Filomat 31:10 (2017), 2951–2956
https://doi.org/10.2298/FIL1710951C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Suzuki-Type Common Fixed Point Theorem

N. Chandra[a], M.C. Arya[a], Mahesh C. Joshi[a]

[a] *Dept. of Mathematics, Kumaun University, Nainital, India*

Abstract. In this paper we establish a common fixed point theorem for two maps under the generalized contractive condition in a complete metric spaces.

1. Introduction

Let (X, d) be a metric space and $T : X \to X$ be a mapping. If there exists a nonnegative real number α such that $\alpha < 1$ and $d(Tx, Ty) \leq \alpha\, d(x, y)$ for all $x, y \in X$, then T is called the contraction mapping. In 1922, Banach [1] proved a very fascinated theorem which is known as "Banach Contraction Principle" and stated as "*A contraction map on a complete metric space has a unique fixed point*". This theorem is a very important because it is a very forceful tool in nonlinear analysis and has wide applications in various areas of study. It has been generalized by many authors in many different directions; see ([2], [3], [4], [13], [15], [16] and references therein). Connell [5] gave an example of a metric space such that X is not complete even every contraction on X has a fixed point. Hence, Banach theorem cannot characterize the metric completeness of X.

Suzuki [16] gave a very interesting result which is the weaker version of Banach contraction principle and also characterizes the completeness of underlying metric spaces. Suzuki proved the following result.

Theorem 1.1. *Let (X, d) be a complete metric space, T be a mapping on X. Define a non-increasing function $\theta : [0, 1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}} \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases} \tag{1}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r) d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq r\, d(x, y)$$

for each $x, y \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 54H25

Keywords. Fixed point, common fixed point and Suzuki-type fixed point theorem
Received: 28 September 2016; Accepted: 07 December 2016
Communicated by Dragan Djurčić

Email addresses: cnaveen329@gmail.com (N. Chandra), mcarya1986@gmail.com (M.C. Arya), mcjoshi69@gmail.com (Mahesh C. Joshi)

Suzuki's result gave a new direction to the subject and as a result, researchers made many important contributions in metric fixed point theory. Several authors obtained variations and refinements of the result due to Suzuki; see ([6], [7], [8], [9], [10], [11], [12], [14] and references therein).

Recently, Kim et al. [12] obtained a common fixed point theorem for two maps T and S on a complete metric spaces X satisfying the following generalized contractive condition:

$$\theta(r) min\{d(x, Tx), d(x, Sx)\} \leq d(x, y) \text{ implies}$$

$$max\,\{d(Sx, Sy), d(Tx, Ty), d(Sx, Ty), d(Sy, Tx)\} \leq r\, d(x, y).$$

for each $x, y \in X$, where θ is defined as in (1).

Now, we prove our main result by taking a new generalized contractive condition for two maps in complete metric spaces.

2. Main Results

Theorem 2.1. *Let (X, d) be a complete metric space. Let $T, S : X \to X$ be two self maps and $\theta : [0, 1) \to (\frac{1}{2}, 1]$ be defined by (1). Assume that there exists $r \in [0, \frac{1}{2})$ such that*

$$\theta(r) min\{d(x, Tx), d(x, Sx)\} \leq d(x, y) \text{ implies}$$

$$max\left\{d(Sx, Sy), d(Tx, Ty), \frac{1}{2}[d(Sx, Ty) + d(Sy, Tx)]\right\} \leq r\, d(x, y), \tag{2}$$

for each $x, y \in X$. Then there exists a unique common fixed point of T and S.

Proof. First, we prove that if z is a fixed point of T, then it is also fixed point of S and vice versa. Let z be a fixed point of T. Taking $x = z$ and $y = Tz$ in (2), we get

$$0 = \theta(r) min\{d(z, Tz), d(z, Sz)\} \leq d(z, Tz) \text{ implies}$$

$$max\left\{d(Sz, STz), d(Tz, T^2z), \frac{1}{2}[d(Sz, T^2z) + d(STz, Tz)]\right\} \leq r\, d(z, Tz) = 0.$$

Hence

$$max\{d(Sz, Sz), d(z, z), \frac{1}{2}[d(Sz, z) + d(Sz, z)]\} \leq 0$$

implies $d(Sz, z) = 0$, i.e., z is a fixed point of S also. Similarly, we can show that if z is a fixed point of S, then it is also fixed point of T. Now to prove our theorem, it is enough to show that T or S has a fixed point. Putting $y = Sx$ in (2), we get,

$$\theta(r) min\{d(x, Tx), d(x, Sx)\} \leq d(x, Sx)$$

implies

$$max\left\{d(Sx, S^2x), d(Tx, TSx), \frac{1}{2}[d(Sx, TSx) + d(S^2x, Tx)]\right\} \leq r\, d(x, Sx) \qquad \forall x \in X.$$

Hence,

$$\frac{1}{2} d(Sx, TSx) \leq \frac{1}{2}[d(Sx, TSx) + d(S^2x, Tx)] \leq r\, d(x, Sx). \tag{3}$$

Now, putting $y = Tx$ in (2), we get

$$\theta(r) min\{d(x, Tx), d(x, Sx)\} \leq d(x, Tx)$$

implies

$$max\left\{d(Sx,STx),d(Tx,T^2x),\frac{1}{2}[d(Sx,T^2x)+d(STx,Tx)]\right\}\leq r\,d(x,Tx)\qquad \forall x\in X.$$

Hence, we have

$$d(Tx,T^2x)\leq r\,d(x,Tx), \tag{4}$$

and

$$\frac{1}{2}d(Tx,STx)\leq\frac{1}{2}[d(Sx,T^2x)+d(STx,Tx)]\leq r\,d(x,Tx). \tag{5}$$

Now let us take an arbitrary $x_0 \in X$. Construct a sequence $\{x_n\}$ such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for $n \in \mathbb{N}\cup\{0\}$. From (5), we get

$$\begin{aligned} d(x_{2n},x_{2n+1}) &= d(Tx_{2n-1},STx_{2n-1})\\ &\leq 2r\,d(x_{2n-1},Tx_{2n-1})\\ &= 2r\,d(x_{2n-1},x_{2n}) \end{aligned}$$

And also, from (3), we get

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(Sx_{2n},TSx_{2n})\\ &\leq 2r\,d(x_{2n},Sx_{2n})\\ &= 2r\,d(x_{2n},x_{2n+1}) \end{aligned}$$

Therefore, for each $n \in \mathbb{N}\cup\{0\}$, we have

$$d(x_n,x_{n+1})\leq 2r\,d(x_{n-1},x_n). \tag{6}$$

For each $m \geq n$, we have

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{m-1},x_m)\\ &\leq (2r)^n d(x_0,x_1)+(2r)^{n+1}d(x_0,x_1)+\ldots+(2r)^{m-1}d(x_0,x_1)\\ &= (2r)^n(1+2r+\ldots+(2r)^{m-n-1})d(x_0,x_1)\\ &\leq \frac{(2r)^n}{1-2r}d(x_0,x_1)\to 0\quad \text{as } n\to\infty. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $z \in X$ such that $\{x_n\}$ converges to a point $z \in X$, i.e.

$$\begin{aligned} \lim_{n\to\infty} Sx_{2n} &= \lim_{n\to\infty} x_{2n+1} = z\\ \text{and } \lim_{n\to\infty} Tx_{2n+1} &= \lim_{n\to\infty} x_{2n+2} = z. \end{aligned}$$

Now we consider $x, z \in X$ such that $x \neq z$. As $\lim_{n\to\infty} d(x_{2n+1}, Tx_{2n+1}) = 0$ and $\lim_{n\to\infty} d(x_{2n+1}, x) \neq 0$, therefore there exists some $x_{2n_k+1} \in X$ such that

$$\theta(r)min\{d(x_{2n_k+1},Sx_{2n_k+1}),d(x_{2n_k+1},Tx_{2n_k+1})\}\leq d(x_{2n_k+1},x)$$

implies

$$\begin{aligned} max\{d(Sx_{2n_k+1},Sx),d(Tx_{2n_k+1},Tx),\frac{1}{2}[d(Sx_{2n_k+1},Tx)+d(Sx,Tx_{2n_k+1})]\}\leq r\,d(x_{2n_k+1},x)\\ \Rightarrow\qquad d(Tx_{2n_k+1},Tx)\leq r\,d(x_{2n_k+1},x). \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$d(z,Tx)=\lim_{n\to\infty}d(Tx_{2n_k+1},Tx)\leq r\,lim_{n\to\infty}d(x_{2n_k+1},x)=r\,d(z,x).$$

Therefore, for each $x \neq z$,

$$d(z, Tx) \leq r\, d(z, x). \tag{7}$$

Now by induction, we prove that

$$d(T^n z, z) \leq d(Tz, z) \quad \forall\ n \in \mathbb{N}. \tag{8}$$

For $n = 1$, the inequality is obvious. Suppose, the inequality (8) is true for some $m \in \mathbb{N}$, i.e., $d(T^m z, z) \leq d(Tz, z)$. Now, for $n = m + 1$, if $T^m z = z$, then

$$d(T^{m+1} z, z) = d(Tz, z). \tag{9}$$

If $T^m z \neq z$, then by (7),

$$d(T^{m+1} z, z) \leq r\, d(T^m z, z) \leq r\, d(Tz, z) < d(Tz, z). \tag{10}$$

Thus, (9) and (10) imply

$$d(T^{m+1} z, z) \leq d(Tz, z).$$

Hence, the inequality (8) holds for each $n \in \mathbb{N}$.

Let us assume that $Tz \neq z$. Now, we prove the following inequality by the principle of mathematical induction,

$$d(T^n z, Tz) \leq r\, d(Tz, z) \tag{11}$$

for each $n \in \mathbb{N}$. For $n = 1$, it is obvious. Further, from (4) inequality (11) holds for $n = 2$. Suppose (11) holds for some $n > 2$, then we have

$$\begin{aligned} d(Tz, z) &\leq d(z, T^n z) + d(T^n z, Tz) \\ &\leq d(z, T^n z) + r\, d(Tz, z) \end{aligned}$$

$\Rightarrow$

$$(1 - r) d(z, Tz) \leq d(z, T^n z). \tag{12}$$

Since $0 \leq r < \frac{1}{2}$, so $\theta(r) \leq \frac{1-r}{r^2}$. Hence,

$$\begin{aligned} \theta(r) min\{d(ST^n z, T^n z), d(T^n z, T^{n+1} z)\} &\leq \frac{1-r}{r^2} d(T^n z, T^{n+1} z) \\ &\leq \frac{1-r}{r^n} d(T^n z, T^{n+1} z). \end{aligned}$$

Since inequality (11) holds for $n = 2$, so by (12), we have

$$\theta(r) min\{d(ST^n z, T^n z), d(T^n z, T^{n+1} z)\} \leq \frac{1-r}{r^n} .r^n d(z, Tz) = (1 - r) d(z, Tz) \leq d(z, T^n z).$$

Which implies

$$max\{d(ST^n z, Sz), d(T^{n+1} z, Tz), \frac{1}{2}[d(ST^n z, Tz) + d(Sz, T^{n+1} z)]\} \leq r\, d(z, T^n z).$$

Using (8), we obtain

$$d(T^{n+1} z, Tz) \leq r\, d(T^n z, z) \leq r\, d(Tz, z).$$

So, the inequality (11) holds for each $n \in \mathbb{N}$. Now $Tz \neq z$ and (11) implies that $T^n z \neq z$ (if $T^n z = z$, then we find $d(T^n z, Tz) \leq r\, d(z, Tz)$ implies $d(z, Tz) \leq r\, d(z, Tz) < d(z, Tz)$, which is not possible).

Hance, (7) implies that

$$d(z, T^{n+1}z) \quad \leq \quad r\, d(z, T^n z) \leq \ r^2 d(z, T^{n-1}z) \ \leq \ \ldots \ \leq \ r^n d(z, Tz).$$

Hence, $\lim_{n\to\infty} d(z, T^{n+1}z) = 0 \Rightarrow T^n z \to z$. From this and (11), we have

$$\begin{aligned} d(z, Tz) = \lim_{n\to\infty} d(T^n z, Tz) \quad &\leq \quad \lim_{n\to\infty} r\, d(z, Tz) = r\, d(Tz, z) \\ \Rightarrow \qquad d(Tz, z) \quad &= \quad 0. \end{aligned}$$

Which is contrary to our assumption. Hence, $Tz = z$. As already proved z is fixed point of S also. Hence, $Tz = Sz = z$.
Finally, we prove the uniqueness of the fixed point. Let z and z' be two common fixed points of T and S, such that $z \neq z'$. Taking $x = z$ and $y = z'$ in (2), we get

$$0 = \theta(r) min\{d(z, Tz), d(z, Sz)\} \leq d(z, z')$$

implies

$$\begin{aligned} & max\left\{d(Sz, Sz'), d(Tz, Tz'), \frac{1}{2}[d(Sz, Tz') + d(Tz, Sz')]\right\} \leq r\, d(z, z') \\ \Rightarrow \quad & max\left\{d(z, z'), d(z, z'), \frac{1}{2}[d(z, z') + d(z, z')]\right\} \leq r\, d(z, z') \\ \Rightarrow \quad & d(z, z') \leq r\, d(z, z') < d(z, z') \end{aligned}$$

which is not possible. Hence $z = z'$. This completes the proof. □

Taking $S = T$, we get following Suzuki type result [16]:

Corollary 2.2. *Let (X, d) be a complete metric space. If $T : X \to X$ be a self mapping and $\theta : [0, 1) \to (\frac{1}{2}, 1]$ is defined by (1). Assume that there exists $r \in [0, \frac{1}{2})$ such that for each $x, y \in X$,*

$$\theta(r) d(x, Tx) \leq d(x, y) \quad implies \quad d(Tx, Ty) \leq r\, d(x, y).$$

Then T has a unique fixed point $z \in X$.

In place of two maps taking three maps in our main result, we obtain the following result as a consequence of Theorem 2.1 with commutativity as an extra condition imposed on maps.

Corollary 2.3. *Let (X, d) be a complete metric space. If $f, S, T : X \to X$ be three maps and $\theta : [0, 1) \to (\frac{1}{2}, 1]$ be defined by (1). Assume that there exists $r \in [0, \frac{1}{2})$ such that for each $x, y \in X$,*

$$\theta(r)\, min\{d(x, fTx), d(x, fSx)\} \leq d(x, y)$$

implies

$$max\{d(fSx, fSy), d(fTx, fTy), \frac{1}{2}[d(fSx, fTy) + d(fSy, fTx)]\} \leq r\, d(x, y). \tag{13}$$

And, if f is one-one, $fS = Sf$ and $fT = Tf$, then f, T and S have a unique common fixed point $z \in X$.

Proof. Considering fS and fT as two maps in the given contractive condition of Theorem 2.1, we get a unique common fixed point for fS and fT, i.e., $fSz = fTz = z$. Since f is one-one,

$$fSz = fTz = z \ \Rightarrow \ Sz = Tz. \tag{14}$$

From (13), we get

$$0 = \theta(r) min\{d(z, fTz), d(z, fSz)\} \leq d(z, Tz)$$

implies

$$
\begin{aligned}
& max\{d(fSz, fSTz), d(fTz, fT^2z), \frac{1}{2}[d(fSz, fT^2z) + d(fSTz, fTz)]\} \leq r\, d(z, Tz) \\
\Rightarrow\ & max\{d(fSz, SfTz), d(fTz, TfTz), \frac{1}{2}[d(fSz, TfTz) + d(SfTz, fTz)]\} \leq r\, d(z, Tz) \\
\Rightarrow\ & max\{d(z, Sz), d(z, Tz), \frac{1}{2}[d(z, Tz) + d(z, Sz)]\} \leq r\, d(z, Tz) \\
\Rightarrow\ & d(z, Tz) \leq r\, d(z, Tz) \\
\Rightarrow\ & d(Tz, z) = 0.
\end{aligned}
$$

Hence z is a common fixed point of f, S and T. □

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