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Fixed Points of α-Dominated Mappings on Dislocated Quasi Metric Spaces

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Abstract. We prove some common fixed point results for two α-dominated mappings satisfying some restricted contractive conditions on a closed ball of a left (right) K-sequentially complete dislocated quasi metric space. Some examples are given to show the utility of our work. The results of this paper complement, extend and enrich several recent results in the literature.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

It is well known that the Banach contraction principle ensures the existence and uniqueness of a fixed point for a contractive self-mapping T on a complete metric space (X, d), i.e., if the condition

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1}$$

is satisfied for all $x, y \in X$. Many generalizations of this principle exist which also use certain contractive-type conditions which have to be fulfilled on the whole space (see, e.g., Ljubomir Ćirić's papers [10, 11, 13]). From the viewpoint of applications, it is often enough that this condition is not fulfilled just on some subset of the underlying space, e.g., on one of its closed balls. One can obtain fixed point results for such mappings by using suitable conditions. For example, recently, Hussain et al. [18] have proved a result of this kind (see also, [5–7, 38–40]).

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On the other hand, partial metric spaces have applications in theoretical computer science (see [16]). The notion of dislocated topologies has useful applications in the context of logic programming semantics (see [16, 17]). Dislocated metric (metric-like) spaces (see [4, 20, 23, 25, 27, 33, 36, 41]) are generalizations of partial metric spaces. Furthermore, dislocated quasi metric spaces (quasi-metric-like spaces) (see [1, 9, 37, 43, 44]) generalize the idea of dislocated metric spaces and quasi-partial metric spaces (see [24, 26, 38]). In [34] Romaguera gave the idea of 0-complete partial metric space, which generalizes the completeness of a partial metric space.

Samet et al. [35] announced the notion of α-admissible mappings. They weakened and generalized the contractive condition (1.1) and several other known results. The existence of fixed points of α-admissible mappings in complete metric spaces has been studied by several researchers (see [19, 35, 42] and references therein).

In this paper, we discuss common fixed point results for two α-dominated mappings in a closed ball in complete dislocated quasi metric space, under various contractive-type conditions. The given results improve and extend several recent results proved in [5, 7, 9, 38, 39]. One can easily use this method to prove common fixed point results in quasi metric spaces. Moreover, we discuss the relation between the left (right) K-sequentially complete dislocated quasi metric spaces and left (right) K-sequentially 0-complete quasi-partial metric spaces. Examples are provided which illustrate our results and their usefulness.

2. Preliminaries

Definition 2.1. *[24] Let X be a nonempty set. A quasi-partial metric on X is a function $q : X \times X \to \mathbb{R}^+$ satisfying, for all $x, y, z \in X$,*

(i) $0 \le q(x,x) = q(x,y) = q(y,y)$ *implies* $x = y$ *(equality),*
(ii) $q(x,x) \le q(y,x)$ *(small self-distances),*
(iii) $q(x,x) \le q(x,y)$ *(small self-distances),*
(iv) $q(x,y) + q(z,z) \le q(x,z) + q(z,y)$ *(triangle inequality).*

The pair (X,q) is called a quasi-partial metric space.

Definition 2.2. *[43] Let X be a nonempty set. A function $d_q : X \times X \to [0,\infty)$ is called a dislocated quasi metric (or simply d_q-metric) if the following conditions hold for any $x, y, z \in X$:*

(i) *If* $d_q(x,y) = d_q(y,x) = 0$, *then* $x = y$,
(ii) $d_q(x,y) \le d_q(x,z) + d_q(z,y)$.

In this case, the pair (X, d_q) is called a dislocated quasi metric space.

It is clear that, if $d_q(x,y) = d_q(y,x) = 0$, then from (i) we have $x = y$. But, if $x = y$, then $d_q(x,y)$ may not be 0. It can be observed that, if $d_q(x,y) = d_q(y,x)$ for all $x, y \in X$, then (X, d_q) becomes a dislocated metric space (metric-like space) [4, 17]. We will denote by (X, d_l) a dislocated metric space. For $x \in X$ and $\varepsilon > 0$, $\overline{B_{d_q}(x,\varepsilon)} = \{y \in X : d_q(x,y) \le \varepsilon\}$ is a closed ball in (X, d_q). Every quasi-partial metric space is a dislocated quasi metric space, but the converse is not true in general.

Example 2.3. If $X = \mathbb{R}^+ \cup \{0\}$, then $d_q(x,y) = x + \max\{x,y\}$ defines a dislocated quasi metric d_q on X. But, it is not a quasi-partial metric space. Indeed,

$$d_q(2,2) = 4 > d_q(1,2) = 3.$$

Reilly et al. [32] introduced the notion of left (right) K-Cauchy sequence and left (right) K-sequentially complete spaces (see also [14]). We use this concept to establish the following definition.

Definition 2.4. *Let (X, d_q) be a dislocated quasi metric space.*

(a) *A sequence* $\{x_n\}$ *in* (X, d_q) *is called left (right) K-Cauchy if* $\forall\ \varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ *such that* $\forall\ n > m \geq n_0$, $d_q(x_m, x_n) < \varepsilon$ *(respectively* $d_q(x_n, x_m) < \varepsilon$*).*
(b) *A sequence* $\{x_n\}$ *in* (X, d_q) *dislocated quasi-converges (for short* d_q*-converges) to* x *if* $\lim_{n\to\infty} d_q(x_n, x) = \lim_{n\to\infty} d_q(x, x_n) = 0$. *In this case, the point* x *is called a* d_q*-limit of* $\{x_n\}$.
(c) (X, d_q) *is called left (right) K-sequentially complete if every left (right) K-Cauchy sequence in* $\left(X, d_q\right)$ d_q*-converges to a point* $x \in X$ *such that* $d_q(x, x) = 0$.

One can easily observe that every complete dislocated quasi metric space is also left K-sequentially complete dislocated quasi metric space, but the converse is not true in general.

Remark 2.5. *It is easy to see that, if* $x_n \in \overline{B_{d_q}(x_0, r)}$ *for all* $n \in \mathbb{N}$ *and for some* $x_0 \in X$, $r > 0$, *and the sequence* $\{x_n\}$ d_q*-converges to a point* $x^* \in X$, *then* $x^* \in \overline{B_{d_q}(x_0, r)}$.

Definition 2.6. *[38] Let* (X, q) *be a quasi-partial metric space.*

(a) *A sequence* $\{x_n\}$ *in* (X, q) *is called* 0*-Cauchy if* $\lim_{n,m\to\infty} q(x_n, x_m) = 0$ *or* $\lim_{n,m\to\infty} q(x_m, x_n) = 0$.
(b) *The space* (X, q) *is called* 0*-complete if every* 0*-Cauchy sequence in* X *converges to a point* $x \in X$ *such that* $q(x, x) = 0$.

Remark 2.7. *By definitions, one can easily observe that if* X *is a* 0*-complete quasi-partial metric space then it is also a K-sequentially complete dislocated quasi metric space. But a K-sequentially complete dislocated quasi metric space may not be a* 0*-complete quasi-partial metric space (see Example 3.12). Therefore, the results in a K-sequentially complete dislocated quasi metric space are more general than those in a* 0*-complete quasi-partial metric space.*

Recall also the following well-known notions.

Definition 2.8. *Let* X *be a non-empty set and* $T, f : X \to X$ *be two mappings. A point* $y \in X$ *is called a point of coincidence of* T *and* f *if there exists a point* $x \in X$ *such that* $y = Tx = fx$, *here* x *is called a coincidence point of* T *and* f. *The mappings* T, f *are said to be weakly compatible if they commute at their coincidence points (i.e.* $Tfx = fTx$ *whenever* $Tx = fx$*).*

Let Ψ denote the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t \geq 0$, where ψ^n is the nth iterate of ψ. The following lemma is a consequence of definition of Ψ.

Lemma 2.9. *If* $\psi \in \Psi$, *then* $\psi(t) < t$ *for all* $t > 0$.

In the next section, we state our main results.

3. Main Results

First, we introduce some more notions which will be needed in the sequel.

Definition 3.1. *Let* (X, d_q) *be a dislocated quasi metric space,* $A \subseteq X$, $T : X \to X$ *be a self-mapping and* $\alpha : X \times X \to [0, +\infty)$. *Then:*

(i) *The mapping* T *is said to be* α*-dominated on* A, *if* $\alpha(x, Tx) \geq 1$ *for all* $x \in A$.
(ii) *The function* α *is said to be a triangular function on* A, *if* $\alpha(x, y) \geq 1$ *and* $\alpha(y, z) \geq 1$ *implies that* $\alpha(x, z) \geq 1$ *for all* $x, y, z \in A$.
(iii) (X, d_q) *is* α*-regular on* A *if for any sequence* $\{x_n\}$ *in* A *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \geq 0$ *and* $x_n \to u \in A$ *as* $n \to \infty$ *we have* $\alpha(x_n, u) \geq 1$ *for all* $n \geq 0$

It is clear that if T is an α-dominated mapping on X then T is α-dominated on each subset of X, but T can be α-dominated on some $A \subseteq X$, without being α-dominated mapping on X.

Next, we prove the main result of this paper.

Theorem 3.2. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space and $T, S\colon X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0, +\infty)$ such that S and T are α-dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Suppose that $x_0 \in \overline{B_{d_q}(x_0, r)}$ and there exist nonnegative real numbers k, t such that $k + 2t \in (0, 1)$ and the following condition holds: if $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ and $x, y \in \overline{B_{d_q}(x_0, r)}$, then*

$$d_q(Sx, Ty) \leq kd_q(x, y) + t[d_q(x, Sx) + d_q(y, Ty)], \tag{2}$$
$$d_q(Tx, Sy) \leq kd_q(x, y) + t[d_q(x, Tx) + d_q(y, Sy)] \tag{3}$$

and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r, \tag{4}$$

where $\lambda = \frac{k+t}{1-t}$. Suppose that (X, d_q) is α-regular on $\overline{B_{d_q}(x_0, r)}$. Then there exists a common fixed point $x^* \in \overline{B_{d_q}(x_0, r)}$ of S and T. Moreover, $d_q(x^*, x^*) = 0$.

Proof. For the given $x_0 \in X$, define $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence $\{x_n\}$ of points in X, such that

$$x_{2i+1} = Sx_{2i} \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0, 1, 2, \ldots.$$

By mathematical induction, we shall show that

$$\begin{cases} x_{n+1} \in \overline{B_{d_q}(x_0, r)}, \quad \alpha(x_n, x_{n+1}) \geq 1 \text{ and} \\ d_q(x_n, x_{n+1}) \leq \lambda^n d_q(x_0, x_1), \text{ for all } n \in \mathbb{N}. \end{cases} \tag{P_n}$$

Using the inequality (4) and the fact that $0 < \lambda = \frac{k+t}{1-t} < 1$, we have

$$d_q(x_0, x_1) = d_q(x_0, Sx_0) \leq (1 - \lambda)r \leq r.$$

This implies that $x_1 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_0, x_1) \geq 1$. Therefore, using inequality (2) we obtain

$$\begin{aligned} d_q(x_1, x_2) &= d_q(Sx_0, Tx_1) \\ &\leq kd_q(x_0, x_1) + t[d_q(x_0, Sx_0) + d_q(x_1, Tx_1)] \\ &= kd_q(x_0, x_1) + t[d_q(x_0, x_1) + d_q(x_1, x_2)] \end{aligned}$$

which implies that

$$d_q(x_1, x_2) \leq \frac{k+t}{1-t} d_q(x_0, x_1) = \lambda d_q(x_0, x_1). \tag{5}$$

Using inequality(5) we obtain

$$\begin{aligned} d_q(x_0, x_2) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) \leq d_q(x_0, x_1) + \lambda d_q(x_0, x_1) \\ &= (1 + \lambda)d_q(x_0, x_1) \leq (1 + \lambda)(1 - \lambda)r = (1 - \lambda^2)r \leq r. \end{aligned}$$

Therefore, $x_2 \in \overline{B_{d_q}(x_0, r)}$. Again, since T is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_1, Tx_1) = \alpha(x_1, x_2) \geq 1$. Therefore, (P_1) holds. Now, using inequality (3) we obtain

$$\begin{aligned} d_q(x_2, x_3) &= d_q(Tx_1, Sx_2) \\ &\leq kd_q(x_1, x_2) + t[d_q(x_1, Tx_1) + d_q(x_2, Sx_2)] \\ &= kd_q(x_1, x_2) + t[d_q(x_1, x_2) + d_q(x_2, x_3)]. \end{aligned}$$

Using (5) in the above inequality we obtain

$$d_q(x_2, x_3) \le \frac{k+t}{1-t} d_q(x_1, x_2) = \lambda d_q(x_1, x_2) \le \lambda^2 d_q(x_0, x_1). \tag{6}$$

Again, it follows from (5) and (6) that

$$\begin{aligned} d_q(x_0, x_3) &\le d_q(x_0, x_1) + d_q(x_1, x_2) + d_q(x_2, x_3) \\ &\le d_q(x_0, x_1) + \lambda d_q(x_0, x_1) + \lambda^2 d_q(x_0, x_1) \\ &= (1 + \lambda + \lambda^2) d_q(x_0, x_1) = \frac{1-\lambda^3}{1-\lambda} d_q(x_0, x_1) \\ &\le \frac{1-\lambda^3}{1-\lambda}(1-\lambda) r = (1-\lambda^3) r \le r. \end{aligned}$$

Therefore, $x_3 \in \overline{B_{d_q}(x_0, r)}$. Again, since S is α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_2, Sx_2) = \alpha(x_2, x_3) \ge 1$. Therefore, (P_2) holds. Suppose, (P_1), (P_2), $\ldots$, (P_j) be the inductive hypothesis. We shall show that (P_{j+1}) holds. For this, we consider two possible cases. First, suppose that j is even. Then, since $\alpha(x_j, x_{j+1}) \ge 1$ and using inequality (2) we obtain

$$\begin{aligned} d_q(x_{j+1}, x_{j+2}) &= d_q(Sx_j, Tx_{j+1}) \\ &\le k d_q(x_j, x_{j+1}) + t\left[d_q(x_j, Sx_j) + d_q(x_{j+1}, Tx_{j+1})\right] \\ &= k d_q(x_j, x_{j+1}) + t\left[d_q(x_j, x_{j+1}) + d_q(x_{j+1}, x_{j+2})\right]. \end{aligned}$$

Since (P_j) holds, we obtain from the above inequality that

$$d_q(x_{j+1}, x_{j+2}) \le \frac{k+t}{1-t} d_q(x_j, x_{j+1}) = \lambda d_q(x_j, x_{j+1}) \le \lambda^{j+1} d_q(x_0, x_1).$$

Therefore, we have

$$\begin{aligned} d_q(x_0, x_{j+2}) &\le d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_{j+1}, x_{j+2}) \\ &\le (1 + \lambda + \cdots + \lambda^{j+2}) d_q(x_0, x_1) \\ &\le \frac{1-\lambda^{j+2}}{1-\lambda}(1-\lambda) r \le \left(1-\lambda^{j+2}\right) r \le r. \end{aligned}$$

This implies that $x_{j+2} \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_{j+1}, Sx_{j+1}) = \alpha(x_{j+1}, x_{j+2}) \ge 1$. Therefore, (P_{j+1}) holds.

Similarly, one can see that if j is odd, then (P_{j+1}) holds, which completes the inductive proof. Thus, we can write

$$d_q(x_n, x_{n+1}) \le \lambda^n d_q(x_0, x_1) \text{ for all } n \in \mathbb{N}. \tag{7}$$

Next, we shall show that the sequence $\{x_n\}$ is a left K-Cauchy sequence. Indeed, for $n, m \in \mathbb{N}$ with $m > n$ using (7) we have

$$\begin{aligned} d_q(x_n, x_m) &\le d_q(x_n, x_{n+1}) + d_q(x_{n+1}, x_{n+2}) + \cdots + d_q(x_{m-1}, x_m) \\ &\le \lambda^n d_q(x_0, x_1) + \lambda^{n+1} d_q(x_0, x_1) + \cdots + \lambda^{m-1} d_q(x_0, x_1). \end{aligned}$$

This implies that

$$d_q(x_n, x_m) \le \frac{\lambda^n}{1-\lambda} d_q(x_0, x_1) \text{ for all } n, m \in \mathbb{N}, m > n. \tag{8}$$

Since $0 < \lambda = \frac{k+t}{1-t} < 1$, for every $\varepsilon > 0$ we can choose $n_0 \in \mathbb{N}$ such that $\lambda^n < \dfrac{1-\lambda}{d_q(x_0, x_1)}\varepsilon$ for all $n > n_0$. Therefore, it follows from the inequality (8) that

$$d_q(x_n, x_m) < \varepsilon \quad \text{for all} \quad m > n > n_0.$$

Hence, the sequence $\{x_n\}$ is a left K-Cauchy sequence in X. By left K-sequential completeness of X, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d_q(x_n, x^*) = \lim_{n\to\infty} d_q(x^*, x_n) = 0. \tag{9}$$

We shall show that x^* is a common fixed point of the mappings S and T.

By Remark 2.5 we have $x^* \in \overline{B_{d_q}(x_0, r)}$. Now, by the assumption we have $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$, therefore for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d_q(x^*, Sx^*) &\leq d_q(x^*, x_{2n+2}) + d_q(x_{2n+2}, Sx^*) \\ &\leq d_q(x^*, x_{2n+2}) + d_q(Tx_{2n+1}, Sx^*) \\ &\leq d_q(x^*, x_{2n+2}) + kd_q(x_{2n+1}, x^*) + t[d_q(x_{2n+1}, Tx_{2n+1}) + d_q(x^*, Sx^*)] \\ &= d_q(x^*, x_{2n+2}) + kd_q(x_{2n+1}, x^*) + t[d_q(x_{2n+1}, x_{2n+2}) + d_q(x^*, Sx^*)]. \end{aligned}$$

Using the inequalities (8) and (9) in the above inequality, we obtain

$$d_q(x^*, Sx^*) \leq t\, d_q(x^*, Sx^*)$$

and since $0 < t < 1$, the above inequality implies that $d_q(x^*, Sx^*) = 0$. Similarly, one can show that $d_q(Sx^*, x^*) = 0$. Therefore, $d_q(x^*, Sx^*) = d_q(Sx^*, x^*) = 0$, i.e., $x^* = Sx^*$. Similarly, one can show that $x^* = Tx^*$.

Thus, S and T have a common fixed point x^* in $\overline{B(x_0, r)}$. As S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$ we have $\alpha(x^*, Sx^*) = \alpha(x^*, x^*) \geq 1$. Therefore,

$$\begin{aligned} d_q(x^*, x^*) &= d_q(Sx^*, Tx^*) \\ &\leq kd_q(x^*, x^*) + t[d_q(x^*, Sx^*) + d_q(x^*, Tx^*)] \\ &= (k + 2t)d_q(x^*, x^*). \end{aligned}$$

Since $k + 2t \in (0, 1)$, we must have $d_q(x^*, x^*) = 0$ and the proof is complete. □

Example 3.3. Let $X = \mathbb{Q}^+ \cup \{0\}$ and let $d_q : X^2 \times X^2 \to X$ be defined by $d_q((x_1, y_1), (x_2, y_2)) = x_1 + 2y_1 + \dfrac{x_2}{2} + y_2$. Then it is easy to show that (X^2, d_q) is a left K-sequentially complete dislocated quasi metric space. If $(x_0, y_0) = (2, 1)$, $r = 20$, then

$$\overline{B_{d_q}((2, 1), 20)} = \{(x, y) \in X : x + 2y \leq 32\}.$$

In particular, $(2, 1) \in \overline{B_{d_q}((2, 1), 20)}$.

Let $S, T : X^2 \to X^2$ be defined by

$$S(x, y) = \begin{cases} \left(\dfrac{x}{5}, \dfrac{y}{5}\right), & \text{if } x + 2y \leq 32; \\ (4x^2, 5y + 2), & \text{if } x + 2y > 32 \end{cases} \quad \text{and} \quad T(x, y) = \begin{cases} \left(\dfrac{x}{3}, \dfrac{y}{6}\right), & \text{if } x + 2y \leq 32; \\ (3x^2 + 1, y), & \text{if } x + 2y > 32. \end{cases}$$

Also, define $\alpha\colon X^2 \times X^2 \to [0, +\infty)$ by

$$\alpha((x_1, y_1), (x_2, y_2)) = \begin{cases} 1, & \text{if } \dfrac{x_1}{2} + y_1 + x_2 + y_2 \leq 32; \\ 0, & \text{if } \dfrac{x_1}{2} + 2y_1 + x_2 + y_2 > 32. \end{cases}$$

Clearly, S and T are α-dominated mappings on $\overline{B_{d_q}((2,1),20)}$. Let $k = \frac{1}{6}$, $t = \frac{1}{3}$; then $\lambda = \frac{k+t}{1-t} = \frac{3}{4} \in [0,1)$, and

$$(1-\lambda)r = (1-\frac{3}{4})20 = 5,$$

$$d_q((x_0,y_0),S(x_0,y_0)) = d_q((2,1),S(2,1)) = \frac{22}{5} < 5 = (1-\lambda)r.$$

Observe that, for $(33,0) \notin \overline{B_{d_q}((2,1),20)}$, we have

$$d_q(S(33,0),T(33,0)) = d_q((4356,2),(3268,0)) = 5994$$

and $d_q((33,0),(33,0)) = \frac{99}{2}$ and $d_q((33,0),S(33,0)) + d_q((33,0),T(33,0)) = 3880$. Therefore, there are no k,t such that $k+2t \in (0,1)$ and the inequality (2) is satisfied. So the contractive condition does not hold on X^2.

On the other hand, if $(x_1,y_1),(x_2,y_2) \in \overline{B_{d_q}((2,1),20)}$, then

$$\begin{aligned} d_q(S(x_1,y_1),T(x_2,y_2)) &= d_q\left(\left(\frac{x_1}{5},\frac{y_1}{5}\right),\left(\frac{x_2}{3},\frac{y_2}{6}\right)\right) \\ &= \frac{x_1}{5} + \frac{2y_1}{5} + \frac{x_2}{6} + \frac{y_2}{6} \\ &< \frac{1}{6}d_q((x_1,y_1),(x_2,y_2)) + \frac{1}{3}\Big[d_q((x_1,y_1),S(x_1,y_1)) + d_q((x_2,y_2),T(x_2,y_2))\Big]. \end{aligned}$$

Also,

$$\begin{aligned} d_q(T(x_1,y_1),S(x_2,y_2)) &= d_q\left(\left(\frac{x_1}{3},\frac{y_1}{6}\right),\left(\frac{x_2}{5},\frac{y_2}{5}\right)\right) \\ &= \frac{x_1}{3} + \frac{y_1}{3} + \frac{x_2}{10} + \frac{y_2}{5} \\ &< \frac{1}{6}d_q((x_1,y_1),(x_2,y_2)) + \frac{1}{3}\Big[d_q((x_1,y_1),T(x_1,y_1)) + d_q((x_2,y_2),S(x_2,y_2))\Big]. \end{aligned}$$

Therefore, all the conditions of Theorem 3.2 are satisfied. Moreover, $(0,0)$ is the common fixed point of S and T.

If we take $T = S$ in Theorem 3.2, we obtain the following result.

Corollary 3.4. *Let (X,d_q) be a left K-sequentially complete dislocated quasi metric space and $S\colon X \to X$ be a mapping. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0,+\infty)$ such that S is an α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$. Suppose that $x_0 \in \overline{B(x_0,r)}$ and there exist nonnegative real numbers k,t such that $k+2t \in (0,1)$ and the following condition holds: if $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$ and $x,y \in \overline{B_{d_q}(x_0,r)}$, then*

$$d_q(Sx,Sy) \leq kd_q(x,y) + t[d_q(x,Sx) + d_q(y,Sy)],$$

and

$$d_q(x_0,Sx_0) \leq (1-\lambda)r,$$

where $\lambda = \frac{k+t}{1-t}$. If (X,d_q) is α-regular on $\overline{B_{d_q}(x_0,r)}$, then there exists a point x^* in $\overline{B_{d_q}(x_0,r)}$ such that $x^* = Sx^*$ and $d_q(x^*,x^*) = 0$.

Corollary 3.5. *Let (X,d) be a complete dislocated metric space and $S,T\colon X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0,+\infty)$ such that S and T are α-dominated mappings on $\overline{B_d(x_0,r)}$. Suppose that $x_0 \in \overline{B_d(x_0,r)}$ and there exist nonnegative real numbers k,t such that $k+2t \in (0,1)$ and the following condition holds: if $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$ and $x,y \in \overline{B_{d_q}(x_0,r)}$, then*

$$d(Sx,Ty) \leq kd(x,y) + t[d(x,Sx) + d(y,Ty)]$$

and

$$d(x_0, Sx_0) \le (1-\lambda)r,$$

where $\lambda = \frac{k+t}{1-t}$. *If* (x,d) *is* α*-regular on* $\overline{B_d(x_0,r)}$, *then there exists a common fixed point* x^* *of* S *and* T. *Moreover,* $d(x^*,x^*) = 0$.

In the next theorem, we give a sufficient condition for the uniqueness of common fixed point.

Theorem 3.6. *Suppose that all the conditions of Theorem 3.2 are satisfied. In addition suppose that:*

(i) The function α *is a triangular function on* $\overline{B_{d_q}(x_0,r)}$.

(ii) For $x, y \in \overline{B_{d_q}(x_0,r)}$ *there exists* $z_0 \in \overline{B_{d_q}(x_0,r)}$ *such that* $\alpha(x,z_0) \ge 1$, $\alpha(y,z_0) \ge 1$.

(iii) For all $z \in \overline{B_{d_q}(x_0,r)}$ *such that* $\alpha(Sx_0,z) \ge 1$ *the following condition holds*

$$d_q(x_0,Sx_0) + d_q(z,Tz) \le d_q(x_0,z) + d_q(Sx_0,Tz).$$

Then S *and* T *have a unique common fixed point* x^* *in* $\overline{B_{d_q}(x_0,r)}$ *and* $d_q(x^*,x^*) = 0$.

Proof. Define the sequence $\{x_n\}$ as in the proof Theorem 3.2. Then, $\{x_n\}$, d_q-converges to a common fixed point $x^* \in \overline{B_{d_q}(x_0,r)}$ of the mappings S and T such that $\alpha(x_n,x^*) \ge 1$ for all $n \ge 0$, (P_n) holds and $d_q(x^*,x^*) = 0$. In order to prove uniqueness of x^*, suppose that y is another point in $\overline{B_{d_q}(x_0,r)}$ such that $y = Sy = Ty$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$, we have $\alpha(y,Sy) = \alpha(y,y) \ge 1$. Therefore,

$$\begin{aligned} d_q(y,y) &= d_q(Sy,Ty) \\ &\le kd_q(y,y) + t[d_q(y,Ty) + d_q(y,Sy)] \\ &= (k+2t)d_q(y,y). \end{aligned}$$

Since $k+2t \in (0,1)$, the above inequality implies that $d_q(y,y) = 0$.

By assumption, there exists a point $z_0 \in \overline{B_{d_q}(x_0,r)}$ such that $\alpha(x^*,z_0) \ge 1$ and $\alpha(y,z_0) \ge 1$. Define a sequence $\{z_n\}$ in X such that,

$$z_{2i+1} = Tz_{2i}, \text{ and } z_{2i+2} = Sz_{2i+1} \text{ for all } i \ge 0.$$

Using mathematical induction, we shall show that

$$\begin{cases} \alpha(z_n,z_{n+1}) \ge 1,\ \alpha(x_n,z_n) \ge 1 \text{ for all } n \in \mathbb{N}; \\ d_q(z_n,z_{n+1}) \le \lambda^n d_q(z_0,z_1) \text{ for all } n \in \mathbb{N}; \\ d_q(x_n,z_n) \le \lambda^n r,\ z_n \in \overline{B_{d_q}(x_0,r)} \text{ for all } n \in \mathbb{N}. \end{cases} \tag{P'_n}$$

Since T is an α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$ we have $\alpha(z_0,Tz_0) = \alpha(z_0,z_1) \ge 1$. Since α is a triangular function on $\overline{B_{d_q}(x_0,r)}$ and $\alpha(x_n,x^*) \ge 1$, $\alpha(x^*,z_0) \ge 1$ we have $\alpha(x_n,z_0) \ge 1$ for all $n \ge 0$. Therefore, using (iii) we obtain

$$\begin{aligned} d_q(x_1,z_1) = d_q(Sx_0,Tz_0) &\le kd_q(x_0,z_0) + t[d_q(x_0,Sx_0) + d_q(z_0,Tz_0)] \\ &\le kd_q(x_0,z_0) + t[d_q(x_0,z_0) + d_q(Sx_0,Tz_0)] \\ &\le kd_q(x_0,z_0) + t[d_q(x_0,z_0) + d_q(x_1,z_1)] \end{aligned}$$

which implies that

$$d_q(x_1, z_1) \leq \frac{k+t}{1-t} d_q(x_0, z_0) = \lambda d_q(x_0, z_0) \leq \lambda r. \tag{10}$$

Since $z_0 \in \overline{B_{d_q}(x_0, r)}$, using (10) we obtain

$$\begin{aligned} d_q(x_0, z_1) &\leq d_q(x_0, x_1) + d_q(x_1, z_1) \\ &\leq (1-\lambda)r + \lambda d_q(x_0, z_0) \\ &\leq (1-\lambda)r + \lambda r \leq r. \end{aligned}$$

This implies that $z_1 \in \overline{B_{d_q}(x_0, r)}$. Now since $\alpha(z_0, z_1) \geq 1$, by use of (3) one can show that

$$d_q(z_1, z_2) \leq \frac{k+t}{1-t} d_q(z_0, z_1) = \lambda d_q(z_0, z_1).$$

Again, since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$ we have $\alpha(z_1, Sz_1) = \alpha(z_1, z_2) \geq 1$. As, α is a triangular function on $\overline{B_{d_q}(x_0, r)}$ and $\alpha(x_1, z_0) \geq 1$, $\alpha(z_0, z_1) \geq 1$, we have $\alpha(x_1, z_1) \geq 1$. Therefore (P'_1) holds.

Since $\alpha(z_1, z_2) \geq 1$, by use of (2) we have

$$d_q(z_2, z_3) \leq \frac{k+t}{1-t} d_q(z_1, z_2) = \lambda d_q(z_1, z_2) \leq \lambda^2 d_q(z_0, z_1).$$

Again, since $\alpha(x_1, z_1) \geq 1$, we obtain by (3) that

$$\begin{aligned} d_q(x_2, z_2) = d_q(Tx_1, Sz_1) &\leq k d_q(x_1, z_1) + t[d_q(x_1, Tx_1) + d_q(z_1, Sz_1)] \\ &\leq k d_q(x_1, z_1) + t\lambda[d_q(x_0, Sx_0) + d_q(z_0, Tz_0)] \end{aligned}$$

which gives with (iii)

$$\begin{aligned} d_q(x_2, z_2) &\leq k d_q(x_1, z_1) + t\lambda[d_q(x_0, z_0) + d_q(Sx_0, Tz_0)] \\ &\leq (k + t\lambda) d(x_1, z_1) + t\lambda r. \end{aligned}$$

Using (10) and the fact that $z_0 \in \overline{B_{d_q}(x_0, r)}$ in the above inequality we obtain

$$d_q(x_2, z_2) \leq (k + t\lambda)\lambda r + t\lambda r = (k + t\lambda + t)\lambda r = \lambda^2 r.$$

Therefore,

$$\begin{aligned} d_q(x_0, z_2) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + d_q(x_2, z_2) \\ &\leq d_q(x_0, x_1) + \lambda d_q(x_0, x_1) + \lambda^2 \leq r. \end{aligned}$$

Thus, $z_2 \in \overline{B_{d_q}(x_0, r)}$. Again, since T is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$ we have $\alpha(z_2, Tz_2) = \alpha(z_2, z_3) \geq 1$. Also, since $\alpha(x_2, z_0) \geq 1$, $\alpha(z_0, z_1) \geq 1$ and $\alpha(z_1, z_2) \geq 1$, by triangular nature of α, we have $\alpha(x_2, z_2) \geq 1$. Therefore, (P'_2) holds.

Suppose, (P'_1), (P'_2), $\ldots$, (P'_j) is the inductive hypothesis. We shall show that (P'_{j+1}) holds. For this, we consider two possible cases. First, suppose that, j is even. Then, since $\alpha(z_j, z_{j+1}) \geq 1$, by (3) one can show that

$$d_q(z_{j+1}, z_{j+2}) \leq \frac{k+t}{1-t} d_q(z_j, z_{j+1}) \leq \lambda^{j+1} d_q(z_0, z_1).$$

Since $\alpha(x_j, z_j) \geq 1$ we obtain by (2) that

$$\begin{aligned} d_q(x_{j+1}, z_{j+1}) = d_q(Sx_j, Tz_j) &\leq k d_q(x_j, z_j) + t[d_q(x_j, Sx_j) + d_q(z_j, Tz_j)] \\ &\leq k d_q(x_j, z_j) + t\lambda^j[d_q(x_0, Sx_0) + d_q(z_0, Tz_0)] \end{aligned}$$

which gives with (iii) and (P'_j)

$$\begin{aligned} d_q(x_{j+1}, z_{j+1}) &\le k d_q(x_j, z_j) + t\lambda^j [d_q(x_0, z_0) + d_q(Sx_0, Tz_0)] \\ &\le k\lambda^j r + t\lambda^j [r + \lambda r] \\ &= (k + t + \lambda t)\lambda^j r = \lambda^{j+1} r. \end{aligned}$$

Therefore,

$$\begin{aligned} d_q(x_0, z_{j+1}) &\le d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_j, x_{j+1}) + d_q(x_{j+1}, z_{j+1}) \\ &\le d_q(x_0, x_1) + \lambda d_q(x_0, x_1) + \cdots + \lambda^j d_q(x_0, x_1) + \lambda^{j+1} r \\ &\le (1-\lambda) r + \lambda(1-\lambda) r + \cdots + \lambda^j (1-\lambda) r + \lambda^{j+1} r = r. \end{aligned}$$

Thus, $z_{j+1} \in \overline{B_{d_q}(x_0, r)}$. Again, since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$ we have $\alpha(z_{j+1}, Sz_{j+1}) = \alpha(z_{j+1}, z_{j+2}) \ge 1$. Also, since $\alpha(x_{j+1}, z_0) \ge 1$, $\alpha(z_n, z_{n+1}) \ge 1$, $n = 0, 1, \ldots, j+1$, by triangular nature of α, we have $\alpha(x_{j+1}, z_{j+1}) \ge 1$. Therefore, (P'_{j+1}) holds.

Similarly, one can see that if j is odd, then (P'_{j+1}) holds, which completes the inductive proof.

Now, since $\alpha(x^*, z_0) \ge 1$ and $\alpha(z_0, z_{n+1})$ for all $n \ge 0$, by by triangular nature of α, we have $\alpha(x^*, z_n) \ge 1$ for all $n \ge 0$. Therefore, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d_q(x^*, z_{2n}) &= d_q(Tx^*, Sz_{2n-1}) \\ &\le k d_q(x^*, z_{2n-1}) + t[d_q(x^*, Tx^*) + d_q(z_{2n-1}, Sz_{2n-1})] \\ &= k d_q(Sx^*, Tz_{2n-2}) + t d_q(z_{2n-1}, z_{2n}) \quad (\text{since } d_q(x^*, Tx^*) = 0) \\ &\le k^2 d_q(x^*, z_{2n-2}) + kt d_q(z_{2n-2}, z_{2n-1}) + t d_q(z_{2n-1}, z_{2n}) \\ &\;\vdots \\ &\le k^{2n} d_q(x^*, z_0) + k^{2n-1} t d_q(z_0, z_1) + \cdots + kt d_q(z_{2n-2}, z_{2n-1}) + t d_q(z_{2n-1}, z_{2n}). \end{aligned}$$

Since $\dfrac{k}{\lambda} = \dfrac{k(1-t)}{k+t} < 1$, using (P'_n) in the above inequality we obtain

$$\begin{aligned} d_q(x^*, z_{2n}) &\le k^{2n} d_q(x^*, z_0) + k^{2n-1} t d_q(z_0, z_1) + \cdots + kt\lambda^{2n-2} d_q(z_0, z_1) + t\lambda^{2n-1} d_q(z_0, z_1) \\ &= k^{2n} d_q(x^*, z_0) + t d_q(z_0, z_1)\lambda^{2n-1}\left[1 + \frac{k}{\lambda} + \cdots + \left(\frac{k}{\lambda}\right)^{2n-1}\right] \\ &\le k^{2n} d_q(x^*, z_0) + \frac{t d_q(z_0, z_1)\lambda^{2n-1}}{1 - \frac{k}{\lambda}}. \end{aligned}$$

Since $\lambda, k \in [0, 1)$, it follows from the above inequality that

$$\lim_{n\to\infty} d_q(x^*, z_{2n}) = 0. \tag{11}$$

Similarly, we can show that

$$\lim_{n\to\infty} d_q(z_{2n}, x^*) = \lim_{n\to\infty} d_q(z_{2n}, y) = \lim_{n\to\infty} d_q(y, z_{2n}) = 0. \tag{12}$$

Using (11) and (12) we obtain

$$\begin{aligned} d_q(x^*, y) &\le d_q(x^*, z_{2n}) + d_q(z_{2n}, y) \to 0 \quad \text{as} \quad n \to \infty, \\ d_q(y, x^*) &\le d_q(y, z_{2n}) + d_q(z_{2n}, x^*) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus, $d_q(x^*, y) = d_q(y, x^*) = 0$, i.e., $x^* = y$ and the uniqueness follows. □

Corollary 3.7. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space and $T, S\colon X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$, $x_0 \in \overline{B_{d_q}(x_0, r)}$ and there exist nonnegative real numbers k, t such that $k + 2t \in (0, 1)$ and the following conditions hold:*

$$d_q(Sx, Ty) \le kd_q(x, y) + t[d_q(x, Sx) + d_q(y, Ty)],$$
$$d_q(Tx, Sy) \le kd_q(x, y) + t[d_q(x, Tx) + d_q(y, Sy)],$$

for all $x, y \in \overline{B_{d_q}(x_0, r)}$ and

$$d_q(x_0, Sx_0) \le (1 - \lambda)r,$$

where $\lambda = \frac{k+t}{1-t}$. Then there exists a unique point x^* in $\overline{B_{d_q}(x_0, r)}$ such that $x^* = Sx^* = Tx^*$ and $d_q(x^*, x^*) = 0$. Moreover, S and T have no fixed point in $\overline{B_{d_q}(x_0, r)}$ other than x^*.

Proof. The proof follows by the previous results, taking $\alpha\colon X \times X \to [0, \infty)$ with $\alpha(x, y) = 1$ for all $x, y \in X$. □

In Theorem 3.2, the condition (4) is imposed in order to restrict the contractive conditions (2) and (3) to $\overline{B_{d_q}(x_0, r)}$. However, the condition (4) can be relaxed by imposing the conditions (2) and (3) to all elements $x, y \in X$ such that $\alpha(x, y) \ge 1$ or $\alpha(y, x) \ge 1$, as stated in the next theorem.

Theorem 3.8. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space. Suppose, there exist a function $\alpha : X \times X \to [0, +\infty)$ and nonnegative constants k, t such that $k + 2t \in (0, 1)$ and the following conditions hold:*

$$d_q(Sx, Ty) \le kd_q(x, y) + t[d_q(x, Sx) + d_q(y, Ty)],$$
$$d_q(Tx, Sy) \le kd_q(x, y) + t[d_q(x, Tx) + d_q(y, Sy)],$$

for all $x, y \in X$ such that $\alpha(x, y) \ge 1$ or $\alpha(y, x) \ge 1$. If (X, d_q) is α-regular, then there exists a point x^* in X such that $x^* = Sx^* = Tx^*$ and $d_q(x^*, x^*) = 0$.

The presented results can be used for obtaining (unique) common fixed point theorems for three or four mappings. We state here just a unique common fixed point result for four mappings in left K-sequentially complete dislocated quasi metric space in a closed ball. It can be proved by using the technique given in [15].

Theorem 3.9. *Let $\left(X, d_q\right)$ be a dislocated quasi metric space and $S, T, f, g : X \to X$ be four mappings satisfying $SX, TX \subset fX = gX$. Let $x_0 \in X$, $r > 0$, $fx_0 \in \overline{B_{d_q}(fx_0, r)} \subseteq fX$ and there exist nonnegative real numbers k, t such that $k + 2t \in [0, 1)$ and the following conditions hold:*

$$d_q\left(Sx, Ty\right) \le kd_q(fx, gy) + t[d_q(fx, Sx) + d_q(gy, Ty)],$$
$$d_q\left(Tx, Sy\right) \le kd_q(gx, fy) + t[d_q(fx, Tx) + d_q(gy, Sy)]$$

for all $fx, fy \in \overline{B_{d_q}(fx_0, r)}$ and

$$d_q(fx_0, Sx_0) \le (1 - \lambda)r,$$

where $\lambda = \frac{k+t}{1-t}$. If fX is left K-sequentially complete subspace of X and (S, f) and T, g are weakly compatible, then S, T, f and g have a unique common fixed point fz in $\overline{B_{d_q}(fx_0, r)}$. Also $d_q(fz, fz) = 0$.

The study of existence of fixed points in partially ordered sets has been initiated by Ran and Reurings [31]. Agarwal et al. [3], Ćirić et al. [12] and several other researchers presented new results for nonlinear contractions in partially ordered metric spaces and noted that their theorems can be used to investigate a large class of problems (see also [22] and the survey paper [21]). The authors of [2, 8, 28–30] and several other papers gave some fixed point theorems in ordered dislocated metric spaces.

In several cases, fixed point results in spaces equipped with partial order can be obtained as special cases of results using α-compatible and α-dominated mappings.

Recall that, if $(X, \preceq)$ is a pre-ordered set and $T : X \to X$ is such that $Tx \preceq x$ for all $x \in A \subseteq X$, then the mapping T is said to be dominated on A. Define the set ∇ by

$$\nabla = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$

From the previous theorems, as a sample, we derive the following result in pre-ordered left K-sequentially complete dislocated quasi metric space.

Theorem 3.10. *Let $(X, \preceq, d_q)$ be a pre-ordered left K-sequentially complete dislocated quasi metric space, $x_0 \in X$, $r > 0$ and $S, T : X \to X$ be two dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Suppose that there exist nonnegative real numbers k, t such that $k + 2t \in (0, 1)$ and the following conditions hold:*

$$d_q(Sx, Ty) \leq kd_q(x, y) + t[d_q(x, Sx) + d_q(y, Ty)],$$
$$d_q(Tx, Sy) \leq kd_q(x, y) + t[d_q(x, Tx) + d_q(y, Sy)]$$

for all $(x, y) \in (\overline{B_{d_q}(x_0, r)} \times \overline{B_{d_q}(x_0, r)}) \cap \nabla$ and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{k+t}{1-t}$. If for any sequence $\{x_n\}$ in $\overline{B_{d_q}(x_0, r)}$ such that $(x_n, x_{n+1}) \in \nabla$, $x_n \to u$ as $n \to \infty$ implies that $(u, x_n) \in \nabla$ for all $n \geq 0$, then there exists a point x^* in $\overline{B_{d_q}(x_0, r)}$ such that $x^* = Sx^* = Tx^*$ and $d_q(x^*, x^*) = 0$. In addition, suppose that:

(i) $(x, y), (y, z) \in \nabla$ implies $(x, z) \in \nabla$.

(ii) For $x, y \in \overline{B_{d_q}(x_0, r)}$ there exists $z_0 \in \overline{B_{d_q}(x_0, r)}$ such that $(x, z_0), (y, z_0) \in \nabla$.

(iii) For all $z \in \overline{B_{d_q}(x_0, r)}$ such that $(z, Sx_0) \in \nabla$ the following condition holds

$$d_q(x_0, Sx_0) + d_q(z, Tz) \leq d_q(x_0, z) + d_q(Sx_0, Tz).$$

Then, x^* is the unique common fixed point of S and T in $\overline{B_{d_q}(x_0, r)}$.

Proof. This follows from Theorem 3.6 taking $\alpha : X \times X \to [0, +\infty)$ defined as

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \nabla; \\ 0. & \text{otherwise,} \end{cases}$$

□

A corollary similar to Corollary 3.7 can be formulated.

Now we present a common fixed point result using contractive conditions that involve auxiliary function $\psi \in \Psi$ (see Preliminaries).

Theorem 3.11. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space and $S, T \colon X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha \colon X \times X \to [0, \infty)$ such that S and T are α-dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Suppose that $x_0 \in \overline{B_{d_q}(x_0, r)}$ and there exists $\psi \in \Psi$ such that the following condition hold: if $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ and $x, y \in \overline{B_{d_q}(x_0, r)}$, then*

$$\max\{d_q(Sx, Ty), d_q(Tx, Sy)\} \leq \psi(d_q(x, y)) \tag{13}$$

and

$$\sum_{i=0}^{j} \psi^i(d_q(x_0, Sx_0)) \leq r, \text{ for all } j \geq 0. \tag{14}$$

If (X, d_q) is α-regular, then there exists a common fixed point x^* of S and T and $d_q(x^*, x^*) = 0$.

If, in addition, for any two common fixed points x^*, y^* of S and T in $\overline{B_{d_q}(x_0, r)}$ we have $\alpha(x^*, y^*) \geq 1$, then S and T have a unique common fixed point in $\overline{B_{d_q}(x_0, r)}$.

Proof. For the given $x_0 \in X$, define a sequence $\{x_n\}$ of points in X such that,

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0, 1, 2, \ldots.$$

By mathematical induction, we shall show that

$$\begin{cases} x_n \in \overline{B_{d_q}(x_0, r)} & \text{for all } n \in \mathbb{N} \\ d_q(x_n, x_{n+1}) \leq \psi^n(d_q(x_0, x_1)) & \text{for all } n \in \mathbb{N}. \end{cases} \tag{P''_n}$$

By (14), we have

$$\sum_{i=0}^{j} \psi^i(d_q(x_0, Sx_0)) \leq r \text{ for all } j \in \mathbb{N} \cup \{0\}.$$

In particular, for $j = 0$, we obtain $x_1 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$ and $x_0 \in \overline{B_{d_q}(x_0, r)}$ we have $\alpha(x_0, x_1) = \alpha(x_0, Sx_0) \geq 1$. Now,

$$d_q(x_1, x_2) = d_q(Sx_0, Tx_1) \leq \max\{d_q(Sx_0, Tx_1), d_q(Tx_0, Sx_1)\}.$$

The above inequality with (13) gives

$$d_q(x_1, x_2) \leq \psi(d_q(x_0, x_1)).$$

Therefore, (P''_1) holds. Suppose that (P''_1), (P''_2), …, (P''_j) is the inductive hypothesis. We shall show that (P''_{j+1}) holds.

Suppose that j is even. Then using (14) and the induction hypothesis we obtain

$$\begin{aligned} d_q(x_0, x_{j+1}) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_j, x_{j+1}) \\ &\leq d_q(x_0, x_1) + \psi(d_q(x_0, x_1)) + \cdots + \psi^j(d_q(x_0, x_1)) \\ &= \sum_{i=0}^{j} \psi^i(d_q(x_0, x_1)) \leq r. \end{aligned}$$

Therefore, $x_{j+1} \in \overline{B_{d_q}(x_0, r)}$. Again, since T is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$ and $x_j \in \overline{B_{d_q}(x_0, r)}$ we have $\alpha(x_j, x_{j+1}) = \alpha(x_j, Tx_{j+1}) \geq 1$. Now,

$$d_q(x_{j+1}, x_{j+2}) = d_q(Sx_j, Tx_{j+1}) \leq \max\{d_q(Sx_j, Tx_{j+1}), d_q(Tx_j, Sx_{j+1})\}.$$

The above inequality with (13) and (P''_j) gives

$$d_q(x_{j+1}, x_{j+2}) \leq \psi(d_q(x_j, x_{j+1})) \leq \psi^{j+1}(d_q(x_0, x_1)).$$

Therefore, (P''_{j+1}) holds. Similarly, one can see that (P_{j+1}'') holds if j is odd, which completes the inductive proof. Now, we are going to show that the sequence $\{x_n\}$ is a left K-Cauchy sequence.

Let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \geq 0$, let $n(\varepsilon) \in \mathbb{N}$ be such that $\sum_{k \geq n(\varepsilon)} \psi^k(d_q(x_0, x_1)) < \varepsilon$. Then, for $n, m \in \mathbb{N}$ with $m > n > n(\varepsilon)$ we obtain,

$$\begin{aligned} d_q(x_n, x_m) &\leq \sum_{k=n}^{m-1} d_q(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k(d_q(x_0, x_1)) \\ &\leq \sum_{n \geq n(\varepsilon)} \psi^k(d_q(x_0, x_1)) < \varepsilon. \end{aligned}$$

Therefore, the sequence $\{x_n\}$ is a left K-Cauchy sequence in X. By the left K-sequential completeness of X, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d_q(x_n, x^*) = \lim_{n\to\infty} d_q(x^*, x_n) = 0. \tag{15}$$

We shall show that x^* is a common fixed point of the mappings S and T.

By Remark 2.5 we have $x^* \in \overline{B_{d_q}(x_0, r)}$. Now, by assumption we have $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$, and therefore for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d_q(x^*, Sx^*) &\leq d_q(x^*, x_{2n+2}) + d_q(x_{2n+2}, Sx^*) \\ &= d_q(x^*, x_{2n+2}) + d_q(Tx_{2n+1}, Sx^*) \\ &\leq d_q(x^*, x_{2n+2}) + \max\{d_q(Sx_{2n+1}, Tx^*), d_q(Tx_{2n+1}, Sx^*)\} \\ &\leq d_q(x^*, x_{2n+2}) + \psi(d_q(x_{2n+1}, x^*)) \\ &\leq d_q(x^*, x_{2n+2}) + d_q(x_{2n+1}, x^*). \end{aligned}$$

Using (15) in the above inequality we obtain $d_q(x^*, Sx^*) = 0$. Similarly, one can show that $d_q(Sx^*, x^*) = 0$. Therefore, $d_q(x^*, Sx^*) = d_q(Sx^*, x^*) = 0$, i.e., $x^* = Sx^*$. Similarly, one can show that $x^* = Tx^*$. Thus, S and T have a common fixed point x^* in $\overline{B_{d_q}(x_0, r)}$.

The final assertions of this theorem can be proved in the same way as for Theorems 3.2 and 3.6, so the details are omitted. □

Example 3.12. Let $X = \mathbb{Q}^+$ be the set of all nonnegative rational numbers and let $d_q : X \times X \to X$ be defined by:

$$d_q(x, y) = 2x + y \text{ for all } x, y \in X.$$

Then, (X, d_q) is a left K-sequentially complete dislocated quasi metric space. Let $S, T : X \to X$ be defined by

$$Sx = \begin{cases} \frac{x}{4}, & \text{if } x \in [0, 2] \cap \mathbb{Q}^+; \\ 3x, & \text{if } x \in (2, \infty) \cap \mathbb{Q}^+, \end{cases} \qquad Tx = \begin{cases} \frac{2x}{7}, & \text{if } x \in [0, 2] \cap \mathbb{Q}^+; \\ 4x, & \text{if } x \in (2, \infty) \cap \mathbb{Q}^+. \end{cases}$$

Take $x_0 = 1$ and $r = 4$. Then $\overline{B_{d_q}(x_0, r)} = [0, 2] \cap \mathbb{Q}^+$ and $x_0 \in \overline{B_{d_q}(x_0, r)}$. Define a function $\alpha\colon X \times X \to X$ by

$$\alpha(x, y) = |2x - y + 3| \text{ for all } x, y \in X.$$

Clearly, S and T are α-dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Let $\psi \in \Psi$ be given by $\psi(t) = \frac{t}{3}$. Now,

$$\begin{aligned} &d_q(x_0, Sx_0) = d_q(1, S1) = d_q\left(1, \frac{1}{4}\right) = \frac{9}{4}. \\ &\sum_{i=0}^{n} \psi^i(d_q(x_0, Sx_0)) = \frac{9}{4} \sum_{i=0}^{n} \frac{1}{3^n} < \frac{9}{4} \cdot \frac{3}{2} < 4. \end{aligned}$$

First, notice that, although $\alpha(3,3) = 6 > 1$, but the contractive condition (13) does not hold for $x = y = 3$. Indeed, if the contractive condition (13) hold for $x = y = 3$, then we would have

$$\max\{d_q(S3, T3), d_q(T3, S3)\} \leq \psi(d_q(3,3))$$

it implies that $33 \leq \psi(9)$. Since $\psi \in \Psi$, we have $33 \leq \psi(9) < 9$. This contradiction shows that the contractive condition (13) does not hold on X.

On the other hand, if $x, y \in \overline{B_{d_q}(x_0, r)}$ then we consider the following two cases:
Case 1. If $\max\{d_q(Sx, Ty), d_q(Tx, Sy)\} = d_q(Sx, Ty)$, then

$$d_q(Sx, Ty) = d_q\left(\frac{x}{4}, \frac{2y}{7}\right) = 2 \cdot \frac{x}{4} + \frac{2y}{7} \leq 2 \cdot \frac{x}{3} + \frac{y}{3} = \psi(d_q(x, y)).$$

Case 2. If $\max\{d_q(Sx, Ty), d_q(Tx, Sy)\} = d_q(Tx, Sy)$, then

$$d_q(Tx, Sy) = d_q\left(\frac{2x}{7}, \frac{y}{4}\right) = 2 \cdot \frac{2x}{7} + \frac{y}{4} \leq 2 \cdot \frac{x}{3} + \frac{y}{3} = \psi(d_q(x, y)).$$

Thus, the contractive condition (13) holds on $\overline{B_{d_q}(x_0, r)}$. Therefore, all the conditions of Theorem 3.11 are satisfied and S and T have a common fixed point (which is $x^* = 0$).

Taking $T = S$ in Theorem 3.11 we obtain a corollary, similar as Corollary 3.4 of Theorem 3.2.
Taking

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \nabla; \\ 0. & \text{otherwise,} \end{cases}$$

we obtain a corollary, similarly as Theorem 3.10 is derived from Theorem 3.2.
We omit the details.

References

[1] C. T. Aage and J. N. Salunke, The results on fixed points in dislocated metric spaces and dislocated quasi-metric space, Appl. Math. Sci., 2(59), 2941–2948 (2008).

[2] M. Abbas, H. Aydi and S. Radenović, Fixed point of T-Hardy-Rogers contractive mappings in partially ordered partial metric spaces, Int. J. Math. Math. Sci., 2012, Article ID 313675, 11 pages (2012).

[3] R.P. Agarwal, M.A. El-Gebeily and D. O'Regan, Generalized contractions in partially ordered metric spaces, Applicable Anal., 87, 109–116 (2008).

[4] A Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points, Fixed Point Theory Appl. 2012, 2012:204.

[5] M. Arshad, A. Azam, M. Abbas and A. Shoaib, Fixed point results of dominated mappings on a closed ball in ordered partial metric spaces without continuity, U.P.B. Sci. Bull., Series A, 76 (2) (2014).

[6] M. Arshad, A. Shoaib and P. Vetro, Common fixed points of a pair Of Hardy Rogers type mappings on a closed ball in ordered dislocated metric spaces, J. Funct. Spaces Appl., 2013, Article ID 63818 (2013).

[7] M. Arshad, A. Shoaib, M. Abbas and A. Azam, Fixed points of a pair of Kannan type mappings on a closed ball in ordered partial metric spaces, Misk. Math. Notes, 14, 769–784 (2013).

[8] H. Aydi, Some fixed point results in ordered partial metric spaces, J. Nonlinear Sci. Appl., 4, 210–217 (2011).

[9] I. Beg, M. Arshad and A. Shoaib, Fixed point in a closed ball in ordered dislocated quasi metric space, Fixed Point Theory, 16(2), 195–206 (2015).

[10] Lj. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., 45, 267–273 (1974).

[11] Lj. Ćirić, On some nonexpansive type mappings and fixed points, Indian J. Pure Appl. Math. 24 (3), 145–149 (1993).

[12] L. Ćirić, M. Abbas, R. Saadati and N. Hussain, Common fixed points of almost generalized contractive mappings in ordered metric spaces, Appl. Math. Comput., 217, 5784–5789 (2011).

[13] Lj.B. Ćirić, A. Razani, S. Radenović and J.S. Ume, *Common fixed point theorem for families of weakly compatible maps*, Comput. Math. Appl. 55, 2533–2543 (2008).

[14] S. Cobzaş, Functional Analysis in Asymmetric Normed Spaces, Frontiers in Mathematics, Basel: Birkhäuser, 2013.

[15] C. Di Bari and P. Vetro, Common fixed points for three or four mappings via common fixed point for two mappings, arXiv:1302.3816 [math.FA].

[16] P. Hitzler, Generalized Metrics and Topology in Logic Programming Semantics, Ph.D. Thesis, National University of Ireland (University College, Cork), 2001.
[17] P. Hitzler and A. K. Seda, Dislocated topologies, J. Electrical Engineering, 51 (12/s), 3–7 (2000).
[18] N. Hussain, M. Arshad, A. Shoaib and Fahimuddin, Common fixed point results for α-ψ-contractions on a metric space endowed with graph, J. Inequ. Appl., 2014:136 (2014).
[19] N. Hussain, E. Karapinar, P. Salimi and F, Akbar, α-admissible mappings and related fixed point theorems, J. Inequ. Appl. 2013, 2013:114.
[20] N. Hussain, J. R. Roshan, V. Parvaneh and M. Abbas, Common fixed point results for weak contractive mappings in ordered b-dislocated metric spaces with applications, J. Inequ. Appl. 2013, 2013:486.
[21] J. Jachymski, Order-theoretic aspects of metric fixed point theory, in: Kirk, William A. (ed.) et al., Handbook of metric fixed point theory. Dordrecht: Kluwer Academic Publishers. 613-541 (2001).
[22] J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaes, Nonlinear Anal. Theory Methods Appl. 74, 3, 768–774 (2011).
[23] E. Karapınar, H. Piri and H. H. Alsulami, Fixed points of modified F-contractive mappings in complete metric-like spaces, J. Funct. Spaces, 2015, Article ID 270971, 9 pages (2015).
[24] E. Karapınar, İ.M. Erhan and A. Öztürk, Fixed point theorems on quasi-partial metric spaces, Math. Comp. Model. 57(9-10), 2442–2448 (2013).
[25] P. S. Kumari, V. V. Kumar and I. R. Sarma, Common fixed point theorems on weakly compatible maps on dislocated metric spaces, Math. Sci. 2012, 6:71.
[26] H.P. Künzi, H. Pajoohesh and M.P. Schellekens, Partial quasi-metrics, Theoretical Computer Science–Spatial Representation: Discrete vs. Continuous Computational Models, 365(3), 237–246 (2006).
[27] S. K. Malhotra, S. Radenović and S. Shukla, Some fixed point results without monotone property in partially ordered metric-like spaces, Egyptian Math. Soc., 22, 83–89 (2014).
[28] H.K. Nashine, Z. Kadelburg, S. Radenović and J.K. Kim Fixed point theorems under Hardy-Rogers weak contractive conditions on 0-complete ordered partial metric spaces, Fixed Point Theory Appl. 2012:180 (2012).
[29] H.K. Nashine, Z. Kadelburg and S. Radenović, Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces, Math. Comput. Modelling 57, 2355–2365 (2013).
[30] D. Paesano and P. Vetro, Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces, Topology Appl., 159, 911–920 (2012).
[31] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to metrix equations, Proc. Amer. Math. Soc., 132(5), 1435–1443 (2004).
[32] I. L. Reilly , P. V. Semirahmanya and, M. K. Vamanamurthy, Cauchy sequences in quasi-pseudo-metric spaces, Monatsh. Math., 93, 127–140 (1982).
[33] Y. Ren, J. Li and Y. Yu, Common fixed point theorems for nonlinear contractive mappings in dislocated metric Spaces, Abstr. Appl. Anal., 2013, Article ID 483059, 5 pages (2013).
[34] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory Appl. 2010, Article ID 493298, 6 pages (2010).
[35] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal., 75, 2154–2165 (2012).
[36] I. R. Sarma and P. S. Kumari, On dislocated metric spaces, Int. J. Math. Arch., 3(1), 72–77 (2012).
[37] M. Sarwar, Mujeeb Ur Rahman and G. Ali, Some fixed point results in dislocated quasi metric (dq-metric) spaces, J. Inequ. Appl., 2014, 2014:278.
[38] A. Shoaib, M. Arshad and J. Ahmad, Fixed point results of locally cotractive mappings in ordered quasi-partial metric spaces, Sci. World J., 2013, Article ID 194897, 8 pages (2013).
[39] A. Shoaib, M. Arshad and A. Azam, Fixed points of a pair of locally contractive mappings in ordered partial metric spaces, Mat. Vesnik, 67(1), 26–38 (2015).
[40] A. Shoaib, M. Arshad and M. A. Kutbi, Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces, J. Comput. Anal. Appl., 17, 255–264 (2014).
[41] S. Shukla, S. Radenović and V. Ć. Rajić, Some common fixed point theorems in 0-σ-complete metric-like spaces, Vietnam J. Math., 41:341–352 (2013).
[42] S. Shukla, N. Shahzad, Fixed points of α-admissible Prešić type operators, Nonlinear Anal. Modelling Control, 21(3), 424–436 (2016).
[43] F. M. Zeyada, G. H. Hassan and M. A. Ahmed, A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces, Arabian J. Sci. Eng. A, 31(1), 111–114 (2006).
[44] L. Zhu, C. Zhu, C. Chen and Z. Stojanović, Multidimensional fixed points for generalized Ψ-quasi-contractions in quasi-metric-like spaces, J. Inequ. Appl., 2014, 2014:27.