Filomat 31:10 (2017), 2957–2966
https://doi.org/10.2298/FIL1710957M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Matrix Transformations on Mixed Paranorm Spaces

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Abstract. Recently the concept of mixed norm spaces was generalized to that of the mixed paranorm spaces $[\ell(r), \ell_p]^{<k(\nu)>}$. Here we determine the classes of matrix transformations from $[\ell(r), \ell_p]^{<k(\nu)>}$ into the spaces of bounded, convergent and null sequences, and into the spaces of all bounded, convergent and absolutely convergent series. We also obtain many correponding known results for mixed norm spaces as special cases and visualize some neighbourhoods in the spaces $[\ell(r), \ell_p]^{<k(\nu)>}$.

1. Introduction and Notations

Throughout the paper, let $r = (r_\nu)_{\nu=0}^\infty$ be a bounded sequence of positive reals r_ν with $M(r) = \sup_\nu r_\nu$ and $H(r) = \max\{1, M(r)\}$, $1 \le p \le \infty$, and s_ν for $r_\nu > 1$ and q denote the conjugate exponents of r_ν and p, that is, $s_\nu = r_\nu/(r_\nu - 1)$ $(\nu = 0, 1, \dots)$ and $q = \infty$ for $p = 1$, $q = p/(p-1)$ for $1 < p < \infty$ and $q = 1$ for $p = \infty$.

As usual, we write $e = (e_k)_{k=1}^\infty$ and $e^{(n)} = (e_k^{(n)})_{n=1}^\infty$ for $n \in \mathbb{N}$ for the sequences with $e_k = 1$ for all k , and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. We also denote by ℓ_∞, c, c_0 and ϕ the sets of all bounded, convergent, null and finite sequences, respectively, and consider the set

$$\ell(r) = \left\{ x \in \omega : \sum_{k=1}^\infty |x_k|^{r_k} < \infty \right\},$$

which reduces to the well–known set ℓ_r when $(r_\nu)_{\nu=1}^\infty = r \cdot e$ for some constant $r \ge 1$. Furthermore, let cs and bs be the sets of all convergent and bounded sequences.

For any subset X of ω, let $X^\beta = \{a \in \omega : \sum_{k=1}^\infty a_k x_k$ converges for all $x \in X\}$ denote the β–dual of X. Given any infinite matrix $A = (a_{nk})_{n,k=1}^\infty$ of complex entries and any sequence $x = (x_k)_{k=1}^\infty \in \omega$, we write A_n for the sequence in the n^{th} row of A, $A_n x = \sum_{k=1}^\infty a_{nk} x_k$, $Ax = (A_n x)_{n=1}^\infty$ (provided the series $A_n x$ converge for all n), and $X_A = \{a \in \omega : Ax \in X\}$ for the *matrix domain of A in X*. Finally, if X and Y are subsets of ω, then (X, Y) denotes the class of all infinite matrices A for which $X \subset Y_A$, that is, $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all n and $Ax \in Y$ for all $x \in X$.

2010 *Mathematics Subject Classification*. Primary 46B45; Secondary 47B37
Keywords. Mixed panorm sequence spaces, matrix transformations and visualization
Received: 29 July 2016; Accepted: 28 January 2017
Communicated by Dragan Djurčić
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An infinite matrix $T = (t_{nk})_{n,k=1}^{\infty}$ is called a *triangle*, if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all n. Let $B = (b_{nk})_{n,k=1}^{\infty}$ be a positive triangle and $X \subset \omega$. We write $B_n|x| = \sum_{k=1}^{n} b_{nk}|x_k|$ ($x \in \omega$) for each n and $X_{[B]} = \{x \in \omega : B|x| = (B_n|x|)_{n=1}^{\infty} \in X\}$ for the *strong matrix domain of B in X*.

A subset X of ω is said to be *normal* if $x \in X$ and $|y_k| \leq |x_k|$ ($k = 1, 2, \dots$) imply $y = (y_k)_{k=1}^{\infty} \in X$.

An *FK* space X is a Fréchet sequence space with continuous coordinates $P_n : X \to \mathbb{C}$ where $P_n(x) = x_n$ for all $x = (x_k)_{k=1}^{\infty} \in X$ and $n = 1, 2, \dots$. We say that an *FK* space $X \supset \phi$ has *AK* if $x^{[m]} = \sum_{k=1}^{m} x_k e^{(k)} \to x$ ($m \to \infty$); $x^{[m]}$ is called the m–section of the sequence x. A normable *FK* space is said to be a *BK* space. For instance, $\ell(r)$ is an *FK* space with respect to its total paranorm $h_{(r)}$ defined by

$$h_{(r)}(x) = \left(\sum_{k=1}^{\infty} |x_k|^{r_k} \right)^{1/H(r)} \quad \text{for all } x \in \ell(r);$$

ℓ_p ($1 \leq p \leq \infty$), c and c_0 are *BK* spaces with their natural norms $\|\cdot\|_p$ ($1 \leq p < \infty$) and $\|\cdot\|_\infty$ defined by

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \quad \text{for } x \in \ell_p \text{ and } \|x\|_\infty = \sup_k |x_k| \text{ for } x \in c_0, c, \ell_\infty;$$

$\ell(r)$, ℓ_p ($1 \leq p < \infty$) and c_0 have *AK*.

The mixed paranorm spaces $[\ell(r), \ell_p]^{<k(\nu)>}$ were defined in [7] as follows: Let $(k(\nu))_{\nu=0}^{\infty}$ be a sequence of integers with $1 = k(0) < k(1) < \cdots$. By $K^{<k(\nu)>}$ ($\nu = 0, 1, \dots$), we denote the set of all integers k that satisfy the inequality $k(\nu) \leq k \leq k(\nu+1) - 1$ and we write $\sum_\nu = \sum_{k \in K^{<k(\nu)>}}$ and $\max_\nu = \max_{k \in K^{<k(\nu)>}}$. Given any sequence $x = (x_k)_{k=1}^{\infty} \in \omega$, we define the $K^{<k(\nu)>}$ blocks of x by $x^{<\nu>} = \sum_\nu x_k e^{(k)}$ for $\nu = 0, 1, \dots$, and put

$$[\ell(r), \ell_p]^{<k(\nu)>} = \left\{ x \in \omega : \tilde{h}_{(r),p}(x) = \sum_{\nu=0}^{\infty} \|x^{<\nu>}\|_p^{r_\nu} < \infty \right\} \text{ (the sequence } r \text{ may be unbounded).}$$

We also write $[\ell_\infty(r), \ell_p]^{<k(\nu)>} = \{x \in \omega : \sup_\nu \|x^{<\nu>}\|_p^{r_\nu} < \infty\}$. If $(r_\nu)_{\nu=1}^{\infty} = r \cdot e$ for some constant $r \geq 1$, then the spaces $[\ell(r), \ell_p]^{<k(\nu)>}$ reduce to the mixed norm spaces $\ell(r, p)$ introduced by Hedlund [2]. The Cesàro sequence spaces ces_p [1, 3] can also obtained as special cases of the spaces $[\ell(r), \ell_p]^{<k(\nu)>}$.

In this paper, we characterize the classes $([\ell(r), \ell_p]^{<k(\nu)>}, Y)$, that is, we determine necessary and sufficient conditions on the entries of an infinite matrix A to map $[\ell(r), \ell_p]^{<k(\nu)>}$ into Y, when Y is any of the spaces $\ell_\infty, c_0, c, \ell_1, bs$ or cs. Our new results yield many known results as special cases. Finally, we visualize certain neighbourhoods in several of the spaces $[\ell(r), \ell_p]^{<k(\nu)>}$.

2. Some Known Results

For the reader's convenience, we list the results on mixed paranorm spaces from [7] which are needed in the sequel.

The sets $[\ell(r), \ell_p]^{<k(\nu)>}$ are *FK* spaces with *AK* with their natural total paranorms $h_{(r),p}$ defined by

$$h_{(r),p}(x) = \left(\tilde{h}_{(r),p} \right)^{1/H(r)} = \left(\sum_{\nu=0}^{\infty} \|x^{<\nu>}\|_p^{r_\nu} \right)^{1/H(r)} \quad \text{for all } x \in [\ell(r), \ell_p]^{<k(\nu)>}$$

if and only if the sequence r is bounded ([7, Corollary 2.6]).

Let $\mathcal{U}$ be the set of all complex sequences u such that $u_k \neq 0$ for all k. Then we write $u^{-1} * X = \{y \in \omega : u \cdot y = (u_k y_k)_{k=1}^{\infty} \in X\}$. We put

$$M((r), p) = \begin{cases} \bigcup_{N>1} \left(\dfrac{1}{N} e \right)^{-1} * [\ell(s), \ell_q]^{<k(\nu)>} = \bigcup_{N>1} \left\{ a \in \omega : \sum\limits_{\nu=0}^{\infty} \|a^{<\nu>}\|_q^{s_\nu} \cdot N^{-s_\nu} < \infty \right\} & \text{if } r_\nu > 1 \text{ for all } \nu \\ [\ell_\infty(r), \ell_q]^{<k(\nu)>} & \text{if } r_\nu \leq 1 \text{ for all } \nu. \end{cases}$$

Since $s_\nu = 1 + s_\nu/r_\nu$ when $r_\nu > 1$ for all ν, we have $a \in M((r), p)$ if and only if $\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_q^{s_\nu} \cdot N^{-s_\nu/r_\nu} < \infty$, and so $([\ell(r), \ell_p]^{<k(\nu)>})^\beta = M((r), p)$ by [7, Theorem 3.2].

Let $X \subset \omega$ be a linear metric space with respect to the metric d and $a \in X$. Then we denote the closed ball of radius $\delta > 0$ and centre in 0 by $\bar{B}_\delta(0) = \{y \in X : d(x, 0) \leq \delta\}$, and write $\|a\|_\delta^* = \sup\{|\sum_{k=1}^{\infty} a_k x_k| : x \in \bar{B}_\delta(0)\}$ provided the expression on the left hand side exists and is finite, which is the case whenever X is an FK space and $a \in X^\beta$ ([11, Theorem 7.2.9]. It is known that if X is an FK space then $A \in (X, \ell_\infty)$ if and only if

$$\|A\|_\delta^* = \sup_n \|A_n\|_\delta^* < \infty \text{ for some } \delta > 0 \text{ ([9, Theorem 1.23 (b)]).} \tag{1}$$

We also need the following result.

Proposition 2.1. ([8, Theorem 2 (b)]) *Let B be a positive triangle, X be an arbitary subset of ω, and Y be a normal set of sequences. For every $m \in \mathbb{N}$, let $N_m \subset \{1, 2, \ldots, m\}$, $N = (N_m)_{m=1}^{\infty}$ and $\mathcal{N}$ be the set of all such sequences. Given an infinite matrix $A = (a_{nk})_{n,k=1}^{\infty}$, we define the matrix $S^N(A)$ for every $N \in \mathcal{N}$ by*

$$S_m^N(A) = \sum_{n \in N_m} b_{mn} A_n, \text{ that is, } s_{mk}^N(A) = \sum_{n \in N_m} b_{mn} a_{nk} \ (m, k = 1, 2, \ldots).$$

Then we have $A \in (X, Y_{[B]})$ if and only if $S^N(A) \in (X, Y)$ for all sequences $N \in \mathcal{N}$.

Finally we observe that

$$\textstyle\sum_\nu |a_k x_k| = \|a^{<\nu>}\|_q \cdot \|x^{<\nu>}\|_p \text{ for all } \nu \text{ and all sequences } x, y \in \omega. \tag{2}$$

Using (2) and applying the well–known inequality $|b_\nu y_\nu| \leq |b_\nu|^{s_\nu} + |y_\nu|^{r_\nu}$ with $b_\nu = \|a^{<\nu>}\|_q N^{-1/r_\nu}$ and $y = \|x^{<\nu>}\|_p N^{1/r_\nu}$ $(\nu = 0, 1, \ldots)$ where $N \neq 0$, we obtain

$$\textstyle\sum_\nu |a_k x_k| \leq \left\|a^{<\nu>}\right\|_q \cdot \left\|x^{<\nu>}\right\|_p \leq \left\|a^{<\nu>}\right\|_q^{s_\nu} N^{-s_\nu/r_\nu} + N\left\|x^{<\nu>}\right\|_p^{r_\nu} = N\left(\left\|a^{<\nu>}\right\|_q^{s_\nu} N^{-s_\nu} + \left\|x^{<\nu>}\right\|_p^{r_\nu}\right) \text{ for } \nu = 0, 1, \ldots. \tag{3}$$

3. Matrix Transformations

In this section, we establish the main results, namely the characterizations of the classes $([\ell(r), \ell_p]^{<k(\nu)>}, Y)$ when Y is any of the spaces ℓ_∞, c_0, c , ℓ_1, bs and cs. First we characterize the class $([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$.

Theorem 3.1. *(a) Let $r_\nu > 1$ for all ν. Then we have $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$ if and only if*

$$\|A\|_B = \sup_n \tilde{h}_{(s),q}(A_n/B) = \sup_n \sum_{\nu=0}^{\infty} \left\|A_n^{<\nu>}\right\|_q^{s_\nu} B^{-s_\nu} < \infty \textit{ for some integer } B > 1. \tag{4}$$

(b) Let $r_\nu \leq 1$ for all ν. Then we have $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$ if and only if

$$\|A\| = \sup_{n,\nu} \left\|A_n^{<\nu>}\right\|_q^{r_\nu} < \infty. \tag{5}$$

Proof. (a) Let $r_\nu > 1$ for all ν.
(i) First we prove the sufficiency of the condition in (4).
It follows from (4) and [7, Theorem 3.2] that $A_n \in ([\ell(r), \ell_p]^{<k(\nu)>})^\beta$ for all n. Also (3) yields for all $x \in [\ell(r), \ell_p]^{<k(\nu)>}$ and all n

$$|A_n(x)| \leq \sum_{\nu=0}^{\infty} \textstyle\sum_\nu |a_k x_k| \leq B\left(\displaystyle\sum_{\nu=0}^{\infty}\left(\left\|A_n^{<\nu>}\right\|_q^{s_\nu} B^{-s_\nu} + \left\|x^{<\nu>}\right\|_p^{r_\nu}\right)\right) = B\left(\tilde{h}_{(s),q}(A_n/B) + \tilde{h}_{(r),p}(x)\right)$$

$$\leq B\left(\|A\|_B + \tilde{h}_{(r),p}(x)\right) < \infty,$$

hence $Ax \in \ell_\infty$.
Thus we have shown the sufficiency of the condition in (4).

(ii) Now we show the necessity of the condition in (4). We assume that $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$, but

$$\|A\|_B = \infty \text{ for all } B > 1. \tag{6}$$

Since $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$, it follows by (1) that

$$\|A\|_\delta^* < \infty \text{ for some } \delta \text{ with } 0 < \delta < 1. \tag{7}$$

We choose an integer B_0 such that $B_0\delta^{H(r)} > \|A\|_\delta^*$.

Case 1. $S_n(B_0) = h_{(s),q}(A_n/B_0) < \infty$ for all n. By (6), there exists an integer m such that $S_m(B_0) > 2$, and so there exists $\nu_0 > 0$ such that $\sum_{\nu=\nu_0+1}^{\infty} \left\|A_m^{<\nu>}\right\|_q^{s_\nu} B_0^{-s_\nu} < 1$, hence

$$S_m^{(0)}(B_0) = \sum_{\nu=0}^{\nu_0} \left\|A_m^{<\nu>}\right\|_q^{s_\nu} B_0^{-s_\nu} > 1. \tag{8}$$

Case 2. There exists an integer m such that $S_m(B_0) = \infty$. Then we can also choose $\nu_0 > 0$ such that (8) holds. Thus we have (8) in either case.
We write $S_0 = S_m^{(0)}(B_0)$, for short, put

$$D_\nu = \begin{cases} \delta^{H(r)/r_\nu} B_0^{-s_\nu/r_\nu} S_0^{-1} & (0 \leq \nu \leq v_0) \\ 0 & (\nu > \nu_0) \end{cases},$$

and define the matrix $\hat{A}((s), q) = (\hat{a}_{mk}((s), q))_{m,k=1}^\infty$ and the sequences $x(p) = (x_k(p))_{k=1}^\infty$ by

$$\hat{a}_{mk}((s), q) = \begin{cases} \left.\begin{array}{lll} \operatorname{sgn}(a_{mk})|a_{mk}|^{q-1} \left\|A_m^{<\nu>}\right\|_q^{s_\nu - q} & (k \in K^{<k(\nu)>}) & (1 < p \leq \infty) \\ \operatorname{sgn}(a_{mk_0(\nu)})\|A_m^{<\nu>}\|_\infty^{s_\nu-1}, \text{ where } k_0(\nu) & (k = k_0(\nu)) & (p = 1; q = \infty) \\ \text{is the smallest integer in } K^{<k(\nu)>} & & \\ \text{such that } |a_{mk_0(\nu)}| = \|A_m^{<\nu>}\|_\infty & & \end{array}\right\} & (0 \leq \nu \leq \nu_0) \\ 0 & (\text{otherwise}) \end{cases}$$

and $x_k(p) = D_\nu \hat{a}((s), q)_{mk}$ for $k \in K^{<k(\nu)>}$ $(\nu = 0, 1, \dots)$.
Since $S_0 > 1$ and $r_\nu > 1$ for all ν, we have $S_0^{-r_\nu} \leq S_0^{-1}$, hence

$$D_\nu^{r_\nu} \leq \delta^{H(r)} S_0^{-1} B_\nu^{-s_\nu} \text{ for } 0 \leq \nu \leq \nu_0. \tag{9}$$

If $1 < p < \infty$ then $qp - p = q$ and $r_\nu(s_\nu - q + q/p) = s_\nu$, and if $p = 1, \infty$, then $r_\nu(s_\nu - 1) = s_\nu$ for all ν, and so

$$\|\hat{A}_m((s), q)^{<\nu>}\|_p^{r_\nu} \leq \left(\textstyle\sum_\nu |a_{mk}|^{qp-p}\right)^{r_\nu/p} \left\|A_m^{<\nu>}\right\|_q^{r_\nu(s_\nu - q)} = \left\|A_m^{<\nu>}\right\|_q^{r_\nu(s_\nu - q + q/p)} = \left\|A_m^{<\nu>}\right\|_q^{s_\nu} \text{ if } 1 < p < \infty,$$
$$\|\hat{A}_m((s), \infty)^{<\nu>}\|_1^{r_\nu} = \left(\textstyle\sum_\nu |\hat{a}_{mk}|\right)^{r_\nu} \leq \|A_m^{<\nu>}\|_\infty^{r_\nu(s_\nu - 1)} = \|A_m^{<\nu>}\|_\infty^{s_\nu} \text{ if } p = 1,$$
$$\|\hat{A}_m((s), 1)^{<\nu>}\|_\infty^{r_\nu} = \left(\max_\nu |\hat{a}_{mk}|\right)^{r_\nu} \leq \|A_m^{<\nu>}\|_1^{r_\nu(s_\nu - 1)} = \|A_m^{<\nu>}\|_1^{s_\nu} \text{ if } p = \infty,$$

hence

$$\|\hat{A}_m((s), q)^{<\nu>}\|_p^{r_\nu} \leq \left\|A_m^{<\nu>}\right\|_q^{s_\nu} \text{ for } 1 \leq p \leq \infty \text{ and } 0 \leq \nu \leq \nu_0. \tag{10}$$

Now it follows from (9) and (10) that

$$\tilde{h}_{(r),p}(x(p)) = \sum_{\nu=0}^{\infty} \|x_k(p)^{<\nu>}\|_p^{r_\nu} \le \delta^{H(r)} S_0^{-1} \sum_{\nu=0}^{\nu_0} \|\hat{A}_m^{<\nu>}\|_q^{r_\nu} B_0^{-s_\nu} \le \delta^{H(r)} S_0^{-1} \sum_{\nu=0}^{\nu_0} \left\|A_m^{<\nu>}\right\|_q^{s_\nu} B_0^{-s_\nu} \le \delta^{H(r)},$$

that is,

$$x(p) \in \bar{B}_\delta(0) \text{ for } 1 \le p \le \infty. \tag{11}$$

But on the other hand we have,

$$\sum\nolimits_\nu a_{mk}\hat{a}_{mk}((s),q) = \sum\nolimits_\nu |a_{mk}|^q \left\|A_m^{<\nu>}\right\|_q^{s_\nu - q} = \left\|A_m^{<\nu>}\right\|_q^{s_\nu} \text{ if } 1 < p \le \infty,$$
$$\sum\nolimits_\nu a_{mk}\hat{a}_{mk}((s),\infty) = |a_{mk_0(\nu)}| \|A_m^{<\nu>}\|_\infty^{s_\nu - 1} = \|A_m^{<\nu>}\|_\infty^{s_\nu} \text{ if } p = 1,$$

hence

$$\sum\nolimits_\nu a_{mk}\hat{a}_{mk}((s),q) = \left\|A_m^{<\nu>}\right\|_q^{s_\nu} \text{ for } 1 \le p \le \infty \text{ and } 0 \le \nu \le \nu_0. \tag{12}$$

Since $\delta < 1$ and $r_\nu > 1$, we have $\delta^{H(r)/r_\nu} > \delta^{H(r)}$, and it follows from (12) that

$$|A_m x(p)| = \sum_{\nu=0}^{\nu_0} D_\nu \sum\nolimits_\nu \hat{a}_{mk}((s),q) > \delta^{H(r)} S_0^{-1} \sum_{\nu=0}^{\nu_0} \left\|A_m^{<\nu>}\right\|_q^{s_\nu} B^{-s_\nu/r_\nu} = \delta^{H(r)} S_0^{-1} B_0 \sum_{\nu=0}^{\nu_0} \left\|A_m^{<\nu>}\right\|_q^{s_\nu} B^{-s_\nu} = \delta^{H(r)} B_0 > \|A\|_\delta^*,$$

which, in view of (11), is a contradiction to the definition of $\|A\|_\delta^*$.
Thus the assumption that the condition in (4) is not satisfied, has led to a contradiction for all p with $1 \le p \le \infty$. Therefore (4) has to hold. This completes the proof of the necessity of (4).
Thus we have shown Part (a) of the theorem.

(b) Now let $r_\nu \le 1$ for all ν.
(i) First we prove the sufficiency of the condition in (5).
It follows from (4) and [7, Theorem 3.2] that $A_n \in ([\ell_\infty(r), \ell_p]^{<k(\nu)>})^\beta$ for all n. We choose $N \in \mathbb{N}$ such that $N > \|A\|$. Then we have for all $x \in \bar{B}_{1/N}(0)$ by the inequality in (2)

$$\left(\sum\nolimits_\nu |a_{nk}x_k|\right)^{r_\nu} \le \left\|A_n^{<\nu>}\right\|_q^{r_\nu} \left\|x^{<\nu>}\right\|_p^{r_\nu} \le 1 \text{ for all } n \text{ and all } \nu,$$

and so, since $r_\nu \le 1$ for all ν,

$$\begin{aligned} |A_n x| &= \sum_{\nu=0}^{\infty} \sum\nolimits_\nu |a_{nk}x_k| \le \sum_{\nu=0}^{\infty} \left(\sum\nolimits_\nu |a_{nk}x_k|\right)^{r_\nu} \le \sum_{\nu=0}^{\infty} \left\|A_n^{<\nu>}\right\|_q^{r_\nu} \left\|x^{<\nu>}\right\|_p^{r_\nu} \\ &\le \sup_{n,\nu} \left\|A_n^{<\nu>}\right\|_q^{r_\nu} \sum_{\nu=0}^{\infty} \left\|x^{<\nu>}\right\|_p^{r_\nu} = \|A\| \cdot h_{(r),p}(x) \le 1 \text{ for all } n, \end{aligned} \tag{13}$$

hence $Ax \in \ell_\infty$ for all $x \in \bar{B}_{1/N}(0)$.
Now let x be an arbitrary sequence in $[\ell_\infty(r), \ell_p]^{<k(\nu)>}$. Since the space $[\ell_\infty(r), \ell_p]^{<k(\nu)>}$ is paranormed, the set $\bar{B}_{1/N}$ is absorbing, and consequently there exists a positive real $C > 0$ such that $y = C^{-1}x \in \bar{B}_{1/N}$, and (13) yields $|A_n x| = C|A_n y| \le C$ for all n, hence $Ax \in \ell_\infty$.
Thus we have shown the sufficiency of the condition in (5).
(ii) Finally we show the necessity of the condition in (5).
We assume that $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$ but

$$\|A\| = \infty. \tag{14}$$

Again, $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$ implies by [9, Theorem 1.23 (b)] that (7) holds. By (14), there exist $m \in \mathbb{N}$ and $\mu \in \mathbb{N}_0$ such that

$$\|A_m^{<\mu>}\|_q^{r_\mu} \cdot \delta > \|A\|_\delta^* + 1 = S.$$

We define the sequence $x(p) = (x_k(p))_{k=1}^\infty$ by

$$x_k(p) = \begin{cases} \operatorname{sgn}(a_{mk})|a_{mk}|^{q-1}\|A_m^{<\mu>}\|_q^{-q} S^{1/r_\mu} & (k \in K^{<\mu>}) \quad (1 < p \leq \infty) \\ \operatorname{sgn}(a_{mk_0(\mu)})\|A_m^{<\mu>}\|_\infty^{-1}, \text{ where } k_0(\mu) & (k = k_0(\mu)) \quad (p = 1; q = \infty) \\ \text{is the smallest integer in } K^{<\mu>} & \\ \text{such that } |a_{mk_0(\nu)}| = \|A_m^{<\mu>}\|_\infty & \\ & (\nu = \mu) \\ 0 & (\text{otherwise}). \end{cases}$$

Then we obtain

$$h_{(r),p}(x(p)) = \sum_{\nu=0}^\infty \|x(p)^{<\nu>}\|_p^{r_\nu} \leq S\left(\sum_\mu |a_{mk}|^q\right)^{r_\mu(q/p-q)} = S\|A_m^{<\mu>}\|_q^{-r_\mu} < \delta \text{ for } 1 < p < \infty,$$

$$h_{(r),\infty}(x(\infty)) = \sum_{\nu=0}^\infty \|x_k(\infty)^{<\nu>}\|_\infty^{r_\nu} \leq S\|A_m^{<\mu>}\|_1^{-r_\mu} < \delta \text{ for } p = \infty,$$

$$h_{(r),1}(x(\infty)) = \sum_{\nu=0}^\infty \|x_k(1)^{<\nu>}\|_1^{r_\nu} \leq S\|A_m^{<\mu>}\|_\infty^{-r_\mu} < \delta \text{ for } p = 1,$$

hence $x(p) \in \bar{B}_\delta(0)$ for $1 \leq p \leq \infty$, that is, (11) again holds.
But on the other hand, since $S > 1$ and $r_\mu \leq 1$, it follows that $S^{1/r_\mu} \geq S$, and so

$$|A_m(x(p))| = \left|\sum_\mu x_k(p)\right| = S^{1/r_\mu} \geq S > \|A\|_\delta^* \text{ for } 1 < p \leq \infty,$$

$$|A_m(x(1))| = \left|\sum_\mu x_k(1)\right| = |a_{mk_0}(\mu)| \left\|A_m^{<\mu>}\right\|_\infty^{-1} S^{1/r_\mu} \geq S > \|A\|_\delta^* \text{ for } p = 1,$$

which, in view of (11), is a contradiction to the definition of $\|A\|_\delta^*$.
Thus the assumption that the condition in (5) is not satisfied, has led to a contradiction for all p with $1 \leq p \leq \infty$. Therefore (5) has to hold.
This completes the proof of the necessity of (5).
Thus we have shown Part (b) of the theorem.
This completes the proof of the theorem. □

Using Theorem 3.1 and applying [11, 8.3.6] and Proposition 2.1 we obtain the characterizations of the classes $([\ell(r), \ell_p]^{<k(\nu)>}, c)$, $([\ell(r), \ell_p]^{<k(\nu)>}, c_0)$ and $([\ell(r), \ell_p]^{<k(\nu)>}, \ell_1)$.

Corollary 3.2. *(a) We have $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, c)$ if and only if in addition to the conditions in (4) or (5) the following holds*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ exists for each } k. \tag{15}$$

(b) We have $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, c_0)$ if and only if in addition to the conditions in (4) or (5) the following holds

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for each } k. \tag{16}$$

(c) Let $r_\nu > 1$ for all ν. Then we have $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_1)$ if and only if

$$\|\Sigma A\|_B = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\tilde{h}_{(s),q}\left(\sum_{n\in N} A_n/B\right)\right) = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \sum_{\nu=0}^\infty \left\|\left(\sum_{n\in N} A_n\right)^{<\nu>}\right\|_q^{s_\nu} B^{-s_\nu} < \infty \text{ for some integer } B > 1. \tag{17}$$

(d) Let $r_\nu \leq 1$ for all ν. Then we have $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_1)$ if and only if

$$\|\Sigma A\| = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\sup_\nu \left\| \left(\sum_{n \in N} A_n \right)^{<\nu>} \right\|_q^{r_\nu} \right) < \infty. \tag{18}$$

Proof. (a) and (b) Since $([\ell(r), \ell_p]^{<k(\nu)>}$ is a BK space with AK, and c and c_0 are closed subspaces of ℓ_∞, the statements in (a) and (b) are immediate consequences of [11, 8.3.6] and Theorem 3.1.

(c) and (d) Let $\Sigma = (\sigma_{nk})_{n,k=1}^\infty$ denote the positive triangle with $\sigma_{nk} = 1$ for $k = 1, 2, \dots n$ and all $n \geq 1$. Then $\ell_1 = (\ell_\infty)_{[\Sigma]}$ and ℓ_∞ is normal, and applying Proposition 2.1, we obtain $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_1)$ if and only if $S^N(A) \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$ where

$$(S^N(A))_m = \sum_{n \in N_m} \sigma_{mn} A_n, \text{ that is, } (s^N(A))_{m,k} = \sum_{n \in N_m} a_{nk} \text{ for all } n \text{ and } k.$$

Now it follows by (4) and (5) that $S^N(A) \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$ if and only if

$$\begin{aligned} \|S^N(A)\|_B &= \sup_m \left(\max_{N_m \subset \{1, \dots, m\}} \left(\tilde{h}_{(s),q} \left(\sum_{n \in N_m} A_n / B \right) \right) \right) \\ &= \sup_m \left(\max_{N_m \subset \{1, \dots, m\}} \sum_{\nu=0}^\infty \left\| \left(\sum_{n \in N_m} A_n \right)^{<\nu>} \right\|_q^{s_\nu} B^{-s_\nu} \right) < \infty \text{ for some integer } B > 1, \text{ when } r_\nu > 1 \text{ for all } \nu \end{aligned}$$

and

$$\|\Sigma A\| = \sup_m \left(\max_{N_m \subset \{1, \dots, m\}} \left(\sup_\nu \left\| \left(\sum_{n \in N_m} A_n \right)^{\nu} \right\|_q^{r_\nu} \right) \right) < \infty, \text{ when } r_\nu \leq 1 \text{ for all } \nu.$$

It is easy to see that these conditions are equivalent to those in (17) and (18), respectively. □

Finally, we characterize the classes $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, bs)$ and $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, cs)$.

Example 3.3. *We observe that $bs = (\ell_\infty)_\Sigma$ and $cs = c_\Sigma$. Applying* [6, Theorem 1] *with $T = \Sigma$ we obtain $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, bs)$ and $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, cs)$ if and only if $C = \Sigma \cdot A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$ and $C \in ([\ell(r), \ell_p]^{<k(\nu)>}, c)$, where $c_{nk} = \sum_{j=1}^n a_{jk}$ for all n and k. The characterizations of the classes $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, bs)$ and $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, cs)$ are given by replacing the entries of the matrix A in (4), (5) and (15) by the entries of the matrix C.*

4. Applications, Special Cases and Visualizations

First, we apply the results of Section 3 to obtain some known results as special cases.

Example 4.1. *If $1 < m = \inf_\nu r_\nu \leq r_\nu \leq M = \sup_\nu r_\nu < \infty$ for all ν, then the conditions in (4) for $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_\infty)$, $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, c)$ and $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, c_0)$, and in (17) for $A \in ([\ell(r), \ell_p]^{<k(\nu)>}, \ell_1)$ can be replaced by*

$$\|A\|_1 = \sup_n \left(\tilde{h}_{(s),q}(A_n) \right) = \sup_n \sum_{\nu=0}^\infty \|A_n^{<\nu>}\|_q^{s_\nu} < \infty$$

and

$$\|\Sigma A\|_1 = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\tilde{h}_{(s),q} \left(\sum_{n \in N} A_n \right) \right) = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \sum_{\nu=0}^\infty \left\| \left(\sum_{n \in N} A_n \right)^{<\nu>} \right\|_q^{s_\nu} < \infty,$$

respectively.

Proof. Since

$$\underline{t} = \frac{m}{M-1} \leq s_\nu = \frac{r_\nu}{r_\nu - 1} \leq \frac{M}{m-1} = \bar{t} \text{ for all } \nu,$$

it follows that

$$\|A_n^{<\nu>}\|_q^{s_\nu} B^{-\bar{t}} \leq \|A_n^{<\nu>}\|_q^{s_\nu} B^{-s_\nu} \leq \|A_n^{<\nu>}\|_q^{s_\nu} B^{-\underline{t}} \text{ for every integer } B > 1, \text{ and for all } \nu \text{ and } n,$$

and the statement of the example is an immediate consequence. □

Example 4.2. *(a) Let $r_\nu = r \geq 1$ ($\nu = 0, 1, \ldots$) and $1 \leq p \leq \infty$. Then the spaces $[\ell(r), \ell_p]^{<k(\nu)>}$ reduce to the mixed norm spaces $\ell(r, p) = [\ell_r, \ell_p]^{<\nu>}$* ([7, Example 2.8 (a)]).
We obtain the characterizations of the classes $(\ell(1, p), \ell_\infty)$, $(\ell(1, p), c)$ and $(\ell(1, p), c_0)$ by replacing the condition in (5) by

$$\|A\| = \sup_n \left\| \left(\|A_n^{<\nu>}\|_q \right)_{\nu=0}^\infty \right\|_\infty = \sup_{n,\nu} \|A_n^{<\nu>}\|_q < \infty,$$

and of the class $(\ell(1, p), \ell_1)$ by replacing the condition in (18) by

$$\|\Sigma A\| = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left\| \left(\left\| \left(\sum_{n \in N} A_n \right)^{<\nu>} \right\|_q \right)_{\nu=0}^\infty \right\|_\infty = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\sup_\nu \left\| \left(\sum_{n \in N} A_n \right)^{<\nu>} \right\|_q \right) < \infty.$$

For $r > 1$ and $s = r/(r-1)$, we obtain the characterizations of the classes $(\ell(r, p), \ell_\infty)$, $(\ell(r, p), c)$ and $(\ell(r, p), c_0)$ by replacing the condition in (4) by

$$\|A\| = \sup_n \left\| \left(\|A_n^{<\nu>}\|_q \right)_{\nu=0}^\infty \right\|_s^s = \sup_n \sum_{\nu=0}^\infty \|A_n^{<\nu>}\|_q^s < \infty$$

and of the class $(\ell(r, p), \ell_1)$ by replacing the condition in (17) by

$$\|\Sigma A\| = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left\| \left(\left\| \left(\sum_{n \in N} A_n \right)^{<\nu>} \right\|_q \right)_{\nu=0}^\infty \right\|_s^s = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\sum_{\nu=0}^\infty \left\| \left(\sum_{n \in N} A_n \right)^{<\nu>} \right\|_q^s \right) < \infty.$$

The characterizations of the classes $(\ell(r, p), \ell_\infty)$, $(\ell(r, p), c_0)$ and $(\ell(r, p), \ell_1)$ for $1 < r < \infty$ and $1 < p \leq \infty$ were given in [4, Corollary 4.5 **(1.)**, **(2.)** and **(3.)**].
(b) Let $r_\nu > 1$, $k(\nu) = 2^\nu$ for $\nu = 0, 1, \ldots$ and $u = (u_k)_{k=1}^\infty$ be the sequence with $u_k = 2^{-\nu/s_\nu}$ ($k \in K^\nu; \nu = 0, 1, \ldots$). We consider the sets

$$ces(r) = u^{-1} * [\ell(r), \ell_1]^{<k(\nu)>} = \left\{ x \in \omega : \sum_{\nu=0}^\infty 2^{\nu(1-r_\nu)} \left(\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k| \right)^{r_\nu} < \infty \right\},$$

which reduce to the Cesàro sequence spaces ces_r for $r_k = r > 1$ ($k = 1, 2, \ldots$) [3]. *Since, for all $v \in \mathcal{U}$ and all $X, Y \subset \omega$, obviously $A \in (v^{-1} * X, Y)$ if and only if $C \in (X, Y)$, where $c_{nk} = a_{nk}/v_k$ for all n and k, the characterizations of the classes $(ces(r), Y)$ for $Y \in \{\ell_\infty, c.c_0, \ell_1\}$ can immediately be obtained from the corresponding ones for $([\ell(r), \ell_1]^{<k(\nu)>}, Y)$.*

Example 4.3. *Let $r = (r_\nu)_{\nu=0}^\infty$ be any bounded sequence of positive real numbers and $1 \leq p \leq \infty$. If $k(\nu) = \nu + 1$ for $\nu = 0, 1, \ldots$, then we have $[\ell(r), \ell_p]^{<k(\nu)>} = \ell(r)$* ([7, Example 2.9]). *The characterizations of the classes $(\ell(r), \ell_\infty)$, $(\ell(r), c)$ and $(\ell(r), c_0)$ can be obtained by replacing the conditions in (4) for $r_\nu > 1$ and in (5) for $r_\nu \leq 1$ by*

$$\|A\|_B = \sup_k \left(\tilde{h}_{(s)}(A_n/B) \right) = \sup_k \sum_{k=1}^\infty |a_{nk}|^{s_k} B^{-s_k} < \infty \textit{ for some integer } B > 1$$

and $\|A\| = \sup_{n,k} |a_{nk}|^{r_k} < \infty$, *respectively. The characterizations of these classes can be found in* ([5, Theorem 1 and Corollary]). *Finally, the characterization of the class* $(\ell(r), \ell_1)$ *can be obtained by replacing the conditions in (17) for* $r_\nu > 1$ *and in (18) for* $r_\nu \leq 1$ *by*

$$\|\Sigma A\|_B = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\tilde{h}_{(s)}\left(\sum_{n \in N}(A_n/B)\right)\right) = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \sum_{k=1}^{\infty} \left|\sum_{n \in N} a_{nk}\right|^{s_k} B^{-s_k} < \infty \text{ for some integer } B > 1$$

and

$$\|A\| = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\sup_k \left|\sum_{n \in N} a_{nk}\right|^{r_k}\right) < \infty,$$

respectively.

Finally, we visualize the projections on the first three coordinates of some neighbourhoods $\bar{B}_1(0)$ in several spaces $[\ell(r), \ell_p]^{<k(\nu)>}$.

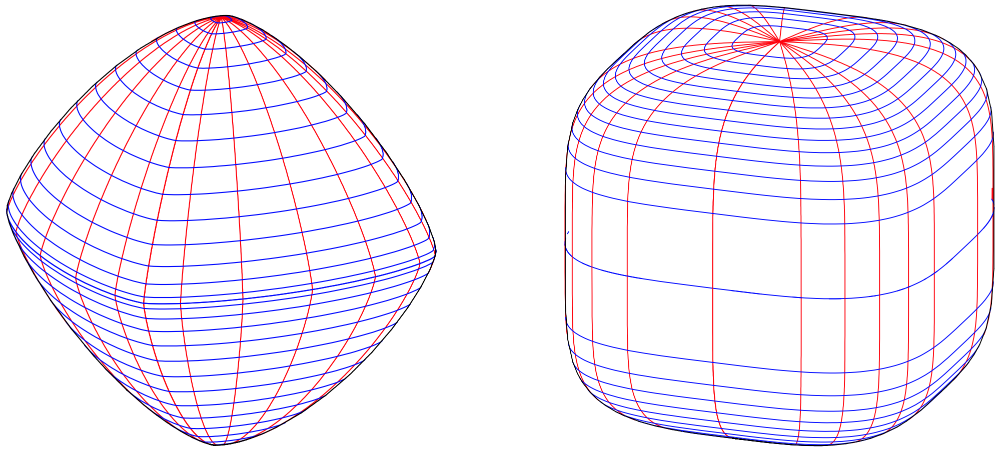

Figure 1: Left $r_0 = 3/4$, $r_1 = 4/5$, $p = 7/5$. Right (dual) $r_0 = 3$, $r_1 = 5$, $p = 7/2$.

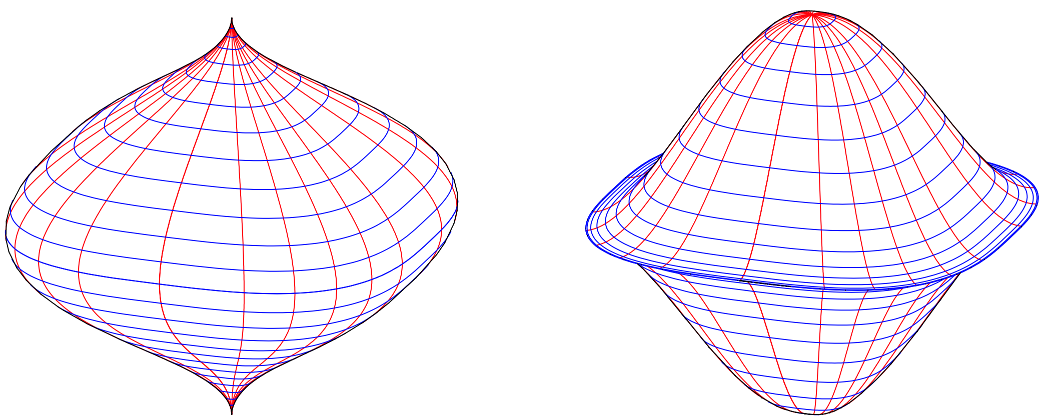

Figure 2: Left $r_0 = 1/2$, $r_1 = 2$, $p = 3$. Right $r_0 = 2$, $r_1 = 1/2$, $p = 3$.

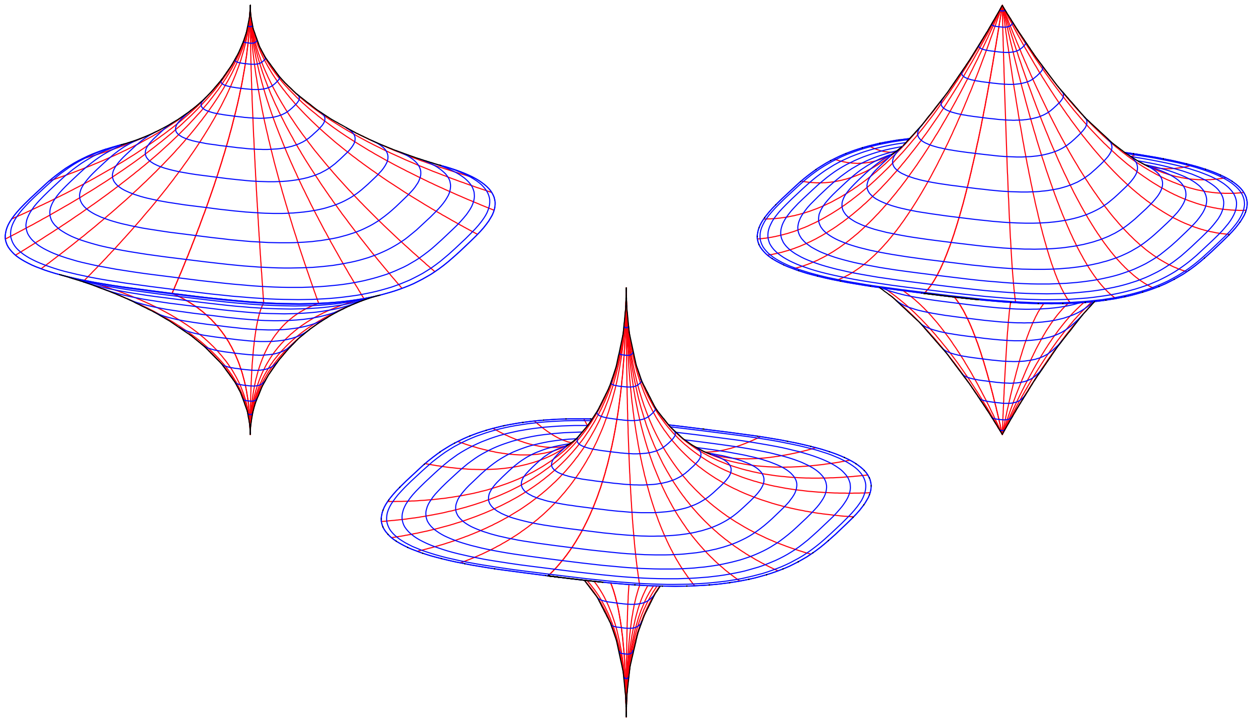

Figure 3: Left $r_0 = 1/2$, $r_1 = 1$, $p = 3$. Right $r_0 = 1$, $r_1 = 1/2$, $p = 3$. Bottom $r_0 = 1/2$, $r_1 = 1/2$, $p = 3$

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