambda, A(B), B)$.

Furthermore, for $B < A_0 < A(B)$, we have

$$\frac{1-A+(1-B)(2\lambda-1)}{A_0-B} \prod_{j=1}^{2} \frac{A_j-B_j}{1-A_j+(1-B_j)(2\lambda-1)}$$
$$> \frac{1-A(B)+(1-B)(2\lambda-1)}{A(B)-B} \prod_{j=1}^{2} \frac{A_j-B_j}{1-A_j+(1-B_j)(2\lambda-1)} = 1.$$

Hence for functions

$$f_j(z) = z^{-p} + \frac{A_j-B_j}{1-A_j+(1-B_j)(2\lambda-1)} z^p \in H_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2)$$

we have $f_1 * f_2 \notin H_{p,k}(\lambda, A_0, B)$.

(iii) If $p(1-A_1)(1-A_2) > (1-B_1)(1-B_2)$ and $1 \le \lambda < \lambda_1$, then we have $h(\lambda) > 0$, $\varphi_1(p) < \psi_1(p+1) = \widetilde{A}(B)$, and $f_1 * f_2 \in H_{p,k}(\lambda, \widetilde{A}(B), B)$. Furthermore, the number $\widetilde{A}(B)$ cannot be decreased as can be seen from the functions f_j defined by

$$f_j(z) = z^{-p} + \frac{p(A_j-B_j)}{((2p+1)\lambda-p)(1-B_j)} z^{p+1} \in H_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2).$$

Theorem 2. Let

$$f_1 \in H_{p,k}(\lambda, A_1, B_1) \quad \textit{and} \quad f_2 \in Q_{p,k}(\lambda, A_2, B_2)$$

with

$$\frac{2p}{k} \in N \quad and -1 \leq B \leq \max\{B_1, B_2\}.$$

Also let $A(B), \widetilde{A}(B)$ and λ_1 be given as in Theorem 1. Then we have the following:

(i) If $p(1 - A_1)(1 - A_2) \leq (1 - B_1)(1 - B_2)$ and $\lambda \geq 1$, then

$$f_1 * f_2 \in Q_{p,k}(\lambda, A(B), B).$$

(ii) If $p(1 - A_1)(1 - A_2) > (1 - B_1)(1 - B_2)$ and $\lambda \geq \lambda_1$, then

$$f_1 * f_2 \in Q_{p,k}(\lambda, A(B), B).$$

(iii) If $p(1 - A_1)(1 - A_2) > (1 - B_1)(1 - B_2)$ and $1 \leq \lambda < \lambda_1$, then

$$f_1 * f_2 \in Q_{p,k}(\lambda, \widetilde{A}(B), B).$$

In all cases (i)-(iii) the numbers $A(B)$ and $\widetilde{A}(B)$ are optimal in the sense that they cannot be decreased for each B.

Proof. Since

$$f_1 \in H_{p,k}(\lambda, A_1, B_1), \;\; 2z^{-p} + \frac{zf_2'(z)}{p} \in H_{p,k}(\lambda, A_2, B_2)$$

(see (1.8)), and

$$f_1(z) * \left(2z^{-p} + \frac{zf_2'(z)}{p}\right) = 2z^{-p} + \frac{z(f_1 * f_2)'(z)}{p} \quad (z \in U_0),$$

the assertion of the theorem follows from Theorem 1.

Next, we denote by λ_2 the root in $(1, +\infty)$ of the equation:

$$h_1(\lambda) = a_1\lambda^2 + b_1\lambda + c_1 = 0,$$

where

$$\begin{cases} a_1 = -(3p + 1)(1 - B_1)(1 - B_2), \\ b_1 = p(p + 1)(1 - B_1)(1 - B_2) - p^2[(1 - B_1)(A_2 - B_2) + (1 - B_2)(A_1 - B_1)], \\ c_1 = p^2(A_1 - B_1)(A_2 - B_2). \end{cases} \tag{2.15}$$

We also denote

$$\widetilde{A_1}(B) = B + \frac{p^2(1 - B)}{(p + 1)((2p + 1)\lambda - p)} \prod_{j=1}^{2} \frac{A_j - B_j}{1 - B_j}. \tag{2.16}$$

Theorem 3. Let

$$f_1 \in H_{p,k}(\lambda, A_1, B_1) \quad and \quad f_2 \in Q_{p,k}(\lambda, A_2, B_2)$$

with

$$\frac{2p}{k} \in N \quad and \quad -1 \leq B \leq \max\{B_1, B_2\}.$$

Then we have the following:

(i) If $p^2(1 - A_1)(1 - A_2) \leq (2p + 1)(1 - B_1)(1 - B_2)$ and $\lambda \geq 1$, then

$$f_1 * f_2 \in H_{p,k}(\lambda, A(B), B).$$

(ii) If $p^2(1 - A_1)(1 - A_2) > (2p + 1)(1 - B_1)(1 - B_2)$ and $\lambda \geq \lambda_2$, then

$$f_1 * f_2 \in H_{p,k}(\lambda, A(B), B).$$

(iii) If $p^2(1-A_1)(1-A_2) > (2p+1)(1-B_1)(1-B_2)$ and $1 \le \lambda < \lambda_2$, then

$$f_1 * f_2 \in H_{p,k}(\lambda, \widetilde{A_1}(B), B).$$

In all cases (i)-(iii) the numbers $A(B)$ and $\widetilde{A_1}(B)$ are optimal in the sense that they cannot be decreased for each B.

Proof. It can be verified that

$$\frac{1-B}{\widetilde{A_1}(B)-B} = \frac{(p+1)((2p+1)\lambda - p)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} > \prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} \ge -\frac{1-B}{2B} > 0$$

and so $B < \widetilde{A_1}(B) < -B$.

In order to prove Theorem 3, we only need to find the smallest A such that

$$\frac{p(1-A)\delta_{n,p,k} + (1-B)(n\lambda + p(\lambda-1))}{p(A-B)} \le \frac{n}{p}\prod_{j=1}^{2}\frac{p(1-A_j)\delta_{n,p,k} + (1-B_j)(n\lambda + p(\lambda-1))}{p(A_j-B_j)} \tag{2.17}$$

for all $n \ge p$.

For $n \ge p$ and $\frac{n+p}{k} \in N$, (2.17) is equivalent to

$$A \ge B + \frac{1-B}{\frac{\lambda n(n+p)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \frac{n}{p}\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda}} = \varphi_2(n). \tag{2.18}$$

Define

$$g(\lambda, x) = \frac{\lambda x(x+p)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \frac{x}{p}\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda} \quad (x \ge p; \lambda \ge 1).$$

Then

$$\begin{aligned}
\frac{\partial g(\lambda, x)}{\partial x} &= \frac{\lambda(2x+p)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \frac{1}{p}\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} \\
&\ge \frac{3\lambda}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \frac{1}{p}\left(\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \prod_{j=1}^{2}\frac{1-A_j}{A_j-B_j} + 1\right) \\
&= \frac{3\lambda-1}{p}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \frac{1}{p} + \frac{1}{p}\prod_{j=1}^{2}\frac{1-A_j}{A_j-B_j} \\
&\ge \frac{3\lambda-1}{p} - \frac{1}{p} > 0 \quad (x \ge p; \lambda \ge 1),
\end{aligned}$$

which implies that $\varphi_2(n)$ defined by (2.18) is decreasing in n $(n \ge p)$. Hence, in view of $\frac{2p}{k} \in N$, we have

$$\begin{aligned}
\varphi_2(n) &\le \varphi_2(p) \\
&= B + \frac{1-B}{2\lambda\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda}} = A(B).
\end{aligned}$$

For $n \ge p$ and $\frac{n+p}{k} \notin N$, (2.17) reduces to

$$A \ge B + \frac{1-B}{\frac{n(n\lambda + p(\lambda-1))}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = \psi_2(n)$$

and, in view of $\frac{2p}{k} \in N$, we have

$$\psi_2(n) \leq \psi_2(p+1) = B + \frac{1-B}{\frac{(p+1)((2p+1)\lambda-p)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}}.$$

Now

$$2\lambda\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} - \sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda} - \frac{(p+1)((2p+1)\lambda-p)}{p^2}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}$$
$$= \frac{h_1(\lambda)}{\lambda p^2(A_1-B_1)(A_2-B_2)}, \tag{2.19}$$

where $h_1(\lambda) = a_1\lambda^2 + b_1\lambda + c_1$ and a_1, b_1, c_1 are given by (2.15). Note that $a_1 < 0$, $h_1(0) = c_1 > 0$ and

$$\begin{aligned} h_1(1) &= 2p^2(1-B_1)(1-B_2) - p^2[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)] \\ &\quad + p^2(A_1-B_1)(A_2-B_2) - (p+1)^2(1-B_1)(1-B_2) \\ &= p^2(1-A_1)(1-A_2) - (2p+1)(1-B_1)(1-B_2). \end{aligned}$$

The remaining part of the proof of Theorem 3 is similar to that as in Theorem 1 and hence we omit it. The proof of the Theorem is completed.

From Theorem 3 we have the following theorem at once.

Theorem 4. Let

$$f_j \in Q_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2)$$

with

$$\frac{2p}{k} \in N \quad \textit{and} \quad -1 \leq B \leq \max\{B_1, B_2\}.$$

Also let $A(B), \widetilde{A_1}(B)$ and λ_2 be given as in Theorem 3. Then we have the following:

(i) If $p^2(1-A_1)(1-A_2) \leq (2p+1)(1-B_1)(1-B_2)$ and $\lambda \geq 1$, then

$$f_1 * f_2 \in Q_{p,k}(\lambda, A(B), B).$$

(ii) If $p^2(1-A_1)(1-A_2) > (2p+1)(1-B_1)(1-B_2)$ and $\lambda \geq \lambda_2$, then

$$f_1 * f_2 \in Q_{p,k}(\lambda, A(B), B).$$

(iii) If $p^2(1-A_1)(1-A_2) > (2p+1)(1-B_1)(1-B_2)$ and $1 \leq \lambda < \lambda_2$, then

$$f_1 * f_2 \in Q_{p,k}(\lambda, \widetilde{A_1}(B), B).$$

In all cases (i)-(iii) the numbers $A(B)$ and $\widetilde{A_1}(B)$ are optimal in the sense that they cannot be decreased for each B.

Finally, we denote by λ_3 the root in $(1, +\infty)$ of the equation:

$$h_2(\lambda) = a_2\lambda^2 + b_2\lambda + c_2 = 0,$$

where

$$\begin{cases} a_2 = -[(p+1)^2(2p+1) - 2p^3](1-B_1)(1-B_2), \\ b_2 = p(p+1)^2(1-B_1)(1-B_2) - p^3[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)], \\ c_2 = p^3(A_1-B_1)(A_2-B_2). \end{cases} \tag{2.20}$$

We also denote

$$\widetilde{A_2}(B) = B + \frac{p^3(1-B)}{(p+1)^2((2p+1)\lambda - p)} \prod_{j=1}^{2} \frac{A_j - B_j}{1 - B_j}. \tag{2.21}$$

Theorem 5. Let

$$f_j \in Q_{p,k}(\lambda, A_j, B_j) \quad (j = 1, 2)$$

with

$$\frac{2p}{k} \in N \quad and \quad -1 \leq B \leq \max\{B_1, B_2\}.$$

Then we have the following:

(i) If $p^3(1-A_1)(1-A_2) \leq (3p^2+3p+1)(1-B_1)(1-B_2)$ and $\lambda \geq 1$, then

$$f_1 * f_2 \in H_{p,k}(\lambda, A(B), B).$$

(ii) If $p^3(1-A_1)(1-A_2) > (3p^2+3p+1)(1-B_1)(1-B_2)$ and $\lambda \geq \lambda_3$, then

$$f_1 * f_2 \in H_{p,k}(\lambda, A(B), B).$$

(iii) If $p^3(1-A_1)(1-A_2) > (3p^2+3p+1)(1-B_1)(1-B_2)$ and $1 \leq \lambda < \lambda_3$, then

$$f_1 * f_2 \in H_{p,k}(\lambda, \widetilde{A_2}(B), B).$$

In all cases (i)-(iii) the numbers $A(B)$ and $\widetilde{A_2}(B)$ are optimal in the sense that they cannot be decreased for each B.

Proof. It can be seen that $B < \widetilde{A_2}(B) < -B$. In order to prove Theorem 5, we only need to find the smallest A such that

$$\frac{p(1-A)\delta_{n,p,k} + (1-B)(n\lambda + p(\lambda-1))}{p(A-B)} \leq \left(\frac{n}{p}\right)^2 \prod_{j=1}^{2} \frac{p(1-A_j)\delta_{n,p,k} + (1-B_j)(n\lambda + p(\lambda-1))}{p(A_j - B_j)} \tag{2.22}$$

for $n \geq p$.

For $n \geq p$ and $\frac{n+p}{k} \in N$, (2.22) can be written as

$$A \geq B + \frac{1-B}{\frac{\lambda n^2(n+p)}{p^3} \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} - \frac{n^2}{p^2} \sum_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + \frac{n^2+p^2}{p\lambda(n+p)}} = \varphi_3(n). \tag{2.23}$$

A simple calculation shows that $\varphi_3(n)$ $(n \geq p, \lambda \geq 1)$ is decreasing in n. Therefore

$$\begin{aligned}\varphi_3(n) \leq \varphi_3(p) &= B + \frac{1-B}{2\lambda \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} - \sum_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda}} \\ &= A(B).\end{aligned}$$

For $n \geq p$ and $\frac{n+p}{k} \notin N$, (2.22) becomes

$$A \geq B + \frac{1-B}{\frac{n^2(n\lambda + p(\lambda-1))}{p^3} \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j}} = \psi_3(n)$$

and we have

$$\psi_3(n) \leq \psi_3(p+1) = B + \frac{1-B}{\frac{(p+1)^2((2p+1)\lambda - p)}{p^3} \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j}}.$$

Now

$$2\lambda\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}-\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j}+\frac{1}{\lambda}-\frac{(p+1)^2((2p+1)\lambda-p)}{p^3}\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}$$
$$=\frac{h_2(\lambda)}{\lambda p^3(A_1-B_1)(A_2-B_2)},$$

where $h_2(\lambda)=a_2\lambda^2+b_2\lambda+c_2$ and a_2,b_2,c_2 are given by (2.20). Note that $a_2<0$, $h_2(0)=c_2>0$ and

$$h_2(1)=p^3(1-A_1)(1-A_2)-(3p^2+3p+1)(1-B_1)(1-B_2).$$

The remaining part of the proof is similar to that of Theorem 1 and thus we omit it.

ACKNOWLEDGEMENT

The authors would like to express sincere thanks to the referee for careful reading and suggestions which helped us to improve the paper.

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