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Regular Functions on Dual Split Quaternions in Clifford Analysis

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Abstract. This paper shows expressions of a power series for the form of dual split quaternions and provides differential operators in dual split quaternions. The paper also represents a power series of dual split regular functions by using a dual split Cauchy-Riemann system in dual split quaternions.

1. Introduction

Cockle [2] introduced a set of split quaternions as

$$\mathcal{S} = \{\lambda = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 : x_m \in \mathbb{R}\ ,\ \ m = 0, 1, 2, 3\},$$

where $\mathbb{R}$ is the set of real numbers and

$$\begin{aligned} &e_1^2 = -1,\ \ e_2^2 = e_3^2 = 1,\ \ e_1 e_2 e_3 = 1, \\ &e_1 e_2 = -e_2 e_1 = e_3,\ \ e_2 e_3 = -e_3 e_2 = -e_1,\ \ e_3 e_1 = -e_1 e_3 = e_2. \end{aligned} \tag{1}$$

A set of split quaternions is non-commutative and it contains zero divisors, nilpotent elements, and non-trivial idempotents (see [17]). To solve split quaternionic equations, split quaternions have been applied to geometric and physical (see [1, 3]). Inoguchi [4] reformulated the Gauss-Codazzi equations in forms consistent with the theory of integrable systems in the Minkowski 3-space for split quaternion numbers.

A dual quaternion can be represented in a form reflecting an ordinary quaternion and a dual symbol. Dual quaternion algebra is constructed from real eight-dimensional vector spaces and an ordered pair of quaternions. In addition, from their algebraic properties, dual quaternions are used in computer vision applications. Kenwright [7] provided the characteristics of dual quaternions and explained how dual number theory can extend quaternions to dual-quaternions and how we can use them to represent rigid transforms (i.e., translations and rotations). Pennestri et al. [18] investigated the development and implementation of algorithms for the solution of linear algebra problems using dual numbers. By using properties of Hamilton operators, Kula et al. [14] defined dual split quaternions and gave some properties of the screw motion in

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the Minkowski 3-space, showing that quaternions have a rotation with unit split quaternions. Song et al. [19] gave the notion of coupled quaternions and studied an iterative solution to coupled quaternion matrix equations.

We [5, 6] proved that any complex-valued harmonic function f_1 in a pseudoconvex domain has a conjugate harmonic function f_2 in D such that the quaternion-valued function $f_1 + f_2 j$ is hyperholomorphic in D. In addition, we [8, 9] provided a new expression of the quaternionic basis and a regular function on reduced quaternions by associating hypercomplex numbers e_1 and e_2. We [10, 12] investigated some regular functions with values in dual quaternions and we [15] researched an extension problem for properties of regular functions with values in dual quaternions and some applications for such problems. Recently, we [11, 13] give special representations, calculations and operators of hypercomplex numbers, by using algebraic properties of cosets.

This paper provides some properties of differential operators and a regular function in dual split quaternions. The paper also researches some equivalent conditions for Cauchy-Riemann systems and expressions of a power series in dual split quaternions from the definition of dual split regular functions on an open set $\Omega \subset \mathbb{C}^2 \times \mathbb{C}^2$, where $\mathbb{C}$ is the set of complex numbers.

2. Preliminaries

A dual number has the form $a + \varepsilon b$, where a and b are real numbers, ε is a dual symbol with the rules $\varepsilon \neq 0,\ \ \varepsilon^2 = 0$ and a split quaternion $\lambda \in \mathcal{S}$ has an expression of the form

$$\lambda = x_0 + x_1e_1 + x_2e_2 + x_3e_3,$$

where $x_m \in \mathbb{R}\ \ (m = 0, 1, 2, 3)$ and $e_r\ \ (r = 1, 2, 3)$ are split quaternionic units satisfying noncommutative multiplication rules: (1).
Similarly, a dual split quaternion p can be written as

$$\mathcal{D}(\mathcal{S}) = \{p | p = p_0 + \varepsilon p_1,\ \ p_r \in \mathcal{S},\ \ r = 0, 1\},$$

where $p_0 = z_1 + z_2e_2$ and $p_1 = w_1 + w_2e_2$ are split quaternion components, $z_1 = x_0 + x_1e_1$, $z_2 = x_2 + x_3e_1$, $w_1 = y_0 + y_1e_1$ and $w_2 = y_2 + y_3e_1$ are usual complex numbers, and $x_m,\ y_m \in \mathbb{R}\ \ (m = 0, 1, 2, 3)$.
The multiplication of split quaternionic units and a dual symbol is commutative:

$$\varepsilon e_r = e_r\varepsilon \quad (r = 1, 2, 3).$$

However, by properties of the split quaternionic unit, we have

$$z_ke_r = e_rz_k\ ,\ \ w_ke_r = e_rw_k \quad (k = 1, 2,\ r = 0, 1)$$

and

$$z_ke_r = e_r\overline{z_k}\ ,\ \ w_ke_r = e_r\overline{w_k} \quad (k = 1, 2,\ r = 2, 3),$$

where $\overline{z_1} = x_0 - x_1e_1$, $\overline{z_2} = x_2 - x_3e_1$, $\overline{w_1} = y_0 - y_1e_1$ and $\overline{w_2} = y_2 - y_3e_1$. For any two elements $p = p_0 + \varepsilon p_1$ and $q = q_0 + \varepsilon q_1$ of $\mathcal{D}(\mathcal{S})$, where $q_0 = \sum_{r=0}^{3} s_re_r$, $q_1 = \sum_{r=0}^{3} t_re_r$ are split quaternion components and $s_r,\ t_r \in \mathbb{R}\ \ (r = 0, 1, 2, 3)$, their non-commutative product is given by

$$pq = (p_0 + \varepsilon p_1)(q_0 + \varepsilon q_1) = p_0q_0 + \varepsilon(p_0q_1 + p_1q_0).$$

The conjugation p^* of p and the corresponding modulus $N(p) = pp^*$ in $\mathcal{D}(\mathcal{S})$ are defined by

$$p^* = p_0^* + \varepsilon p_1^*,$$

and

$$\begin{aligned} N(p) &= pp^* = p^*p = p_0p_0^* + \varepsilon(p_0p_1^* + p_1p_0^*) \\ &= (z_1\overline{z_1} - z_2\overline{z_2}) + 2\varepsilon(z_1\overline{w_1} - z_2\overline{w_2}) \\ &= (x_0^2 + x_1^2 - x_2^2 - x_3^2) + \varepsilon(x_0y_0 + x_1y_1 - x_2y_2 - x_3y_3), \end{aligned}$$

where $p_0^* = \overline{z_1} - z_2e_2$ and $p_1^* = \overline{w_1} - w_2e_2$.
From the above operators, we have a division of dual split quaternions:

$$\frac{p}{q} = \frac{p_0 + \varepsilon p_1}{q_0 + \varepsilon q_1} = \frac{p_0q_0^* + \varepsilon(p_0q_1^* + p_1q_0^*)}{qq^*} = \frac{p_0q_0^* + \varepsilon(p_0q_1^* + p_1q_0^*)}{N(q)},$$

where $N(q) \neq 0$.

Lemma 2.1. *For all $p \in \mathcal{D}(\mathcal{S})$, we have*

$$p^n = \begin{cases} p_0^n + \varepsilon \sum_{k=1}^{n} p_0^{n-k}p_1p_0^{k-1}, & n \in \mathbb{Z}_+, \\ N(p)^n\{(p_0^*)^{-n} + \varepsilon \sum_{k=1}^{-n} (p_0^*)^{-n-k}p_1^*(p_0^*)^{k-1}\}, & n \in \mathbb{Z}_-, \end{cases} \tag{1}$$

where $\mathbb{Z}_+ := \{0, 1, 2, 3, \cdots\}$ and $\mathbb{Z}_- := \{-1, -2, -3, \cdots\}$.

Proof. If $n = 1$, then the equation (1) is trivial. Now suppose that this holds for some $n \in \mathbb{Z}_+$. Then, as desired,

$$\begin{aligned} p^{n+1} &= pp^n = p(p_0^n + \varepsilon \sum_{k=1}^{n} p_0^{n-k}p_1p_0^{k-1}) = p_0^{n+1} + \varepsilon \sum_{k=1}^{n} p_0^{n-k+1}p_1p_0^{k-1} + \varepsilon p_1p_0^n \\ &= p_0^{n+1} + \varepsilon \sum_{k=1}^{n+1} p_0^{n+1-k}p_1p_0^{k-1}. \end{aligned}$$

By the principle of mathematical induction, (1) holds for all $n \in \mathbb{Z}_+$. On the other hands, for $n \in \mathbb{Z}_-$,

$$p^{-1} = \frac{p^*}{N(p)}, \quad p^{-2} = \Big(\frac{p^*}{N(p)}\Big)^2 = N(p)^{-2}(p^*)^2, \quad p^{-3} = \Big(\frac{p^*}{N(p)}\Big)^3 = N(p)^{-3}(p^*)^3.$$

By continuing the above calculations, we obtain

$$p^n = N(p)^n\{(p_0^*)^{-n} + \varepsilon \sum_{k=1}^{-n} (p_0^*)^{-n-k}p_1^*(p_0^*)^{k-1}\}, \quad n \in \mathbb{Z}_-.$$

□

Lemma 2.2. *For all $p \in \mathcal{D}(\mathcal{S})$ and $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and an exponential function on $\mathcal{D}(\mathcal{S})$, denoted by* $\exp(p)$, *we have an Euler formula with a dual part of dual split quaternions:*

$$\exp(\varepsilon p_1) = \cos(\varepsilon p_1) + \sin(\varepsilon p_1) \quad \textit{or} \quad \exp(\varepsilon p_1) = \cosh(\varepsilon p_1) + \sinh(\varepsilon p_1).$$

Proof. From a Taylor series of transcendental functions and properties of the dual symbol ε, we have

$$\exp(\varepsilon p_1) = \sum_{n=0}^{\infty} \frac{(\varepsilon p_1)^n}{n!} = 1 + \varepsilon p_1, \quad \cos(\varepsilon p_1) = \sum_{n=0}^{\infty} \frac{(-1)^n(\varepsilon p_1)^{2n}}{(2n)!} = 1,$$
$$\cosh(\varepsilon p_1) = \sum_{n=0}^{\infty} \frac{(\varepsilon p_1)^{2n}}{(2n)!} = 1, \quad \sin(\varepsilon p_1) = \sum_{n=0}^{\infty} \frac{(-1)^n(\varepsilon p_1)^{2n+1}}{(2n+1)!} = \varepsilon p_1,$$
$$\sinh(\varepsilon p_1) = \sum_{n=0}^{\infty} \frac{(\varepsilon p_1)^{2n+1}}{(2n+1)!} = \varepsilon p_1.$$

Therefore, by comparing above equations, we obtain the result. □

Let Ω be an open subset of $\mathbb{C}^2 \times \mathbb{C}^2$. Then the function $f : \Omega \to \mathcal{D}(\mathcal{S})$ can be expressed as

$$f(p) = f(p_0, p_1) = f_0(p_0, p_1) + \varepsilon f_1(p_0, p_1),$$

where the component functions $f_r : \Omega \to \mathcal{S}$ $(r = 0, 1)$ are split quaternionic-valued functions. The component functions f_r $(r = 0, 1)$ are

$$f_0(p_0, p_1) = f_0(z_1, z_2, w_1, w_2) = g_1(z_1, z_2, w_1, w_2) + g_2(z_1, z_2, w_1, w_2)e_2,$$
$$f_1(p_0, p_1) = f_1(z_1, z_2, w_1, w_2) = h_1(z_1, z_2, w_1, w_2) + h_2(z_1, z_2, w_1, w_2)e_2,$$

where $g_1 = u_0 + u_1e_1$, $g_2 = u_2 + u_3e_1$, $h_1 = v_0 + v_1e_1$ and $h_2 = v_2 + v_3e_1$ are complex-valued functions, and $u_r = u_r(x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3)$ and $v_r = v_r(x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3)$ $(r = 0, 1, 2, 3)$ are real-valued functions.

Now, let differential operators D_1 and D_2 be defined on $\mathcal{D}(\mathcal{S})$ as

$$D_1 := D_{(11)} + \varepsilon D_{(12)}.$$

Also, for efficient representations and calculations of Definition 2.4, Corollaries 3.2 and 3.4, by referring the process of describing complex differential operators with one complex variables, we define the form as follows:

$$D_2 := D_{(21)} + \varepsilon D_{(22)} = \Big(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}e_2\Big) + \varepsilon\Big(\frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}e_2\Big).$$

Then the conjugate operators D_1^* and D_2^* are

$$D_1^* = D_{(11)}^* + \varepsilon D_{(12)}^*, \quad D_2^* = D_{(21)}^* + \varepsilon D_{(22)}^*,$$

where

$$D_{(11)} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \overline{z_2}}e_2, \quad D_{(12)} = \frac{\partial}{\partial w_1} + \frac{\partial}{\partial \overline{w_2}}e_2,$$
$$D_{(21)} = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}e_2, \quad D_{(22)} = \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}e_2$$

and

$$D_{(11)}^* = \frac{\partial}{\partial \overline{z_1}} - \frac{\partial}{\partial \overline{z_2}}e_2, \quad D_{(12)}^* = \frac{\partial}{\partial \overline{w_1}} - \frac{\partial}{\partial \overline{w_2}}e_2,$$
$$D_{(21)}^* = \frac{\partial}{\partial \overline{z_1}} - \frac{\partial}{\partial z_2}e_2, \quad D_{(22)}^* = \frac{\partial}{\partial \overline{w_1}} - \frac{\partial}{\partial w_2}e_2.$$

The above equations act on $\mathcal{D}(\mathcal{S})$. Then, we have the following equations:

$$D^*_{(11)}D_{(11)} = D_{(11)}D^*_{(11)} = \frac{\partial^2}{\partial z_1 \partial \overline{z_1}} - \frac{\partial^2}{\partial z_2 \partial \overline{z_2}}\,, \quad D^*_{(12)}D_{(12)} = D_{(12)}D^*_{(12)} = \frac{\partial^2}{\partial w_1 \partial \overline{w_1}} - \frac{\partial^2}{\partial w_2 \partial \overline{w_2}},$$

$$D^*_{(21)}D_{(21)} = D_{(21)}D^*_{(21)} = \frac{\partial^2}{\partial z_1 \partial \overline{z_1}} - \frac{\partial^2}{\partial z_2 \partial \overline{z_2}}\,, \quad D^*_{(22)}D_{(22)} = D_{(22)}D^*_{(22)} = \frac{\partial^2}{\partial w_1 \partial \overline{w_1}} - \frac{\partial^2}{\partial w_2 \partial \overline{w_2}},$$

called the corresponding Coulomb operators [16] and

$$D^*_{(1)}D_{(1)} = D_{(1)}D^*_{(1)} = D_{(11)}D^*_{(11)} + \varepsilon(D_{(11)}D^*_{(12)} + D_{(12)}D^*_{(11)})$$

and

$$D^*_{(2)}D_{(2)} = D_{(2)}D^*_{(2)} = D_{(21)}D^*_{(21)} + \varepsilon(D_{(21)}D^*_{(22)} + D_{(22)}D^*_{(21)})$$

called the analogous Coulomb operators on $\mathcal{D}(\mathcal{S})$ which are similar to the Laplacian operator in complex analysis.

Remark 2.3. *From the definition of differential operators on* $\mathcal{D}(\mathcal{S})$,

$$\begin{aligned} D_r f &= \Big(D_{(r1)} + \varepsilon D_{(r2)}\Big)(f_0 + \varepsilon f_1) = D_{(r1)} f_0 + \varepsilon(D_{(r1)} f_1 + D_{(r2)} f_0), \\ D^*_r f &= \Big(D^*_{(r1)} + \varepsilon D^*_{(r2)}\Big)(f_0 + \varepsilon f_1) = D^*_{(r1)} f_0 + \varepsilon(D^*_{(r1)} f_1 + D^*_{(r2)} f_0), \end{aligned}$$

where $r = 1,\ 2$.

Definition 2.4. *Let* Ω *be an open set in* $\mathbb{C}^2 \times \mathbb{C}^2$. *A function* $f = f_0 + \varepsilon f_1$ *is called an* L_r *(resp.* R_r*)-regular function* $(r = 1, 2)$ *on* Ω *if the following two conditions are satisfied*:
(i) f_k $(k = 0, 1)$ *are continuously differential functions on* Ω, *and*
(ii) $D^*_r f(p) = 0$ (*resp.* $f(p)D^*_r = 0$) *on* Ω $(r = 1, 2)$.

In particular, the equation $D^*_1 f(p) = 0$ of Definition 2.4 is equivalent to

$$D^*_{(11)} f_0 = 0 \quad \text{and} \quad D^*_{(12)} f_0 + D^*_{(11)} f_1 = 0.$$

Then,

$$\begin{cases} \dfrac{\partial g_1}{\partial \overline{z_1}} - \dfrac{\partial \overline{g_2}}{\partial \overline{z_2}} = 0\,, \quad \dfrac{\partial g_2}{\partial \overline{z_1}} - \dfrac{\partial \overline{g_1}}{\partial \overline{z_2}} = 0, \\[2ex] \dfrac{\partial h_1}{\partial \overline{z_1}} + \dfrac{\partial g_1}{\partial \overline{w_1}} - \dfrac{\partial \overline{h_2}}{\partial \overline{z_2}} - \dfrac{\partial \overline{g_2}}{\partial \overline{w_2}} = 0, \\[2ex] \dfrac{\partial h_2}{\partial \overline{z_1}} + \dfrac{\partial g_2}{\partial \overline{w_1}} - \dfrac{\partial \overline{h_1}}{\partial \overline{z_2}} - \dfrac{\partial \overline{g_1}}{\partial \overline{w_2}} = 0. \end{cases} \tag{2}$$

The above system (2) is a corresponding Cauchy-Riemann system in $\mathcal{D}(\mathcal{S})$.
Similarly, the equation $D^*_2 f(p) = 0$ of Definition 2.4 is equivalent to

$$D^*_{(21)} f_0 = 0 \quad \text{and} \quad D^*_{(22)} f_0 + D^*_{(21)} f_1 = 0.$$

Then,

$$\begin{cases} \dfrac{\partial g_1}{\partial \overline{z_1}} - \dfrac{\partial \overline{g_2}}{\partial z_2} = 0\,, \ \dfrac{\partial g_2}{\partial \overline{z_1}} - \dfrac{\partial \overline{g_1}}{\partial z_2} = 0, \\ \dfrac{\partial h_1}{\partial \overline{z_1}} + \dfrac{\partial g_1}{\partial \overline{w_1}} - \dfrac{\partial \overline{h_2}}{\partial z_2} - \dfrac{\partial \overline{g_2}}{\partial w_2} = 0, \\ \dfrac{\partial h_2}{\partial \overline{z_1}} + \dfrac{\partial g_2}{\partial \overline{w_1}} - \dfrac{\partial \overline{h_1}}{\partial z_2} - \dfrac{\partial \overline{g_1}}{\partial w_2} = 0. \end{cases} \tag{3}$$

The above system (3) is a corresponding Cauchy-Riemann system in $\mathcal{D}(\mathcal{S})$.

On the other hand, for an R_r-regular function on Ω, the equation $f(p)D_1^* = 0$ of Definition 2.4 is equivalent to

$$f_0 D^*_{(11)} = 0 \ \text{ and } \ f_0 D^*_{(12)} = -f_1 D^*_{(11)},$$

then we have

$$\begin{cases} \dfrac{\partial g_1}{\partial \overline{z_1}} - \dfrac{\partial g_2}{\partial z_2} = 0\,, \ \dfrac{\partial g_2}{\partial z_1} - \dfrac{\partial g_1}{\partial \overline{z_2}} = 0, \\ \dfrac{\partial g_1}{\partial \overline{w_1}} - \dfrac{\partial g_2}{\partial w_2} + \dfrac{\partial h_1}{\partial \overline{z_1}} - \dfrac{\partial h_2}{\partial z_2} = 0, \\ \dfrac{\partial g_2}{\partial w_1} - \dfrac{\partial g_1}{\partial \overline{w_2}} + \dfrac{\partial h_2}{\partial z_1} - \dfrac{\partial h_1}{\partial \overline{z_2}} = 0. \end{cases} \tag{4}$$

Similarly, the equation $f(p)D_2^* = 0$ of Definition 2.4 is equivalent to

$$f_0 D^*_{(21)} = 0 \ \text{ and } \ f_0 D^*_{(22)} = -f_1 D^*_{(21)}.$$

Then,

$$\begin{cases} \dfrac{\partial g_1}{\partial \overline{z_1}} - \dfrac{\partial g_2}{\partial \overline{z_2}} = 0\,, \ \dfrac{\partial g_2}{\partial z_1} - \dfrac{\partial g_1}{\partial z_2} = 0, \\ \dfrac{\partial g_1}{\partial \overline{w_1}} - \dfrac{\partial g_2}{\partial \overline{w_2}} + \dfrac{\partial h_1}{\partial \overline{z_1}} - \dfrac{\partial h_2}{\partial \overline{z_2}} = 0, \\ \dfrac{\partial g_2}{\partial w_1} - \dfrac{\partial g_1}{\partial w_2} + \dfrac{\partial h_2}{\partial z_1} - \dfrac{\partial h_1}{\partial z_2} = 0. \end{cases} \tag{5}$$

From the systems (2), (3), (4) and (5), it is obtained that the equations $D_r^* f(p) = 0$ and $f(p)D_r^* = 0$ $(r = 1, 2)$ are different. Now properties of the L_r-regular function $(r = 1, 2)$ with values in $\mathcal{D}(\mathcal{S})$ are considered.

3. Properties of L_r-Regular Functions $(r = 1, 2)$ with Values in $\mathcal{D}(\mathcal{S})$

Consider properties of L_r-regular functions $(r = 1, 2)$ with values in $\mathcal{D}(\mathcal{S})$.

Theorem 3.1. *Let Ω be an open set in $\mathbb{C}^2\times\mathbb{C}^2$ and $f = f_0+\varepsilon f_1 = (g_1+g_2e_2)+\varepsilon(h_1+h_2e_2)$ be an L_1-regular function defined on Ω. Then*

$$D_1 f = \Big\{2\big(\frac{\partial}{\partial \overline{z_2}} + \varepsilon\frac{\partial}{\partial \overline{w_2}}\big)e_2 - \big(\frac{\partial}{\partial x_1} + \varepsilon\frac{\partial}{\partial y_1}\big)e_1\Big\}f.$$

Proof. By the system (2), we have

$$\begin{aligned} D_1 f &= \big(\frac{\partial g_1}{\partial \overline{z_1}} + \frac{\partial u_1}{\partial x_1} - \frac{\partial u_0}{\partial x_1}e_1 + \frac{\partial \overline{g_2}}{\partial \overline{z_2}}\big) + \big(\frac{\partial g_2}{\partial \overline{z_1}} + \frac{\partial u_3}{\partial x_1} - \frac{\partial u_2}{\partial x_1}e_1 + \frac{\partial \overline{g_1}}{\partial \overline{z_2}}\big)e_2 \\ &\quad +\varepsilon\big(\frac{\partial g_1}{\partial \overline{w_1}} + \frac{\partial u_1}{\partial y_1} - \frac{\partial u_0}{\partial y_1}e_1 + \frac{\partial \overline{g_2}}{\partial \overline{w_2}} + \frac{\partial h_1}{\partial \overline{z_1}} + \frac{\partial v_1}{\partial x_1} - \frac{\partial v_0}{\partial x_1}e_1 + \frac{\partial \overline{h_2}}{\partial \overline{z_2}}\big) \\ &\quad +\varepsilon\big(\frac{\partial g_2}{\partial \overline{w_1}} + \frac{\partial u_3}{\partial y_1} - \frac{\partial u_2}{\partial y_1}e_1 + \frac{\partial \overline{g_1}}{\partial \overline{w_2}} + \frac{\partial h_2}{\partial \overline{z_1}} + \frac{\partial v_3}{\partial x_1} - \frac{\partial v_2}{\partial x_1}e_1 + \frac{\partial \overline{h_1}}{\partial \overline{z_2}}\big)e_2 \\ &= 2\frac{\partial}{\partial \overline{z_2}}e_2(g_2e_2 + g_1 + \varepsilon h_2e_2 + \varepsilon h_1) + \varepsilon\frac{\partial}{\partial \overline{w_2}}e_2(g_2e_2 + g_1 + \varepsilon h_2e_2 + \varepsilon h_1) \\ &\quad +\frac{\partial}{\partial x_1}e_1\{(-u_1e_1 - u_0 - u_3e_3 - u_2e_2) + \varepsilon(-v_1e_1 - v_0 - v_3e_3 - v_2e_2)\} \\ &\quad +\varepsilon\frac{\partial}{\partial y_1}e_1\{(-u_1e_1 - u_0 - u_3e_3 - u_2e_2) + \varepsilon(-v_1e_1 - v_0 - v_3e_3 - v_2e_2)\} \\ &= \Big\{2\big(\frac{\partial}{\partial \overline{z_2}} + \varepsilon\frac{\partial}{\partial \overline{w_2}}\big)e_2 - \big(\frac{\partial}{\partial x_1} + \varepsilon\frac{\partial}{\partial y_1}\big)e_1\Big\}f. \end{aligned}$$

Therefore, we obtain

$$D_1 f = \Big\{2\big(\frac{\partial}{\partial \overline{z_2}} + \varepsilon\frac{\partial}{\partial \overline{w_2}}\big)e_2 - \big(\frac{\partial}{\partial x_1} + \varepsilon\frac{\partial}{\partial y_1}\big)e_1\Big\}f.$$

□

Corollary 3.2. *Let Ω be an open set in $\mathbb{C}^2\times\mathbb{C}^2$ and f be an L_2-regular function defined on Ω. Then*

$$D_2 f = \Big\{2\big(\frac{\partial}{\partial z_2} + \varepsilon\frac{\partial}{\partial w_2}\big)e_2 - \big(\frac{\partial}{\partial x_1} + \varepsilon\frac{\partial}{\partial y_1}\big)e_1\Big\}f.$$

Proof. By the system (3), we have

$$\begin{aligned} D_2 f &= \big(\frac{\partial g_1}{\partial \overline{z_1}} + \frac{\partial u_1}{\partial x_1} - \frac{\partial u_0}{\partial x_1}e_1 + \frac{\partial \overline{g_2}}{\partial z_2}\big) + \big(\frac{\partial g_2}{\partial \overline{z_1}} + \frac{\partial u_3}{\partial x_1} - \frac{\partial u_2}{\partial x_1}e_1 + \frac{\partial \overline{g_1}}{\partial z_2}\big)e_2 \\ &\quad +\varepsilon\big(\frac{\partial g_1}{\partial \overline{w_1}} + \frac{\partial u_1}{\partial y_1} - \frac{\partial u_0}{\partial y_1}e_1 + \frac{\partial \overline{g_2}}{\partial w_2} + \frac{\partial h_1}{\partial \overline{z_1}} + \frac{\partial v_1}{\partial x_1} - \frac{\partial v_0}{\partial x_1}e_1 + \frac{\partial \overline{h_2}}{\partial z_2}\big) \\ &\quad +\varepsilon\big(\frac{\partial g_2}{\partial \overline{w_1}} + \frac{\partial u_3}{\partial y_1} - \frac{\partial u_2}{\partial y_1}e_1 + \frac{\partial \overline{g_1}}{\partial w_2} + \frac{\partial h_2}{\partial \overline{z_1}} + \frac{\partial v_3}{\partial x_1} - \frac{\partial v_2}{\partial x_1}e_1 + \frac{\partial \overline{h_1}}{\partial z_2}\big)e_2 \\ &= \Big\{2\big(\frac{\partial}{\partial z_2} + \varepsilon\frac{\partial}{\partial w_2}\big)e_2 - \big(\frac{\partial}{\partial x_1} + \varepsilon\frac{\partial}{\partial y_1}\big)e_1\Big\}f. \end{aligned}$$

Therefore, we obtain the result. □

Theorem 3.3. *Let Ω be an open set in $\mathbb{C}^2 \times \mathbb{C}^2$ and the function f be L_1-regular on Ω. Then, for $n \in \mathbb{N}$,*

$$D_1^n f = \begin{cases} \Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)f + \varepsilon\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)f, & \text{if } n = 1; \\ A^k f + k\varepsilon A^{k-1}Bf, & \text{if } n = 2k; \\ \Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)A^k f + \varepsilon\Big\{k\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)B\Big\} + \Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)A\Big\}A^{k-1}f, & \text{if } n = 2k+1; \end{cases} \tag{6}$$

where

$$A = 4\frac{\partial^2}{\partial x_2^2} + 4\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1^2} \text{ and } B = 2\Big(4\frac{\partial^2}{\partial x_2 \partial y_2} + 4\frac{\partial^2}{\partial x_3 \partial y_3} - \frac{\partial^2}{\partial x_1 \partial y_1}\Big).$$

Proof. Because f is an L_1-regular function on Ω,

$$D_1 f = \Big\{\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big) + \varepsilon\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)\Big\}f,$$

and a function

$$\Big\{\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big) + \varepsilon\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)\Big\}f$$

is also an L_1-regular function on Ω. Therefore,

$$\begin{aligned} D_1(D_1 f) = D_1^2 f &= D_1\Big\{\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big) + \varepsilon\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)\Big\}f \\ &= \Big\{\Big(4\frac{\partial^2}{\partial x_2^2} + 4\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1^2}\Big) + 2\varepsilon\Big(4\frac{\partial^2}{\partial x_2 \partial y_2} + 4\frac{\partial^2}{\partial x_3 \partial y_3} - \frac{\partial^2}{\partial x_1 \partial y_1}\Big)\Big\}f. \end{aligned}$$

Let

$$A := \Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)^2 = 4\frac{\partial^2}{\partial z_2 \partial \overline{z_2}} - \frac{\partial^2}{\partial x_1^2} = 4\frac{\partial^2}{\partial x_2^2} + 4\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1^2}$$

and

$$\begin{aligned} B := &\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big) + \Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big) \\ = &2\Big(4\frac{\partial^2}{\partial x_2 \partial y_2} + 4\frac{\partial^2}{\partial x_3 \partial y_3} - \frac{\partial^2}{\partial x_1 \partial y_1}\Big) \end{aligned}$$

for representation convenience. Then

$$D_1^2 f = (A + \varepsilon B)f.$$

Similarly, we have

$$D_1(D_1^2 f) = D_1^3 f = \Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)Af + \varepsilon\Big\{\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)B + \Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)A\Big\}f$$

and

$$\begin{aligned} D_1(D_1^3 f) &= D_1^4 f = \Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)Af + \varepsilon\Big\{\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)B \\ &\quad + \Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)A\Big\}f + \varepsilon\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)Af \\ &= (A^2 + 2\varepsilon AB)f. \end{aligned}$$

Through the continuous calculation of this process, we obtain the following forms for $n \in \mathbb{N}$,

$$D_1 f = \Big\{\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big) + \varepsilon\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)\Big\} f,$$

$$D_1^{2n} f = A^n f + n\varepsilon A^{n-1} B f$$

and

$$D_1^{2n+1} f = \Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)A^n f + \varepsilon\Big\{n\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)B + \Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}\Big)A\Big\}A^{n-1} f.$$

Therefore, the result (6) is obtained. □

Corollary 3.4. *Let Ω be an open set in $\mathbb{C}^2 \times \mathbb{C}^2$ and the function f be L_2-regular on Ω. Then, for $n \in \mathbb{N}$,*

$$D_2^n f = \begin{cases} \Big(2\frac{\partial}{\partial z_2}e_2 - \frac{\partial}{\partial x_1}e_1\Big)f + \varepsilon\Big(2\frac{\partial}{\partial w_2}e_2 - \frac{\partial}{\partial y_1}e_1\Big)f, & \text{if } n = 1; \\ A^k f + k\varepsilon A^{k-1} B f, & \text{if } n = 2k; \\ \Big(2\frac{\partial}{\partial z_2}e_2 - \frac{\partial}{\partial x_1}e_1\Big)A^k f + \varepsilon\Big\{k\Big(2\frac{\partial}{\partial z_2}e_2 - \frac{\partial}{\partial x_1}e_1\Big)B + \Big(2\frac{\partial}{\partial w_2}e_2 - \frac{\partial}{\partial y_1}e_1\Big)A\Big\}A^{k-1} f, & \text{if } n = 2k+1. \end{cases} \tag{7}$$

Proof. Since we have

$$D_2 f = \Big\{2\Big(\frac{\partial}{\partial z_2}e_2 - \frac{\partial}{\partial x_1}e_1\Big) + \varepsilon\Big(\frac{\partial}{\partial w_2}e_2 - \frac{\partial}{\partial y_1}e_1\Big)\Big\} f$$

and a function $D_2 f$ is also an L_2-regular function on Ω, we obtain the equations (7) for $n \in \mathbb{N}$. □

Theorem 3.5. *Let Ω be an open set in $\mathbb{C}^2 \times \mathbb{C}^2$ and the function f be L_1-regular on Ω. Then, for $n \in \mathbb{N}$*

$$\begin{aligned} f(p) = {} & \Big\{f(0) + \Big(2e_2\frac{\partial f(0)}{\partial z_2} - e_1\frac{\partial f(0)}{\partial x_1}\Big)p_0 + \sum_{\substack{n=2 \\ n:even}}^{\infty} \frac{1}{n!}\Big(A^{\frac{n}{2}} f(0)\Big)p_0^n + \sum_{\substack{n=3 \\ n:odd}}^{\infty} \frac{1}{n!}\Big(\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)A^{[\frac{n}{2}]} f(0)\Big)p_0^n\Big\} \\ & +\varepsilon\Big\{\Big(2e_2\frac{\partial f(0)}{\partial w_2} - e_1\frac{\partial f(0)}{\partial y_1}\Big)p_0 + \Big(2e_2\frac{\partial f(0)}{\partial z_2} - e_1\frac{\partial f(0)}{\partial x_1}\Big)p_1 + \sum_{\substack{n=2 \\ n:even}}^{\infty} \frac{1}{n!}\Big(\frac{n}{2}\Big(A^{\frac{n}{2}} B f(0)\Big)p_0^n \\ & + \sum_{k=1}^{n}\Big(A^{\frac{n}{2}} f(0)\Big)p_0^{n-k} p_1 p_0^{k-1}\Big) + \sum_{\substack{n=3 \\ n:odd}}^{\infty} \frac{1}{n!}\Big(\Big(\Big[\frac{n}{2}\Big]\Big(2e_2\frac{\partial}{\partial z_2} - e_1\frac{\partial}{\partial x_1}\Big)BA^{[\frac{n}{2}]-1} f(0) \\ & +\Big(2e_2\frac{\partial}{\partial w_2} - e_1\frac{\partial}{\partial y_1}\Big)A^{[\frac{n}{2}]} f(0)\Big)p_0^n + \sum_{k=1}^{n}\Big(\Big(2e_2\frac{\partial}{\partial z_2} - e_1\frac{\partial}{\partial x_1}\Big)A^{[\frac{n}{2}]} f(0)\Big)p_0^{n-k} p_1 p_0^{k-1}\Big)\Big\}, \end{aligned}$$

where $[\frac{n}{2}]$ is the greatest integer less than or equal to $\frac{n}{2}$ $(n \in \mathbb{N})$.

Proof. From Theorem 3.3 and the properties of dual split quaternions,

$$\begin{aligned} f(p) &= f(p_0,p_1) = f(0) + \Big\{\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)f(0) + \varepsilon\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)f(0)\Big\}(p_0+\varepsilon p_1) \\ &+ \sum_{\substack{n=2\\ n:\text{even}}}^{\infty} \frac{1}{n!}\Big(A^{\frac{n}{2}}f(0) + \frac{n}{2}\varepsilon A^{\frac{n}{2}}Bf(0)\Big)\Big(p_0^n + \varepsilon\sum_{k=1}^{n} p_0^{n-k}p_1p_0^{k-1}\Big) \\ &+ \sum_{\substack{n=3\\ n:\text{odd}}}^{\infty} \frac{1}{n!}\Big\{\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)A^{[\frac{n}{2}]}f(0) + \varepsilon\Big(\Big[\frac{n}{2}\Big]\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)B \\ &+\Big(2\frac{\partial}{\partial \overline{w_2}}e_2 - \frac{\partial}{\partial y_1}e_1\Big)A\Big)A^{[\frac{n}{2}]-1}f(0)\Big\}\Big(p_0^n + \varepsilon\sum_{k=1}^{n} p_0^{n-k}p_1p_0^{k-1}\Big). \end{aligned}$$

Then, we have

$$\begin{aligned} f(p) &= f(0) + \Big(2e_2\frac{\partial f(0)}{\partial z_2} - e_1\frac{\partial f(0)}{\partial x_1}\Big)p_0 + \varepsilon\Big\{\Big(2e_2\frac{\partial f(0)}{\partial w_2} - e_1\frac{\partial f(0)}{\partial y_1}\Big)p_0 + \Big(2e_2\frac{\partial f(0)}{\partial z_2} - e_1\frac{\partial f(0)}{\partial x_1}\Big)p_1\Big\} \\ &+ \sum_{\substack{n=2\\ n:\text{even}}}^{\infty} \frac{1}{n!}\Big(\Big(A^{\frac{n}{2}}f(0)\Big)p_0^n + \varepsilon\Big\{\Big(\frac{n}{2}A^{\frac{n}{2}}Bf(0)\Big)p_0^n + \sum_{k=1}^{n}\Big(A^{\frac{n}{2}}f(0)\Big)p_0^{n-k}p_1p_0^{k-1}\Big\}\Big) \\ &+ \sum_{\substack{n=3\\ n:\text{odd}}}^{\infty} \frac{1}{n!}\Big(\Big(\Big(2\frac{\partial}{\partial \overline{z_2}}e_2 - \frac{\partial}{\partial x_1}e_1\Big)A^{[\frac{n}{2}]}f(0)\Big)p_0^n + \varepsilon\Big\{\Big(\Big[\frac{n}{2}\Big]\Big(2e_2\frac{\partial}{\partial z_2} - e_1\frac{\partial}{\partial x_1}\Big)BA^{[\frac{n}{2}]-1}f(0) \\ &+\Big(2e_2\frac{\partial}{\partial w_2} - e_1\frac{\partial}{\partial y_1}\Big)A^{[\frac{n}{2}]}f(0)\Big)p_0^n + \sum_{k=1}^{n}\Big(\Big(2e_2\frac{\partial}{\partial z_2} - e_1\frac{\partial}{\partial x_1}\Big)A^{[\frac{n}{2}]}f(0)\Big)p_0^{n-k}p_1p_0^{k-1}\Big\}\Big). \end{aligned}$$

Therefore, the result is obtained. □

Corollary 3.6. *Let Ω be an open set in $\mathbb{C}^2\times\mathbb{C}^2$ and the function f be L_2-regular on Ω. Then, for $n\in\mathbb{N}$*

$$\begin{aligned} f(p) &= \Big\{f(0) + \Big(2e_2\frac{\partial f(0)}{\partial \overline{z_2}} - e_1\frac{\partial f(0)}{\partial x_1}\Big)p_0 + \sum_{\substack{n=2\\ n:even}}^{\infty} \frac{1}{n!}\Big(A^{\frac{n}{2}}f(0)\Big)p_0^n + \sum_{\substack{n=3\\ n:odd}}^{\infty} \frac{1}{n!}\Big(2\frac{\partial}{\partial z_2}e_2 - \frac{\partial}{\partial x_1}e_1\Big)\Big(A^{[\frac{n}{2}]}f(0)\Big)p_0^n\Big\} \\ &+\varepsilon\Big\{\Big(2e_2\frac{\partial f(0)}{\partial \overline{w_2}} - e_1\frac{\partial f(0)}{\partial y_1}\Big)p_0 + \Big(2e_2\frac{\partial f(0)}{\partial \overline{z_2}} - e_1\frac{\partial f(0)}{\partial x_1}\Big)p_1 + \sum_{\substack{n=2\\ n:even}}^{\infty} \frac{1}{n!}\Big(\frac{n}{2}\Big(A^{\frac{n}{2}}Bf(0)\Big)p_0^n \\ &+ \sum_{k=1}^{n}\Big(A^{\frac{n}{2}}f(0)\Big)p_0^{n-k}p_1p_0^{k-1}\Big) + \sum_{\substack{n=3\\ n:odd}}^{\infty} \frac{1}{n!}\Big(\Big(\Big[\frac{n}{2}\Big]\Big(2e_2\frac{\partial}{\partial \overline{z_2}} - e_1\frac{\partial}{\partial x_1}\Big)BA^{[\frac{n}{2}]-1}f(0) \\ &+\Big(2e_2\frac{\partial}{\partial \overline{w_2}} - e_1\frac{\partial}{\partial y_1}\Big)A^{[\frac{n}{2}]}f(0)\Big)p_0^n + \sum_{k=1}^{n}\Big(\Big(2e_2\frac{\partial}{\partial \overline{z_2}} - e_1\frac{\partial}{\partial x_1}\Big)A^{[\frac{n}{2}]}f(0)\Big)p_0^{n-k}p_1p_0^{k-1}\Big)\Big\}. \end{aligned}$$

Proof. From Theorems 3.3 and 3.5, we have

$$\begin{aligned} f(p) &= f(p_0, p_1) = f(0) + \Big\{\Big(\frac{\partial}{\partial z_2}e_2 - \frac{\partial}{\partial x_1}e_1\Big)f(0) + \varepsilon\Big(\frac{\partial}{\partial w_2}e_2 - \frac{\partial}{\partial y_1}e_1\Big)f(0)\Big\}(p_0 + \varepsilon p_1) \\ &+ \sum_{\substack{n=2 \\ n:\text{even}}}^{\infty} \frac{1}{n!}\Big(A^{\frac{n}{2}} f(0) + \frac{n}{2}\varepsilon A^{\frac{n}{2}} B f(0)\Big)(p_0^n + \varepsilon \sum_{k=1}^{n} p_0^{n-k} p_1 p_0^{k-1}) \\ &+ \sum_{\substack{n=3 \\ n:\text{odd}}}^{\infty} \frac{1}{n!}\Big\{\Big(\frac{\partial}{\partial z_2}e_2 - \frac{\partial}{\partial x_1}e_1\Big)A^{[\frac{n}{2}]}f(0) + \varepsilon\Big(\Big[\frac{n}{2}\Big]\Big(\frac{\partial}{\partial z_2}e_2 - \frac{\partial}{\partial x_1}e_1\Big)B \\ &+ \Big(\frac{\partial}{\partial w_2}e_2 - \frac{\partial}{\partial y_1}e_1\Big)A\Big)A^{[\frac{n}{2}]-1}f(0)\Big\}(p_0^n + \varepsilon \sum_{k=1}^{n} p_0^{n-k} p_1 p_0^{k-1}). \end{aligned}$$

Therefore, by rearranging the terms of the above equation, the result is also obtained. □

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