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The Actions on the Generating Functions for the Family of the Generalized Bernoulli Polynomials

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Abstract.
In this paper, we study the generalization Bernoulli numbers and polynomials attached to a periodic group homomorphism from a finite cyclic group to the set of complex numbers and derive new periodic group homomorphism by using a fixed periodic group homomorphism. Hence, we obtain not only multiplication formulas, but also some new identities for the generalized Bernoulli polynomials attached to a periodic group homomorphism.

1. Introduction

The theory of the family of the Bernoulli polynomials and numbers have played a very important role in many branches of mathematics such as analytic number theory, combinatorics, special functions and in the other sciences such as , engineering, computer, geometric design and mathematical physics.

Recently in [1], the authors study the periodic function to decompose the q–Eulerian numbers and polynomials. This decomposition provided us to compute q-Apostol-type Frobenius-Euler polynomials and numbers more easily. Moreover, in [5], Cangul *et al* and in [10], Cevik *et al* studied some new relationship between the subgroup and monoid presentation and special generating functions (such as Stirling numbers, Array polynomials etc.).

There are many useful Raabe type or multiplication formula for Bernoulli numbers and polynomials in the literature (*cf.* [24], [25]- [26]). Unfortunately so far we have not find any very useful Raabe type formulas related to the generalized Bernoulli numbers and polynomials attached to a character. Therefore, in this paper, our main aim is to obtain some multiplication formulas for the generalized Bernoulli polynomials attached to a periodic group homomorphism. For this aim, we define new generating functions for the generalized Bernoulli numbers and polynomials attached to a periodic group homomorphism and investigative the relations among them. These relations let us verify some identities among the Bernoulli numbers, the Euler numbers, the Apostol-Bernoulli polynomials and the Frobenius-Euler number and polynomials. Moreover, we decompose the generalized Bernoulli numbers and polynomials attached to a periodic group homomorphism and as a results of them, we drive some equations to more easily compute of the nth generalized Bernoulli polynomials attached to a periodic group homomorphism.

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Throughout our paper, we use the following standard notations:$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of natural numbers, the set of integer numbers, the set of real numbers and the set of complex numbers respectively.

Now we recall the following well-known generating functions:

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions, respectively (*cf.* [1]-[23]):

$$F(t,x) = \frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad (|t| < 2\pi) \tag{1}$$

and

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad (|t| < \pi)\,.$$

The classical Bernoulli numbers B_n and the classical Euler numbers E_n of order n are defined by

$$B_n = B_n(0)$$

and

$$E_n = E_n(0),$$

respectively (*cf.* [1]-[23]). By using Equation (1), we easily have

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k. \tag{2}$$

Apostol [3] gave some interesting analogues of the classical Bernoulli polynomials and numbers. Here we also recall Apostol-Bernoulli polynomials $B_n(x,\lambda)$ which are given by means of the following generating functions:

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(x,\lambda)\frac{t^n}{n!} \tag{3}$$

($|t| < 2\pi$ if $\lambda = 1$; $|t| < \left|\log\lambda\right|$ if $\lambda \neq 1$ and $\lambda \in \mathbb{C}$). Then

$$B_n(x) = B_n(x,1)$$

and

$$B_n(\lambda) = B_n(0,\lambda),$$

where $B_n(\lambda)$ denotes the so-called Apostol-Bernoulli Bernoulli polynomials (*cf.* [1]-[23]). The Frobenius-Euler numbers $H_n(u)$ are defined by means of the following generating function:

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!} \tag{4}$$

where u is an algebraic number (*cf.* [23]). If we substitute $u = (-1)$ into Equation (4), we have the following well-known result:

$$E_n = H_n(-1).$$

Following [2], we recall a Dirichlet character χ module a natural number f which is a function from the set of integers to the set of complex numbers and satisfies the following two conditions:

i) χ is a multiplicative function with the period $f \in \mathbb{N}$;

ii) if $(a, f) = 1$ where $a, f \in \mathbb{N}$, then $\chi(a)$ is non zero and if $(a, f) \neq 1$ then $\chi(a) = 0$.

In the literature, the generalized Bernoulli numbers $B_{n,\chi}$ attached to a Dirichlet character χ module a natural number f are defined by the following generating function:

$$F_{\chi}(t) = \sum_{j=0}^{f-1} \frac{\chi(j)te^{jt}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_{n,\tau}\frac{t^n}{n!} \quad (|t| < 2\pi)\,. \tag{5}$$

In this paper, we modify Equation (5) with a group homomorphism τ instead of a Dirichlet character since the properties of τ allows us to get new relations among the Bernoulli , Euler and Apostol-Bernoulli numbers and also obtain some multiplication formula for the generalized Bernoulli numbers attached to a group homomorphism. A group homomorphism τ with the period f is from the additive integer group $(\mathbb{Z}, +)$ to the multiplicative complex group $(\mathbb{C} - \{0\}, \cdot)$. Now, we note the following fundamental properties of a group homomorphism τ;

i) $\tau(0) = 1$,

ii) $\tau(a) = \tau(1)^a$ for a integer a,

iii) $\tau(f) = \tau(0)^f = 1$ and $\tau(0)$ is a roof of unity.

It is clear that there exist a group homomorphism with a period f which is not a character and so our generating function in Equation (6) is clearly different to the generating function in Equation (5).

2. The Generalized Bernoulli Numbers and Polynomials Attached to Any Group Homomorphism

We define a new family of the generalized Bernoulli numbers $B_{n,\tau}$ attached to a group homomorphism τ with the period $f \in \mathbb{N}$ by the following generating functions:

$$F_{\tau}(t) = \sum_{j=0}^{f-1} \frac{\tau(j)te^{jt}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_{n,\tau}\frac{t^n}{n!}, \quad (|t| < 2\pi)\,. \tag{6}$$

The generalized Bernoulli polynomials $B_{n,\tau}(x)$ attached to a group homomorphism τ with the period $f \in \mathbb{N}$ are defined by the following generating functions:

$$F_{\tau}(t, x) = F_{\tau}(t)e^{xt} = \sum_{n=0}^{\infty} B_{n,\tau}(x)\frac{t^n}{n!}. \tag{7}$$

By using the second property of a group homomorphism τ and applying the finite geometric series to Equation (6), we modify Equation (7) and so obtain our first relation:

$$F_{\tau}(t) = \frac{t}{e^{ft}-1}\sum_{j=0}^{f-1} \tau(j)e^{jt} = \frac{t}{\tau(1)e^t - 1}. \tag{8}$$

By combining Equation (8) with Equation (3) for $x = 0$, we arrive at the following proposition which gives us a relation between the generalized Bernoulli numbers attached to the group homomorphism τ and Apostol-Bernoulli numbers :

Proposition 2.1. *Let $n \in \mathbb{N}$. Then*

$$B_{n,\tau} = B_n(\tau(1)),$$

where $B_n(\tau(1))$ denotes the Apostol-Bernoulli numbers.

Since

$$1 = \tau(0) = \tau(1)\tau(-1),$$

we get the following

$$F_\tau(t) = \frac{t}{\tau(1)(1-\tau(-1))}\frac{(1-\tau(-1))}{(e^t-\tau(-1))}.$$

By combining Equations (8) and (4) with the above equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\tau}\frac{t^n}{n!} &= \frac{t}{\tau(1)(1-\tau(-1))}\sum_{n=0}^{\infty} H_n(\tau(-1))\frac{t^n}{n!} \\ &= \frac{1}{\tau(1)-1}\sum_{n=0}^{\infty} H_n(\tau(-1))\frac{t^{n+1}}{n!},\end{aligned}$$

where $H_n(\tau(-1))$ is the nth Frobenius-Euler numbers. By comparing the coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, we find the relation between the generalized Bernoulli numbers attached to the group homomorphism τ and Frobenius-Euler numbers by the following theorem:

Theorem 2.2. *Let n be a positive integer and for $n \geq 1$. Then we have*

$$B_{n,\tau} = \frac{nH_{n-1}(\tau(-1))}{\tau(1)-1}. \tag{9}$$

It is easy to get the following well known result:

$$B_{n,\tau} = f^{n-1}\sum_{a=0}^{f-1}\tau(a)B_n\left(\frac{a}{f}\right) \tag{10}$$

where $B_n(x)$ is nth Bernoulli polynomial.

Combining Equation (9) and Equation (10), we get

$$f^{n-1}\sum_{n=0}^{f-1}\tau(a)B_n\left(\frac{a}{f}\right) = \frac{1}{\tau(1)-1}nH_{n-1}(\tau(-1)).$$

Therefore, we find the relation between the Bernoulli polynomials attached to the group homomorphism τ and Frobenius-Euler number by the following theorem:

Theorem 2.3. *Let n be a positive integer and for $n \geq 1$. Then we have*

$$H_{n-1}(\tau(-1)) = \frac{(\tau(1)-1)\,f^{n-1}}{n}\sum_{a=0}^{f-1}\tau(a)B_n\left(\frac{a}{f}\right).$$

As a results of Theorem 2.3, we derive a relation between the classical Euler numbers and the classical Bernoulli numbers in the following result:

Corollary 2.4. *Let n be a positive integer and for $n \geq 1$. Then we have*

$$E_{n-1} = \frac{2^n}{n}\left(B_n\left(\frac{1}{2}\right) - B_n\right) \tag{11}$$

Proof. If the period τ is 2, then $\tau(-1) = \tau(1) = -1$. Then by Equation (4), it follows that

$$H_{n-1}(\tau(-1)) = E_{n-1}.$$

Therefore, by using Theorem 2.3, we get the assertion Equation (11). □

We know the multiplication formula (or Rabbe's formula) for the Bernoulli polynomials given by

$$B_n(rx) = r^{n-1}\sum_{j=0}^{r-1} B_n\left(x + \frac{j}{r}\right). \tag{12}$$

Thus, the Bernoulli polynomials $B_n(x)$ are the normalized polynomials, which have many applications in analytic number theory. Substituting $r = 2$ and $x = 0$ into Equation (12), we get the following

$$B_n = 2^{n-1}\left(B_n + B_n\left(\frac{1}{2}\right)\right).$$

Therefore, it follows that

$$B_n\left(\frac{1}{2}\right) = (2^{1-n} - 1)B_n$$

and so we get the following *known* result:

Corollary 2.5. *Let n be a positive integer and for $n \geq 1$. Then we have*

$$E_{n-1} = \frac{1}{n}(2 - 2^{n+1})B_n.$$

3. Some Properties of the Generating Functions

In this section, we define new group homomorphisms and obtain the relations between the generalized Bernoulli numbers attached to these group homomorphisms which are obtained from τ. Moreover, we compare two generalized Bernoulli numbers attached to group homomorphisms with the period f. We also find sum of the generalized Bernoulli numbers attached to different group homomorphisms with the period f.

Theorem 3.1. *Let Ω be a periodic group homomorphism with the period f and k a positive integer. Then $\Omega_k(j) := \Omega(kj)$ is a periodic group homomorphism with the period $p = \frac{f}{(f,k)}$ where (f,k) is $gcd(f,k)$.*

Proof. Let k be a positive integer and $(f,k) = d$. Then $f = pd$ and $k = p'd$ for some $p, p\prime \in \mathbb{N}$.
Clearly, Ω_k is a group homomorphism. For $z, j \in \mathbb{N}$, it is proved that

$$\Omega_k(j+z) = \Omega(k(j+z)) = \Omega(kj+kz) = \Omega(kj)\Omega(kz) = \Omega(kj) = \Omega_k(j)$$

if and only if $\Omega(kz) = 1$ if and only if f divides kz.
If the period of Ω_k is z then f divides kz and so

$$p'dz = kz = fn = pdn$$

for some $n \in \mathbb{N}$ and so $p'z = pn$. This means that p divides z since $(p, p') = 1$.
On the other hand, using the equality $kp = fp'$, we get

$$\Omega_k(j+p) = \Omega(k(j+p)) = \Omega(kj+kp) = \Omega(kj)\Omega(kp) = \Omega(kj) = \Omega_k(j).$$

Therefore, z dives p and so $z = p$. □

Let h be a positive integer and $(f,h)=d$ and so $f=f'd$ for $f'\in\mathbb{N}$. By using Equation (8) and Theorem 3.1, we investigate the relations between $F\tau$ and $F\tau_h$;

$$F_{\tau_h}(t)=\left(\frac{\tau(1)e^t-1}{\tau(h)e^t-1}\right)F_\tau(t). \tag{13}$$

We note that if $h\equiv 1 \bmod f$ then , it is easy to observe that

$$F_{\tau_h}(t)=F_\tau(t)$$

by applying Equation (13).

Using Theorem 3.1 and Equation (13), our aim is to find relations between two different generating functions for the generalized Bernoulli numbers and polynomials attached to different group homomorphisms with the period f.

Let Δ be a group homomorphism with $\Delta(a)=\exp\left(\frac{2\pi ia}{f}\right)$ for all $a\in\mathbb{Z}$. If $\tau(1)=\exp\left(\frac{2\pi iv}{f}\right)$ for $v\in\mathbb{Z}$, then we get $(f,v)=1$. Therefore, we get

$$\tau(1)=\Delta_v(1).$$

Using Equation (13), we have the following relation;

$$F_\tau(t)=F_{\Delta_v}(t)=\left(\frac{\Delta(1)e^t-1}{\Delta(v)e^t-1}\right)F_\Delta(t). \tag{14}$$

Let $F_\mu(t)$ be the generating function for the generalized Bernoulli numbers attached to the group homomorphism μ with the period f and

$$\mu(1)=\exp\left(\frac{2\pi iy}{f}\right).$$

Then we arrive at the following result:

Theorem 3.2. *With the above notations, we have that*

$$F_\tau(t)=\left(\frac{\Delta(y)e^t-1}{\Delta(v)e^t-1}\right)F_\mu(t). \tag{15}$$

Let $\mathcal{L}$ be the set of group homomorphisms with the period f. Now, we are ready to give sum of all generating functions for the generalized Bernoulli numbers with the period f. By using Equation (15), we get the following theorem:

Theorem 3.3. *With the above notations, we have that for all $n\in\mathbb{N}$,*

$$\sum_{\eta\in\mathcal{L}}B_{n,\eta}=\frac{1}{n-1}\sum_{r=0}^{n}\binom{n}{r}B_{r-1,\Delta}\left(\sum_{(v,f)=1,v\le f}\left(\Delta(1)B_{n-r,\Delta(v)}(1)-B_{n-r,\Delta(v)}\right)\right).$$

Proof. By using Equation (14), we get

$$\begin{aligned}\sum_{\eta\in\mathcal{L}}F_\eta(t) &= \sum_{(v,f)=1,v\le f}F_{\Delta_v}(t)=\sum_{(v,f)=1,v\le f}\left(\frac{\Delta(1)e^t-1}{\Delta(v)e^t-1}\right)F_\Delta(t)\\ &= \sum_{(v,f)=1,v\le f}\left(\frac{\Delta(1)te^t}{\Delta(v)e^t-1}-\frac{t}{\Delta(v)e^t-1}\right)\frac{1}{t}F_\Delta(t)\\ &= \sum_{(v,f)=1,v\le f}\frac{1}{t}F_\Delta(t)\left[\Delta(1)F_{\Delta_v}(t,1)-F_{\Delta_v}(t)\right]\end{aligned}$$

where $F_{\Delta_v}(t,1)$, $F_{\Delta_v}(t)$ are the generating functions for the Apostol-Bernoulli polynomials and numbers. Then, applying Cauchy product in the above equations, we obtain that

$$\begin{aligned}
\sum_{\eta\in\mathcal{L}} F_\eta(t) &= \sum_{(v,f)=1, v\leq f} \left(\sum_{n=0}^{\infty} B_{n,\Delta}\frac{t^{n-1}}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\Delta(1)B_{n,\Delta(v)}(1) - B_{n,\Delta(v)}\right)\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{(v,f)=1,v\leq f}\frac{1}{n-1}\sum_{r=0}^{n}\binom{n}{r}B_{r-1,\Delta}\left(\Delta(1)B_{n-r,\Delta(v)}(1) - B_{n-r,\Delta(v)}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

Hence, comparing the coefficient $\frac{t^n}{n!}$ on both sides yields the assertion of the desired results. □

Now we want to decompose the generalized Bernoulli numbers attached to a group homomorphism with the period f with respect to the integer k where $f = pk$ and $(p,k) = 1$. The decomposition is that

$$\begin{aligned}
F_\tau(t) &= \sum_{j=0}^{p-1}\sum_{r=0}^{k-1}\frac{\tau(kj+r)te^{(kj+r)t}}{e^{ft}-1} \qquad (16)\\
&= \frac{1}{k}\left(\sum_{r=0}^{k-1}\tau(r)e^{rt}\right)F_{\tau_k}(kt).
\end{aligned}$$

If $f = 2p$ then using the finite geometric sequence, Equation (16) make efficient the following result.

Theorem 3.4. *Let n be a positive integer and τ a group homomorphism with the period $f = 2p$. Then we have*

$$B_{n,\tau} = 2^{n-1}\left(B_{n,\tau_2} + \tau(1)\sum_{j=0}^{n}\binom{n}{j}B_{(n-j),\tau_2}\right).$$

Proof. Using Equation (16), we get

$$F_\tau(t) = \frac{1}{2}F_{\tau_2}(2t)\left(1+\tau(1)e^t\right).$$

Hence applying the Cauchy product, it follows that

$$\begin{aligned}
F_\tau(t) &= \left(\sum_{n=0}^{\infty}(2^{n-1}B_{n,\tau_2})\frac{t^n}{n!}\right)\left(1+\tau(1)\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)\\
&= \sum_{n=0}^{\infty}\left(2^{n-1}B_{n,\tau_2} + \tau(1)\sum_{j=0}^{n}(\binom{n}{j})(2^{n-1}B_{(n-j),\tau_2})\right)\frac{t^n}{n!}
\end{aligned}$$

By comparing the coefficient of $\frac{t^n}{n!}$ on both sides above, we get the desired results. □

On the other hand, from Equation (16), we obtain a formula for the *nth* generalized Bernoulli numbers.

Theorem 3.5. *Let $f = pk$ for some $p,k \in Z$ such that $(p,k) = 1$. Then for a positive integer n, we have*

$$B_{n,\tau} = \sum_{v=0}^{n}\binom{n}{v}B_{v,\tau_k}k^{v-1}\sum_{r=0}^{k-1}\tau(r)r^{n-v}. \qquad (17)$$

Proof. By using Equation (16), we get

$$\begin{aligned} F_{\tau}(t) &= \frac{1}{k}\sum_{r=0}^{k-1}\tau(r)e^{rt}F_{\tau_k}(kt) = \frac{1}{k}\sum_{r=0}^{k-1}\tau(r)\left(\sum_{n=0}^{\infty} r^n\frac{t^n}{n!}\right)\left(\sum_{n=0} B_{n,\tau_k}k^v\frac{t^n}{n!}\right) \\ &= \frac{1}{k}\sum_{n=0}^{\infty}\left(\sum_{v=0}^{n}\binom{n}{v}\sum_{r=0}^{k-1}\tau(r)r^{n-v}k^vB_{v,\tau_k}\right)\frac{t^v}{v!}. \end{aligned}$$

Hence, comparing the coefficient $\frac{t^v}{v!}$ on both sides yields the assertion of this theorem. □

Now, we try to find the similar relations for the generalized Bernoulli polynomials. Firstly, by using Equation (16), we observe the following;

$$\begin{aligned} F_{\tau}\left(\frac{t}{k}, kx\right) &= \sum_{n=0}^{\infty} B_{n,\tau}(kx)k^{-n}\frac{t^n}{n!} = \left(\frac{1}{k}\sum_{r=0}^{k-1}\tau(r)F_{\tau_k}\left(t, x+\frac{r}{k}\right)\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{k}\sum_{r=0}^{k-1}\tau(r)B_{n\tau_k}\left(x+\frac{r}{k}\right)\frac{t^n}{n!} \end{aligned} \tag{18}$$

By comparing the coefficient of the last Equation, we have the following results which are similar to the multiplication formula:

Theorem 3.6. *Let $f = pk$ for some $p, k \in Z$ such that $(p, k) = 1$. Then for all positive integer n, we have that*

$$B_{n,\tau}(kx) = k^{n-1}\sum_{r=0}^{k-1}\tau(r)B_{n,\tau_k}(x+\frac{r}{k}). \tag{19}$$

For $B_{n\tau}(kx)$, we also find the different formula from Equation (19) in the following Theorem.

Theorem 3.7. *Let $f = pk$ for some $p, k \in \mathbb{N}$ such that $(p, k) = 1$. Then for all positive integer n, we have*

$$B_{n,\tau}(kx) = \sum_{l=0}^{n}\binom{n}{l}B_{l,\tau_k}(x)k^{l-1}\sum_{i=0}^{k-1}\tau(i)i^{n-l}. \tag{20}$$

Proof. For the generalized Bernoulli polynomials, we have the following the generating function;

$$\begin{aligned} F_{\tau}(t,x) &= \sum_{i=0}^{k-1}\frac{1}{k}\tau(i)e^{it}F_{\tau_k}(kt, \frac{x}{k}) = \left(\sum_{i=0}^{k-1}\frac{1}{k}\tau(i)\sum_{n=0}^{\infty} i^n\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}B_{n,\tau_k}(\frac{x}{k})k^n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\sum_{i=0}^{k-1}\frac{1}{k}\tau(i)i^{n-l}B_{l,\tau_k}\left(\frac{x}{k}\right)k^l\right)\frac{t^n}{n!}. \end{aligned}$$

By the comparing of the coefficient in above Equation, we get the result. □

The classical multiplicative formula for the generalized Bernoulli polynomials has been given in Equation (12). Unfortunately so far, we have found some Formulas in Equations (19-20) under some conditions. And now, our aim is to obtain more general formula. For this aim, we need to define new group homomorphisms and study the generating functions for the generalized Bernoulli polynomials attached to these group homomorphisms.

Let h be a positive integer. We define the *group homomorphism* $\rho(v) = \exp(\frac{2\pi i v}{h})$ with the period h for an integer v.

Let us define the function λ from $\mathbb{Z}$ to $\mathbb{C}$ such that $\lambda(j) := \tau(j)\rho(j)$ for all integer j. It is clearly a group homomorphism and also its period is $lcm(f, h)$. Then we have the following result.

Theorem 3.8. *Let h be a positive integer. Then for all integer j, $\lambda(j) := \tau(j)\rho(j)$ is a group homomorphism with the period f where $\rho(v) = \exp(\frac{2\pi i v}{h})$.*

Now, we show the relations between the generalized Bernoulli numbers attached to the group homomorphisms λ and τ_h. Then we try to obtain the formula like the multiplicative formula for the generalized Bernoulli polynomials attached group homomorphisms τ and τ_h.

Firstly, we note that $lcm(f,h)\gcd(f,h) = fh$ and so $f = f'\gcd(f,h)$ for $f' \in \mathbb{N}$.

Let $H = lcm(f,h)$ and so $H = f'h$. Then by Theorem 3.1, the period of τ_h is f' and so we have the following;

$$F_\lambda(t,x) = F_\lambda(t)e^{xt} = \sum_{j=0}^{f'-1}\sum_{r=0}^{h-1} \frac{\tau(hj)\tau(r)\rho(r)te^{(hj)t}e^{rt+xt}}{e^{Ht}-1} \tag{21}$$

Therefore, by Equation (21), we arrive at the following theorem.

Theorem 3.9. *For a positive integer n, with the above notations, we have*

$$B_{n,\lambda}(hx) = h^{n-1}\sum_{r=0}^{h-1}\lambda(r)B_{n,\tau_h}(x+\frac{r}{h}). \tag{22}$$

Proof. By using Equation (21), we have the following;

$$F_\lambda(t,x) = \frac{1}{h}\sum_{r=0}^{h-1}\lambda(r)F\tau_h(ht,\frac{x+r}{h}).$$

Then we get the generalized Bernoulli polynomials and so the results by comparing the coefficient of $\frac{t^n}{n!}$ in the following equation;

$$\sum_{n=0}^{\infty} B_{n\lambda}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(h^{n-1}\sum_{r=0}^{h-1}\lambda(r)B_{n,\tau_h}(\frac{x+r}{h})\right)\frac{t^n}{n!}.$$

□

To find the relations between the generalized Bernoulli polynomials attached to group homomorphisms τ and λ directly, we may focus on the Equation (13) and (21). Thus we get interesting multiplication formula.

$$F_\lambda(t,x) = \left(\sum_{r=0}^{h-1}\lambda(r)\left(\frac{\tau(1)e^t-1}{\tau(h)e^t-1}\right)F_\tau\left(ht,\frac{x+r}{h}\right)\right).$$

Maybe, the above equation is not useful in general case, but if $h \equiv 1 \bmod f$, then we get

$$\sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!} = \sum_{r=0}^{h-1}\lambda(r)F_\tau\left(ht,\frac{x+r}{h}\right) = \sum_{n=0}^{\infty}\sum_{r=0}^{h-1}\lambda(r)h^nB_{n,\tau}\left(\frac{x+r}{h}\right)\frac{t^n}{n!}.$$

Therefore, by comparing the coefficient of $\frac{t^n}{n!}$ on both sides in Equation (21), we get the following theorem which gives us the modification of Equation (22).

Theorem 3.10. *Let h be a positive integer and $h \equiv 1 \bmod f$. Then for all $n \in \mathbb{N}_0$, we have*

$$B_{n,\lambda}(hx) = h^n\sum_{r=0}^{h-1}\lambda(r)B_{n,\tau}\left(x+\frac{r}{h}\right).$$

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