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# Existence of Solutions of Cantilever Beam Problem via ($\alpha$-$\beta$-$FG$)-Contractions in $b$-Metric-Like Spaces

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**Abstract.** In this work, our intention is to introduce the notion of rational ($\alpha$-$\beta$-$FG$)-contraction mapping in $b$-metric-like spaces, and produce relevant fixed point and periodic point results for weakly $\alpha$-admissible mappings. Ulam-Hyers stability of this problem is also investigated. To illustrate our results, we give throughout the paper some examples, in particular in order to justify the use of rational terms. As an application, we obtain sufficient conditions for the existence of solutions for Cantilever Beam Problem.

*To the memory of Professor Lj. Ćirić (1935–2016)*

## 1. Introduction

The Banach Contraction Principle (BCP) is the most famous, simplest and one of the most versatile elementary result in fixed point theory in metric space structure. A huge amount of literature witnesses applications, generalizations and extensions of this principle carried out by several authors in different directions, e.g., by weakening the hypothesis, using different setups, considering different mappings and generalized form of metric spaces. In this context, the work of Ljubomir Ćirić plays one of the central roles, see, e.g., the papers [3, 4].

The study of new classes of spaces and their basic properties are always favorite topics of interest among the mathematical research community. Recently, some authors have introduced some generalizations of metric spaces in several ways and have studied fixed point problems in these classes, as well as their applications. In this context, Matthews [15] introduced the notion of a partial metric space as a part of the study of denotational semantics of data-flow networks. He showed that BCP can be generalized to the partial metric context for applications in program verifications. Note that in partial metric spaces, self-distance of an arbitrary point need not be equal to zero.

Hitzler and Seda [10], resp. Amini-Harandi [2] made a further generalization under the name of dislocated, resp. metric-like space, also having the property of "non-zero self-distance". Amini-Harandi defined

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$\sigma$-completeness of these spaces. Further, Shukla *et al*. introduced in [22] the notion of 0-$\sigma$-complete metric-like space and proved some fixed point theorems in such spaces, as improvements of Amini-Harandi's results. This concept was further extended by Alghamdi *et al.* [1] under the name of $b$-metric-like space. They established some existence and uniqueness results in $b$-metric-like spaces and in partially ordered $b$-metric-like spaces.

It is also important to study stability problems for functional equations, known as Ulam-Hyers stability (see [13, 26]). This concept has influenced a number of mathematicians studying the stability problems not only for functional equations but also for fixed point problems. In particular, there exist a number of results which extend Ulam-Hyers stability for fixed point problems in the papers by Felhi *et al*., [7], Phiangsungnoen *et al.* [19], Sintunavarat [23, 24] (see also the reference cited therein).

With the above discussion in mind, we introduce in this paper the notion of rational ($\alpha$-$\beta$-$FG$)-contractive mapping in a $b$-metric-like space and derive some fixed point and periodic point results. Further, we give some examples in order to justify the use of rational terms and counterexamples to illustrate the applicability and effectiveness of the results compared with existing results in metric and $b$-metric spaces. The considered ($\alpha$-$\beta$-$FG$)-contraction condition not only generalizes the known ones but also includes the contraction conditions considered in [9, 17, 18, 27] and many other papers.

Further, Ulam-Hyers stability of the investigated fixed point problem is discussed. In the final section, we apply the given results to obtain sufficient conditions for the existence of solutions of a fourth-order two-point boundary value problem for a nonlinear ordinary differential equation, known as Cantilever Beam Problem, as well as the Ulam-Hyers stability of this problem.

## 2. Preliminaries

First, we recall some definitions and facts which will be used throughout the paper.

**Definition 2.1.** *[1] Let $\mathcal{X}$ be a nonempty set and a real number $s \geq 1$ be given. A function $\sigma_b : \mathcal{X}^2 \to [0, +\infty)$ is $b$-metric-like if for all $u, v, z \in \mathcal{X}$, the following assertions hold:*

($\sigma_b$1) $\sigma_b(u, v) = 0$ *implies* $u = v$,
($\sigma_b$2) $\sigma_b(u, v) = \sigma_b(v, u)$,
($\sigma_b$3) $\sigma_b(u, v) \leq s[\sigma_b(u, z) + \sigma_b(z, v)]$.

*The pair $(\mathcal{X}, \sigma_b)$ is called a $b$-metric-like space, and $s$ is its coefficient.*

In a $b$-metric-like space $(\mathcal{X}, \sigma_b)$, if $u, v \in \mathcal{X}$ and $\sigma_b(u, v) = 0$, then $u = v$, but the converse may not be true and $\sigma_b(u, u)$ may be positive for $u \in \mathcal{X}$. Clearly, every $b$-metric (Czerwik [5]) and every partial $b$-metric are $b$-metric-like with the same coefficient $s$. However, the converses of these facts need not hold [22].

Every $b$-metric-like $\sigma_b$ on $\mathcal{X}$ generates a topology $\tau_{\sigma_b}$ on $\mathcal{X}$ whose base is the family of all open $\sigma_b$-balls $\{B_{\sigma_b}(u, \delta) : u \in \mathcal{X}, \delta > 0\}$, where $B_{\sigma_b}(u, \delta) = \{v \in \mathcal{X} : |\sigma_b(u, v) - \sigma_b(u, u)| < \delta\}$, for $u \in \mathcal{X}$ and $\delta > 0$.

**Proposition 2.2.** *[11] Let $(\mathcal{X}, \sigma)$ be a metric-like space and $\sigma_b(x, y) = [\sigma(x, y)]^p$, where $p > 1$ is a real number. Then $\sigma_b$ is a $b$-metric-like with coefficient $s = 2^{p-1}$.*

**Example 2.3.** Let $\mathcal{X} = [0, \infty)$ and $p > 1$ be a constant. Define a function $\sigma_b : \mathcal{X}^2 \to [0, \infty)$ by $\sigma_b(x, y) = (x + y)^p$ or $\sigma_b(x, y) = (\max\{x, y\})^p$. Then $(\mathcal{X}, \sigma_b)$ is a $b$-metric-like space with constant $s = 2^{p-1}$. Clearly, $(\mathcal{X}, \sigma_b)$ is neither a $b$-metric, nor metric-like, nor partial $b$-metric space.

**Example 2.4.** [11] Let $\mathcal{X} = [0, 1]$ and a mapping $\sigma_b : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be defined by $\sigma_b = (x + y - xy)^p$, where $p > 1$ is a real number. Then $\sigma_b$ is a $b$-metric-like on $\mathcal{X}$ with coefficient $s = 2^{p-1}$.

Now, we define the concepts of Cauchy sequence and convergent sequence in a $b$-metric-like space.

**Definition 2.5.** *[1, 11] Let $(\mathcal{X}, \sigma_b)$ be a $b$-metric-like space with coefficient $s \geq 1$, let $\{u_n\}$ be any sequence in $\mathcal{X}$ and $u \in \mathcal{X}$. Then* *(i) The sequence $\{u_n\}$ is called convergent to $u$ w.r.t. $\tau_{\sigma_b}$, if $\lim_{n\to\infty}\sigma_b(u_n,u)=\sigma_b(u,u)$;*

*(ii) The sequence $\{u_n\}$ is called a Cauchy sequence in $(\mathcal{X},\sigma_b)$ if $\lim_{n,m\to\infty}\sigma_b(u_n,u_m)$ exists (and is finite).*

*(iii) The space $(\mathcal{X},\sigma_b)$ is called complete if for every Cauchy sequence $\{u_n\}$ in $\mathcal{X}$ there exists $u\in\mathcal{X}$ such that*

$$\lim_{n,m\to\infty}\sigma_b(u_n,u_m)=\lim_{n\to\infty}\sigma_b(u_n,u)=\sigma_b(u,u). \tag{1}$$

*(iv) A function $\mathcal{J}:\mathcal{X}\to\mathcal{X}$ is said to be continuous if*

$$\lim_{n\to\infty}\sigma_b(x_n,x)=\sigma_b(x,x) \text{ implies } \lim_{n\to\infty}\sigma_b(\mathcal{J}x_n,\mathcal{J}x)=\sigma_b(\mathcal{J}x,\mathcal{J}x).$$

It is clear that the limit of a sequence is usually not unique in a $b$-metric-like space (already partial metric spaces have this property).

**Lemma 2.6.** *[11] Let $(\mathcal{X},\sigma_b)$ be a b-metric-like space with coefficient $s>1$ and assume that $\{u_n\}$ and $\{v_n\}$ are sequences in $\mathcal{X}$ such that $u_n\to u$ and $v_n\to v$. Then we have*

$$\frac{1}{s^2}\sigma_b(u,v)-\frac{1}{s}\sigma_b(u,u)-\sigma_b(v,v)\le\liminf_{n\to\infty}\sigma_b(u_n,v_n)\le\limsup_{n\to\infty}\sigma_b(u_n,v_n)\le s\sigma_b(u,u)+s^2\sigma_b(v,v)+s^2\sigma_b(u,v).$$

Contraction-type mappings have been also generalized in several directions. In a series of generalizations, starting with Samet *et al.* [21], the concept of $\alpha$-admissible mappings and $\alpha$-$\psi$-contractive mappings were introduced, thus generalizing BCP. Recently, Sintunavarat [24] introduced the notion of weakly $\alpha$-admissible mapping and discussed respective fixed point results in metric spaces.

**Definition 2.7.** *For a nonempty set $\mathcal{X}$, let $\alpha:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ and $f:\mathcal{X}\to\mathcal{X}$ be two mappings. Then $f$ is said to be:*

*(i) [21] $\alpha$-admissible if: $x,y\in\mathcal{X}$ with $\alpha(x,y)\ge1\Rightarrow\alpha(fx,fy)\ge1$.*

*(ii) [24] weakly $\alpha$-admissible if: $x\in\mathcal{X}$ with $\alpha(x,fx)\ge1\Rightarrow\alpha(fx,ffx)\ge1$.*

In what follows, we use the following terminology from the paper [25]. For a nonempty set $\mathcal{X}$ and a mapping $\alpha:\mathcal{X}\times\mathcal{X}\to[0,\infty)$, we use $\mathcal{A}(\mathcal{X},\alpha)$ and $\mathcal{WA}(\mathcal{X},\alpha)$ to denote the collection of all $\alpha$-admissible mappings on $\mathcal{X}$ and the collection of all weakly $\alpha$-admissible mappings on $\mathcal{X}$, respectively. Obviously,

$$\mathcal{A}(\mathcal{X},\alpha)\subset\mathcal{WA}(\mathcal{X},\alpha)$$

and, by [24, Example 2.1], the inclusion can be strict.

In the paper [27], Wardowski introduced a new type of contractions which he called $F$-contractions. He used the family $\mathfrak{F}$ of functions $F:\mathbb{R}^+\to\mathbb{R}$ with the following properties:

(F1) $F$ is strictly increasing;

(F2) for each sequence $\{t_n\}$ of positive numbers,

$$\lim_{n\to\infty}t_n=0 \text{ if and only if } \lim_{n\to\infty}F(t_n)=-\infty.$$

(F3) There exists $k\in(0,1)$ such that $\lim_{t\to0^+}t^kF(t)=0$.

**Definition 2.8.** *[27] Let $(\mathcal{X},d)$ be a metric space. A self-mapping $f$ on $\mathcal{X}$ is called an F-contraction if there exist $F\in\mathfrak{F}$ and $\tau\in\mathbb{R}^+$ such that*

$$\tau+F(d(fx,fy))\le F(d(x,y)),$$

*for all $x,y\in\mathcal{X}$ with $d(fx,fy)>0$.*

It was proved in [27] that each $F$-contraction in a complete metric space has a unique fixed point and so a genuine generalization of BCP was obtained.

Following this direction of research, Gopal *et al.* [9] introduced the concept of $\alpha$-type $F$-contractive mappings and gave relevance to fixed point and periodic point theorems. Hussain and Salimi [12] introduced an $\alpha$-$GF$-contraction with respect to a general family of functions $G$ and established Wardowski type fixed point results in metric and ordered metric spaces. Parvaneh *et al.* [18] used slightly modified family of functions, denoted by $\Delta_{G,\beta}$. Then they introduced an $\alpha$-$\beta$-$FG$-contraction and generalized the Wardowski fixed point results in $b$-metric and ordered $b$-metric spaces.

In Parvaneh *et al.*'s approach, the following set of functions is used:

**Definition 2.9.** *Denote by $\Delta_F$ the family of all functions $F:\mathbb{R}^+\to\mathbb{R}$ with the following properties:*

($\Delta_1$) *$F$ is continuous and strictly increasing;*

($\Delta_2$) *for each sequence $\{t_n\}\subseteq\mathbb{R}^+$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} F(t_n) = -\infty$.*

*$\Delta_{G,\beta}$ denotes the set of pairs $(G,\beta)$, where $G:\mathbb{R}^+\to\mathbb{R}$ and $\beta:[0,\infty)\to[0,1)$, such that*

($\Delta_3$) *for each sequence $\{t_n\}\subseteq\mathbb{R}^+$, $\limsup_{n\to\infty} G(t_n)\ge 0$ if and only if $\limsup_{n\to\infty} t_n\ge 1$.*

($\Delta_4$) *for each sequence $\{t_n\}\subseteq[0,\infty)$, $\limsup_{n\to\infty}\beta(t_n)=1$ implies $\lim_{n\to\infty} t_n = 0$;*

($\Delta_3$) *for each sequence $\{t_n\}\subseteq\mathbb{R}^+$, $\sum_{n=1}^{\infty} G(\beta(t_n)) = -\infty$.*

**Example 2.10.** *(i) If $F(t)=G(t)=\ln t$ and $\beta(t)=k\in(0,1)$ then $F\in\Delta_F$ and $(G,\beta)\in\Delta_{G,\beta}$. (ii) Let $F(t)=-1/\sqrt{t}$, $G(t)=\ln t$ and $\beta(t)=\frac{1}{s}e^{-t}$ for $t>0$ and $\beta(0)=0$. Then $F\in\Delta_F$ and $(G,\beta)\in\Delta_{G,\beta}$.*

## 3. Fixed Point Results for Rational ($\alpha$-$\beta$-$FG$)-Contraction Mappings

We introduce now the notion of rational ($\alpha$-$\beta$-$FG$)-contraction in a $b$-metric-like space as follows.

**Definition 3.1.** *Let $(\mathcal{X},\sigma_b)$ be a b-metric-like space with coefficient $s\ge 1$ and $\alpha\colon\mathcal{X}\times\mathcal{X}\to[0,\infty)$. A self-mapping $\mathcal{J}$ on $\mathcal{X}$ is called a rational ($\alpha$-$\beta$-FG)-contraction, if there exist $F\in\Delta_F$ and $(G,\beta)\in\Delta_{G,\beta}$ such that*

$$\begin{aligned}&u,v\in\mathcal{X} \text{ with } \alpha(u,v)\ge 1 \text{ and } \sigma_b(\mathcal{J}u,\mathcal{J}v)>0 \text{ implies}\\ &F(s\sigma_b(\mathcal{J}u,\mathcal{J}v))\le F(\Theta_{\mathcal{J}}(u,v))+G(\beta(\Theta_{\mathcal{J}}(u,v))),\end{aligned}\tag{2}$$

*where*

$$\Theta_{\mathcal{J}}(u,v)=\max\left\{\begin{array}{c}\sigma_b(u,v),\sigma_b(u,\mathcal{J}u),\sigma_b(v,\mathcal{J}v),\dfrac{\sigma_b(u,\mathcal{J}v)+\sigma_b(v,\mathcal{J}u)}{4s},\\ \dfrac{\sigma_b(u,\mathcal{J}u)\sigma_b(v,\mathcal{J}v)}{1+\sigma_b(u,v)},\dfrac{\sigma_b(u,\mathcal{J}u)\sigma_b(v,\mathcal{J}v)}{1+\sigma_b(\mathcal{J}u,\mathcal{J}v)}\end{array}\right\}.\tag{3}$$

*We denote by $\Upsilon(\mathcal{X},\alpha,FG)$ the collection of all rational ($\alpha$-$\beta$-FG)-contraction mappings on $(\mathcal{X},\sigma_b)$.*

If we take $F(t)=G(t)=\ln t$ and $\beta(t)=k\in(0,1)$, we see that every rational $\alpha$-contraction is also a rational ($\alpha$-$\beta$-$FG$)-contraction in a $b$-metric-like space. However, for other functions $F\in\Delta_F$ and $(G,\beta)\in\Delta_{G,\beta}$, new conditions can be obtained (see further Remark 3.5).

We are equipped now to state our first main result.

**Theorem 3.2.** *Let $(\mathcal{X},\sigma_b)$ be a complete b-metric-like space with coefficient $s\ge 1$ and let $\alpha:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ and $\mathcal{J}:\mathcal{X}\to\mathcal{X}$ be given mappings. Suppose that the following conditions hold:*

(FG1) *$\mathcal{J}\in\Upsilon(\mathcal{X},\alpha,FG)\cap\mathcal{WA}(\mathcal{X},\alpha)$;*

(FG2) *there exists $u_0\in\mathcal{X}$ such that $\alpha(u_0,\mathcal{J}u_0)\ge 1$;*

(*FG*3) *$\alpha$ has a transitive property, that is, for $u, v, w \in \mathcal{X}$,*

$$\alpha(u, v) \geq 1 \text{ and } \alpha(v, w) \geq 1 \Rightarrow \alpha(u, w) \geq 1;$$

(*FG*4) *$\mathcal{J}$ is $\sigma_b$-continuous.*

*Then $\mathcal{J}$ has a fixed point $u^* \in \mathcal{X}$ such that $\sigma_b(u^*, u^*) = 0$ provided $\alpha(u^*, u^*) \geq 1$.*

*Proof.* Starting from the given $u_0 \in \mathcal{X}$ satisfying $\alpha(u_0, \mathcal{J}u_0) \geq 1$, define a sequence $\{u_n\}$ in $\mathcal{X}$ by $u_{n+1} = \mathcal{J}u_n$ for $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$. If there exists $n_0 \in \mathbb{N}^*$ such that $u_{n_0} = u_{n_0+1}$, then $u_{n_0} \in Fix(\mathcal{J})$ and hence the proof is completed. Therefore, we will assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}^*$ and let $\varrho_n = \sigma_b(u_n, u_{n+1})$ for $n \in \mathbb{N}^*$. Then $\varrho_n > 0$ for all $n \in \mathbb{N}^*$. We will prove that $\lim_{n\to\infty} \varrho_n = 0$.

Using that $\mathcal{J} \in \mathcal{WA}(\mathcal{X}, \alpha)$ and $\alpha(u_0, \mathcal{J}u_0) \geq 1$, we have

$$\alpha(u_1, u_2) = \alpha(\mathcal{J}u_0, \mathcal{J}\mathcal{J}u_0) \geq 1.$$

Repeating this process, we obtain

$$\alpha(u_{n+1}, u_{n+2}) \geq 1, \quad \forall n \in \mathbb{N}^*.$$

It follows from $\mathcal{J} \in \Upsilon(\mathcal{X}, \alpha, FG)$ that

$$F(\varrho_n) = F(\sigma_b(x_{n+1}, x_n)) \leq F(s\sigma_b(\mathcal{J}x_n, \mathcal{J}x_{n-1})) \leq F(\Theta_{\mathcal{J}}(x_n, x_{n-1})) + G(\beta(\Theta_{\mathcal{J}}(x_n, x_{n-1}))), \tag{4}$$

where

$$\begin{aligned}
\Theta_{\mathcal{J}}(x_n, x_{n-1}) &= \max\left\{\begin{array}{c} \sigma_b(x_n, x_{n-1}), \sigma_b(x_n, fx_n), \sigma_b(x_{n-1}, fx_{n-1}), \frac{\sigma_b(x_n, fx_{n-1})+\sigma_b(x_{n-1}, fx_n)}{4s}, \\ \frac{\sigma_b(x_n, fx_n)\sigma_b(x_{n-1}, fx_{n-1})}{1+\sigma_b(x_n, x_{n-1})}, \frac{\sigma_b(x_n, fx_n)\sigma_b(x_{n-1}, fx_{n-1})}{1+\sigma_b(fx_n, fx_{n-1})} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} \sigma_b(x_n, x_{n-1}), \sigma_b(x_n, x_{n+1}), \sigma_b(x_{n-1}, x_n), \frac{\sigma_b(x_n, x_n)+\sigma_b(x_{n-1}, x_{n+1})}{4s}, \\ \frac{\sigma_b(x_n, x_{n+1})\sigma_b(x_{n-1}, x_n)}{1+\sigma_b(x_n, x_{n-1})}, \frac{\sigma_b(x_n, x_{n+1})\sigma_b(x_{n-1}, x_n)}{1+\sigma_b(x_{n+1}, x_n)} \end{array}\right\} \\
&\leq \max\left\{\sigma_b(x_n, x_{n-1}), \sigma_b(x_n, x_{n+1}), \frac{3s\sigma_b(x_n, x_{n+1})+s\sigma_b(x_{n-1}, x_n)}{4s}\right\} \\
&\leq \max\left\{\sigma_b(x_n, x_{n-1}), \sigma_b(x_n, x_{n+1})\right\} \\
&= \max\left\{\varrho_{n-1}, \varrho_n\right\}.
\end{aligned}$$

If $\varrho_{n-1} \leq \varrho_n$ for some $n \in \mathbb{N}$, then from (4) we have

$$F(\varrho_n) \leq F(\varrho_n) + G(\beta(\varrho_n)).$$

Therefore $G(\beta(\varrho_n)) \geq 0$, which yields that $\beta(\varrho_n) \geq 1$, which is a contradiction. Thus $\varrho_{n-1} > \varrho_n$ for all $n \in \mathbb{N}$ and so from (4) we have

$$F(\varrho_n) \leq F(\varrho_{n-1}) + G(\beta(\varrho_{n-1})).$$

Therefore we derive

$$\begin{aligned}
F(\varrho_n) &\leq F(\varrho_{n-1}) + G(\beta(\varrho_{n-1})) \\
&\leq F(\varrho_{n-2}) + G(\beta(\varrho_{n-1})) + G(\beta(\varrho_{n-2})) \\
&\ \vdots \\
&\leq F(\varrho_0) + \sum_{i=1}^{n} G(\beta(\varrho_{n-1})),
\end{aligned}$$

that is,

$$F(\varrho_n) \leq F(\varrho_0) + \sum_{i=1}^{n} G(\beta(\varrho_{n-1})) \text{ for all } n \in \mathbb{N}. \tag{5}$$

Owing to the properties of $(G, \beta) \in \Delta_{G,\beta}$ and from (5), we get $F(\varrho_n) \to -\infty$ as $n \to \infty$. Thus, from the property $(\Delta_2)$, we have

$$\lim_{n\to\infty} \varrho_n = 0. \tag{6}$$

Next, we have to show that $\{u_n\}$ is a $\sigma_b$-Cauchy sequence in $(\mathcal{X}, \sigma_b)$. Suppose the contrary; then, there exist $\epsilon > 0$ and two subsequences $\{u_{m(r)}\}$ and $\{u_{n(r)}\}$ of $\{u_n\}$ such that $m(r) > n(r) > r$ and

$$\sigma_b(u_{m(r)}, u_{n(r)}) \geq \epsilon. \tag{7}$$

We may also assume

$$\sigma_b(u_{n(r)}, u_{m(r)-1}) < \epsilon, \tag{8}$$

choosing $m(r)$ to be the smallest index exceeding $n(r)$ for which (7) holds. Then we get

$$\begin{aligned}\epsilon \leq \sigma_b(u_{m(r)}, u_{n(r)}) &\leq s\sigma_b(u_{n(r)}, u_{m(r)-1}) + s\sigma_b(u_{m(r)-1}, u_{m(r)})\\ &< s\epsilon + s\sigma_b(u_{m(r)-1}, u_{m(r)}).\end{aligned} \tag{9}$$

Passing to the upper limit in (8) as $r \to \infty$, obtain

$$\frac{\epsilon}{s} \leq \liminf_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)-1}) \leq \limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)-1}) \leq \epsilon. \tag{10}$$

Also, from (9), (10), we obtain

$$\epsilon \leq \limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)-1}) \leq s\epsilon.$$

Due to $(\sigma_b 3)$, we get

$$\begin{aligned}\sigma_b(u_{n(r)+1}, u_{m(r)}) &\leq s\sigma_b(u_{n(r)+1}, u_{n(r)}) + s\sigma_b(u_{n(r)}, u_{m(r)})\\ &\leq s\sigma_b(u_{n(r)+1}, u_{n(r)}) + s^2\sigma_b(u_{n(r)}, u_{m(r)-1}) + s^2\sigma_b(u_{m(r)-1}, u_{m(r)})\\ &\leq s\sigma_b(u_{n(r)+1}, u_{n(r)}) + s^2\epsilon + s^2\sigma_b(u_{m(r)-1}, u_{m(r)}),\end{aligned} \tag{11}$$

and passing to the upper limit in (11) as $r \to \infty$, we obtain

$$\limsup_{r\to\infty} \sigma_b(u_{n(r)+1}, u_{m(r)}) \leq s^2\epsilon.$$

Finally,

$$\begin{aligned}\sigma_b(u_{n(r)+1}, u_{m(r)-1}) &\leq s\sigma_b(u_{n(r)+1}, u_{n(r)}) + s\sigma_b(u_{n(r)}, u_{m(r)-1})\\ &\leq s\sigma_b(u_{n(r)+1}, u_{n(r)}) + s\epsilon.\end{aligned} \tag{12}$$

Also, passing to the upper limit as $r \to \infty$ in (12), we get

$$\limsup_{r\to\infty} \sigma_b(u_{n(r)+1}, u_{m(r)-1}) \leq s\epsilon.$$

Hence,

$$\frac{\epsilon}{s} \le \liminf_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)-1}) \le \limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)-1}) \le \epsilon. \tag{13}$$

Similarly,

$$\limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)}) \le s\epsilon, \tag{14}$$

$$\frac{\epsilon}{s} \le \limsup_{r\to\infty} \sigma_b(u_{n(r)+1}, u_{m(r)}), \tag{15}$$

$$\limsup_{r\to\infty} \sigma_b(u_{n(r)+1}, u_{m(r)-1}) \le s\epsilon. \tag{16}$$

On using (2) we get

$$\begin{aligned} F(s\sigma_b(u_{n(r)+1}, u_{m(r)})) &= F(s\sigma_b(\mathcal{J}u_{n(r)}, \mathcal{J}u_{m(r)-1})) \\ &\le F(\Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1})) + G(\beta(\Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1}))), \end{aligned} \tag{17}$$

where

$$\begin{aligned} &\Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1}) \\ &\quad = \max \left\{ \begin{matrix} \sigma_b(u_{n(r)}, u_{m(r)-1}), \sigma_b(u_{n(r)}, \mathcal{J}u_{n(r)}), \sigma_b(u_{m(r)-1}, \mathcal{J}u_{m(r)-1}), \frac{\sigma_b(u_{n(r)}, \mathcal{J}u_{m(r)-1}) + \sigma_b(u_{m(r)-1}, \mathcal{J}u_{n(r)})}{4s}, \\ \frac{\sigma_b(u_{n(r)}, \mathcal{J}u_{n(r)})\sigma_b(u_{m(r)-1}, \mathcal{J}u_{m(r)-1})}{1+\sigma(u_{n(r)}, u_{m(r)-1})}, \frac{\sigma_b(u_{n(r)}, \mathcal{J}u_{n(r)})\sigma_b(u_{m(r)-1}, \mathcal{J}u_{m(r)-1})}{1+\sigma(\mathcal{J}u_{n(r)}, \mathcal{J}u_{m(r)-1})} \end{matrix} \right\} \\ &\quad = \max \left\{ \begin{matrix} \sigma_b(u_{n(r)}, u_{m(r)-1}), \sigma_b(u_{n(r)}, u_{n(r)+1}), \sigma_b(u_{m(r)-1}, u_{m(r)}), \frac{\sigma_b(u_{n(r)}, u_{m(r)}) + \sigma_b(u_{m(r)-1}, u_{n(r)+1})}{4s}, \\ \frac{\sigma_b(u_{n(r)}, u_{n(r)+1})\sigma_b(u_{m(r)-1}, u_{m(r)})}{1+\sigma(u_{n(r)}, u_{m(r)-1})}, \frac{\sigma_b(u_{n(r)}, u_{n(r)+1})\sigma_b(u_{m(r)-1}, u_{m(r)})}{1+\sigma(u_{n(r)+1}, u_{m(r)})} \end{matrix} \right\}. \end{aligned} \tag{18}$$

Passing to the upper limit in (18) as $r \to \infty$, and using (6), (13), (14) and (16) we obtain

$$\begin{aligned} &\limsup_{r\to\infty} \Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1}) \\ &\quad = \max \left\{ \begin{matrix} \limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)-1}), \limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{n(r)+1}), \limsup_{r\to\infty} \sigma_b(u_{m(r)-1}, u_{m(r)}), \\ \dfrac{\limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)}) + \limsup_{r\to\infty} \sigma_b(u_{m(r)-1}, u_{n(r)+1})}{4s}, \\ \dfrac{\limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{n(r)+1}) \limsup_{r\to\infty} \sigma_b(u_{m(r)-1}, u_{m(r)})}{1 + \limsup_{r\to\infty} \sigma(u_{n(r)}, u_{m(r)-1})}, \\ \dfrac{\limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{n(r)+1}) \limsup_{r\to\infty} \sigma_b(u_{m(r)-1}, u_{m(r)})}{1 + \limsup_{r\to\infty} \sigma(u_{n(r)+1}, u_{m(r)})} \end{matrix} \right\} \\ &\quad = \max \left\{ \begin{matrix} \limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)-1}), 0, 0, \\ \dfrac{\limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)}) + \limsup_{r\to\infty} \sigma_b(u_{m(r)-1}, u_{n(r)+1})}{4s}, 0, 0 \end{matrix} \right\} \\ &\quad \le \max\left\{\epsilon, \frac{\epsilon}{2}\right\} = \epsilon. \end{aligned} \tag{19}$$

Next, passing to the upper limit in (17) as $r \to \infty$, and using (15), (19) we get

$$\begin{aligned} F(s\frac{\epsilon}{s}) &\le F(\limsup_{r\to\infty} s\sigma_b(u_{n(r)+1}, u_{m(r)})) \\ &\le F(\limsup_{r\to\infty} \Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1})) + \limsup_{r\to\infty} G(\beta(\Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1}))) \\ &\le F(\epsilon) + \limsup_{r\to\infty} G(\beta(\Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1}))), \end{aligned}$$

implying that

$$\limsup_{r\to\infty} G(\beta(\Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1}))) \ge 0,$$

which implies that $\limsup_{r\to\infty} \beta(\Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1})) \ge 1$ and since $\beta(t) < 1$ for all $t \ge 0$, we have

$$\limsup_{r\to\infty} \beta(\Theta_{\mathcal{J}}(u_{n(r)}, u_{m(r)-1})) = 1.$$

Therefore by using (17) we obtain

$$\limsup_{r\to\infty} \sigma_b(u_{n(r)}, u_{m(r)-1}) = 0,$$

which is a contradiction with (13). Thus, $\{u_n\}$ is a $\sigma_b$-Cauchy sequence in the $b$-metric-like space $(\mathcal{X}, \sigma_b)$.

Since $(\mathcal{X}, \sigma_b)$ is $\sigma_b$-complete, there exists $u^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \sigma_b(u_n, u^*) = \lim_{n,m\to\infty} \sigma_b(u_n, u_m) = \sigma_b(u^*, u^*) = 0.$$

By $(\sigma_b 3)$, we obtain

$$\sigma_b(u^*, \mathcal{J}u^*) \le s\sigma_b(u^*, \mathcal{J}u_n) + s\sigma_b(\mathcal{J}u_n, \mathcal{J}u^*). \tag{20}$$

So using the continuity of $\mathcal{J}$ and passing to the limit in (20) as $n \to \infty$, we get

$$\sigma_b(u^*, \mathcal{J}u^*) \le s\lim_{n\to\infty} \sigma_b(u^*, u_{n+1}) + s\lim_{n\to\infty} \sigma_b(\mathcal{J}u_n, \mathcal{J}u^*) = s\sigma_b(\mathcal{J}u^*, \mathcal{J}u^*).$$

Since $\alpha(u^*, u^*) \ge 1$ and using (2) we get

$$F(\sigma_b(u^*, \mathcal{J}u^*)) \le F(s\sigma_b(\mathcal{J}u^*, \mathcal{J}u^*)) \le F(\Theta_{\mathcal{J}}(u^*, u^*)) + G(\beta(\Theta_{\mathcal{J}}(u^*, u^*))),$$

where

$$\begin{aligned} \Theta_{\mathcal{J}}(u^*, u^*) &= \max\left\{ \begin{array}{l} \sigma_b(u^*, u^*), \sigma_b(u^*, \mathcal{J}u^*), \sigma_b(u^*, \mathcal{J}u^*), \frac{\sigma_b(u^*,\mathcal{J}u^*)+\sigma_b(u^*,\mathcal{J}u^*)}{4s}, \\ \frac{\sigma_b(u^*,\mathcal{J}u^*)\sigma_b(u^*,\mathcal{J}u^*)}{1+\sigma_b(u^*,u^*)}, \frac{\sigma_b(u^*,\mathcal{J}u^*)\sigma_b(u^*,\mathcal{J}u^*)}{1+\sigma_b(\mathcal{J}u^*,\mathcal{J}u^*)} \end{array} \right\} \\ &< \sigma_b(z, \mathcal{J}u^*). \end{aligned}$$

Therefore

$$F(\sigma_b(\mathcal{J}u^*, \mathcal{J}u^*)) \le F(\sigma_b(u^*, \mathcal{J}u^*)) + G(\beta(\sigma_b(u^*, \mathcal{J}u^*))),$$

which implies that $G(\beta(\sigma_b(u^*, \mathcal{J}u^*))) \ge 0$ and so $\beta(\sigma_b(u^*, \mathcal{J}u^*)) \ge 1$, which is not true, so $\sigma_b(u^*, \mathcal{J}u^*) = 0$ and $\mathcal{J}u^* = u^*$. This proves that $u^*$ is a fixed point of $\mathcal{J}$ such that $\sigma_b(u^*, u^*) = 0$. □

We note that the previous result can still be valid for $\mathcal{J}$ not necessarily $\sigma_b$-continuous. We have the following result.

**Theorem 3.3.** *Let* $(\mathcal{X}, \sigma_b)$ *be a complete b-metric-like space with coefficient* $s \geq 1$ *and let* $\alpha : \mathcal{X}^2 \to [0, \infty)$, *and* $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ *be given mappings. Suppose that the assumptions* (FG1)–(FG3) *of Theorem 3.2 hold, as well as:*

(FG4′) $\mathcal{X}$ *is* $\alpha$*-regular, i.e., if* $\{u_n\}$ *is a sequence in* $\mathcal{X}$ *with* $\alpha(u_n, u_{n+1}) \geq 1$ *for* $n \in \mathbb{N}$ *and* $u_n \to u^*$ *as* $n \to \infty$, *then* $\alpha(u_n, u^*) \geq 1$ *for* $n \in \mathbb{N}$.

*Then* $\mathcal{J}$ *has a fixed point* $u^* \in \mathcal{X}$ *such that* $\sigma_b(u^*, u^*) = 0$.

*Proof.* Following the lines of proof of Theorem 3.2, the sequence $\{u_n\}$ defined by $u_{n+1} = \mathcal{J}u_n, \forall\ n \in \mathbb{N}$ is a $\sigma_b$-Cauchy sequence in the $\sigma_b$-complete $b$-metric-like space $(\mathcal{X}, \sigma_b)$. From the completeness of $(\mathcal{X}, \sigma_b)$, it follows that there exists $u^* \in \mathcal{X}$ such that that

$$\lim_{n\to\infty} \sigma_b(u_n, u^*) = \lim_{n,m\to\infty} \sigma_b(u_n, u_m) = \sigma_b(u^*, u^*) = 0.$$

Now, by using (2) and $\alpha(u_n, u^*) \geq 1, \forall\ \ n \in \mathbb{N}$, we have

$$F(s\sigma_b(u_{n+1}, \mathcal{J}u^*)) \leq F(\Theta_{\mathcal{J}}(u_n, u^*)) + G(\beta(\Theta_{\mathcal{J}}(u_n, u^*))), \tag{21}$$

where

$$\begin{aligned}\Theta_{\mathcal{J}}(u_n, u^*) &= \max\left\{\begin{array}{c} \sigma_b(u_n, u^*), \sigma_b(u_n, \mathcal{J}u_n), \sigma_b(u^*, \mathcal{J}u^*), \frac{\sigma_b(u_n, \mathcal{J}u^*) + \sigma_b(\mathcal{J}u_n, u^*)}{4s}, \\ \frac{\sigma_b(u_n, \mathcal{J}u_n)\sigma_b(u^*, \mathcal{J}u^*)}{1+\sigma_b(u_n, u^*)}, \frac{\sigma_b(u_n, \mathcal{J}u_n)\sigma_b(u^*, \mathcal{J}u^*)}{1+\sigma_b(\mathcal{J}u_n, \mathcal{J}u^*)} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} \sigma_b(u_n, u^*), \sigma_b(u_n, u_{n+1}), \sigma_b(u^*, \mathcal{J}u^*), \frac{\sigma_b(u_n, \mathcal{J}u^*) + \sigma_b(u_{n+1}, u^*)}{4s}, \\ \frac{\sigma_b(u_n, u_{n+1})\sigma_b(u^*, u^*)}{1+\sigma_b(u_n, u^*)}, \frac{\sigma_b(u_n, u_{n+1})\sigma_b(u^*, u^*)}{1+\sigma_b(u_{n+1}, u^*)} \end{array}\right\}.\end{aligned} \tag{22}$$

Passing to the limit as $n \to \infty$ in (22) and using Lemma 2.6, we get

$$\begin{aligned}\frac{\sigma_b(u^*, \mathcal{J}u^*)}{4s^2} &= \min\left\{\sigma_b(u^*, \mathcal{J}u^*), \frac{\frac{\sigma_b(u^*, \mathcal{J}u^*)}{s}}{4s}\right\} \\ &\leq \liminf_{n\to\infty} \Theta_{\mathcal{J}}(u_n, u^*) \leq \limsup_{n\to\infty} \Theta_{\mathcal{J}}(u_n, u^*) \\ &\leq \max\left\{\sigma_b(u^*, \mathcal{J}z), \frac{s\sigma_b(u^*, \mathcal{J}u^*)}{4s}\right\} = \sigma_b(u^*, \mathcal{J}u^*).\end{aligned}$$

Again, by using (21) and taking the upper limit as $n \to \infty$ and using Lemma 2.6, we get

$$\begin{aligned}F(\sigma_b(u^*, \mathcal{J}u^*)) &= F(s\frac{1}{s}\sigma_b(u_{n+1}, \mathcal{J}u^*)) \\ &\leq F(s \limsup_{n\to\infty} \sigma_b(u_{n+1}, \mathcal{J}u^*)) \\ &\leq F(\limsup_{n\to\infty} \Theta_{\mathcal{J}}(u_n, u^*)) + \limsup_{n\to\infty} G(\beta(\Theta_{\mathcal{J}}(u_n, u^*))) \\ &\leq F(\sigma_b(u^*, \mathcal{J}u^*)) + \limsup_{n\to\infty} G(\beta(\Theta_{\mathcal{J}}(u_n, u^*))).\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} G(\beta(\Theta_{\mathcal{J}}(u_n, u^*))) \geq 0.$$

Hence $\limsup_{n\to\infty} \beta(\Theta_{\mathcal{J}}(u_n, z)) \geq 1$. By the property $(\Delta_4)$, we have $\limsup_{n\to\infty} \Theta_{\mathcal{J}}(u_n, z) = \sigma_b(u^*, \mathcal{J}u^*)) = 0$, a contradiction. Therefore $u^* = \mathcal{J}u^*$. Hence $u^*$ is a fixed point of $\mathcal{J}$. $\square$

To ensure the uniqueness of the fixed point, we will consider the following hypothesis.

$(H0)$ : for all $x, y \in Fix(\mathcal{J})$, $\alpha(x, y) \geq 1$.

**Theorem 3.4.** *Adding condition* (H0) *to the hypotheses of Theorem 3.2 (respectively, Theorem 3.3) then* $\mathcal{J}u^* = u^*$, $\mathcal{J}v^* = v^*$ *and* $\sigma_b(u^*, u^*) = \sigma_b(v^*, v^*) = 0$ *imply that* $u^* = v^*$.

*Proof.* Suppose that $\mathcal{J}u^* = u^*$, $\mathcal{J}v^* = v^*$, $\sigma_b(u^*, u^*) = \sigma_b(v^*, v^*) = 0$, and, to the contrary, $u^* \neq v^*$, hence $\sigma_b(u^*, v^*) > 0$. By the assumption, we can replace $u$ by $u^*$ and $v$ by $v^*$ in the condition (2), and we get

$$F(\sigma_b(u^*, v^*)) = F(\sigma_b(\mathcal{J}u^*, \mathcal{J}v^*)) \leq F(\Theta_{\mathcal{J}}(u^*, v^*)) + G(\beta(\Theta_{\mathcal{J}}(u^*, v^*))),$$

where

$$\begin{aligned}\Theta_{\mathcal{J}}(u^*, v^*) &= \max\left\{\begin{array}{c}\sigma_b(u^*, v^*), \sigma_b(u^*, \mathcal{J}u^*), \sigma_b(v^*, \mathcal{J}v^*), \frac{\sigma_b(u^*, \mathcal{J}v^*) + \sigma_b(v^*, \mathcal{J}u^*)}{4s}, \\ \frac{\sigma_b(u^*, \mathcal{J}u^*)\sigma_b(v^*, \mathcal{J}v^*)}{1+\sigma_b(u^*, v^*)}, \frac{\sigma_b(u^*, \mathcal{J}u^*)\sigma_b(v^*, \mathcal{J}v^*)}{1+\sigma_b(\mathcal{J}u^*, \mathcal{J}v^*)}\end{array}\right\} \\ &\leq \max\left\{\sigma_b(u^*, v^*), \frac{\sigma_b(u^*, v^*)}{2s}\right\} \\ &= \sigma_b(u^*, v^*).\end{aligned}$$

Therefore, we have

$$F(\sigma_b(u^*, v^*)) \leq F(\sigma_b(u^*, v^*)) + G(\beta(\sigma_b(u^*, v^*)))$$

which implies $G(\beta(\sigma_b(u^*, v^*))) \geq 0$, that is, $\beta(\sigma_b(u^*, v^*)) \geq 1$, a contradiction and eventually $u^* = v^*$. □

**Remark 3.5.** *Considering a range of concrete functions* $F \in \Delta_F$ *and* $(G, \beta) \in \Delta_{G,\beta}$ *in the condition* (2) *of Theorems 3.2–3.4, we can get various classes of rational* ($\alpha$-$\beta$-*FG*)*-contractive conditions in a b-metric-like space. We state just a few examples (recall that* $\Theta_{\mathcal{J}}(u, v)$ *is defined in* (3)*).*

(I) *Taking* $G(t) = \ln t$ $(t > 0)$, $\beta(t) = \lambda \in (0, 1)$ *and* $\tau = -\ln\lambda > 0$, *we have Wardowski-type [27] condition*

$$\begin{aligned}&u, v \in \mathcal{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma_b(\mathcal{J}u, \mathcal{J}v) > 0 \text{ implies} \\ &\tau + F(s\sigma_b(\mathcal{J}u, \mathcal{J}v)) \leq F(\Theta_{\mathcal{J}}(u, v)).\end{aligned}$$

(II) *Taking* $F(t) = G(t) = \ln t$ $(t > 0)$, $\beta(t) = \lambda \in (0, 1)$, *we have Banach-type contraction condition*

$$\begin{aligned}&u, v \in \mathcal{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma_b(\mathcal{J}u, \mathcal{J}v) > 0 \text{ implies} \\ &s\sigma_b(\mathcal{J}u, \mathcal{J}v) \leq \lambda\Theta_{\mathcal{J}}(u, v).\end{aligned}$$

(III) *Taking* $F(t) = G(t) = \ln t$ $(t > 0)$, *we have Geraghty-type [6, 8] condition*

$$\begin{aligned}&u, v \in \mathcal{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma_b(\mathcal{J}u, \mathcal{J}v) > 0 \text{ implies} \\ &s\sigma_b(\mathcal{J}u, \mathcal{J}v) \leq \beta(\Theta_{\mathcal{J}}(u, v))\Theta_{\mathcal{J}}(u, v).\end{aligned} \tag{23}$$

(IV) *Taking* $F(t) = -\frac{1}{\sqrt{t}}$, $G(t) = \ln t$ $(t > 0)$, *the condition is*

$$\begin{aligned}&u, v \in \mathcal{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma_b(\mathcal{J}u, \mathcal{J}v) > 0 \text{ implies} \\ &s\sigma_b(\mathcal{J}u, \mathcal{J}v) \leq \frac{\Theta_{\mathcal{J}}(u, v)}{[1 - \sqrt{\Theta_{\mathcal{J}}(u, v)}\ln(\beta(\Theta_{\mathcal{J}}(u, v)))]^2}.\end{aligned}$$

(V) *Taking* $F(t) = -\frac{1}{\sqrt{t}}$, $G(t) = \ln t$ $(t > 0)$ *and* $\beta(t) = \lambda \in (0, 1)$, $\tau = -\ln\lambda > 0$, *we have the condition*

$$\begin{aligned}&u, v \in \mathcal{X} \text{ with } \alpha(u, v) \geq 1 \text{ and } \sigma_b(\mathcal{J}u, \mathcal{J}v) > 0 \text{ implies} \\ &s\sigma_b(\mathcal{J}u, \mathcal{J}v) \leq \frac{\Theta_{\mathcal{J}}(u, v)}{\left(1 + \tau\sqrt{\Theta_{\mathcal{J}}(u, v)}\right)^2}.\end{aligned}$$

## 4. Illustration of Results

The first example demonstrates a possible usage of Theorem 3.4 with the involvement of rational terms in contractive condition (2).

**Example 4.1.** Consider $\mathcal{X} = \{0, 1, 2\}$ and let $\sigma_b : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be defined by

$$\sigma_b(0,0) = 0 \qquad \sigma_b(1,1) = \frac{1}{2} \qquad \sigma_b(2,2) = \frac{15}{4},$$
$$\sigma_b(0,1) = \sigma_b(1,0) = \frac{3}{4}, \quad \sigma_b(0,2) = \sigma_b(2,0) = \frac{3}{2}, \quad \sigma_b(1,2) = \sigma_b(2,1) = 3.$$

It is clear that $(\mathcal{X}, \sigma_b)$ is a complete $b$-metric like space with constant $s = \frac{4}{3}$ ($(\mathcal{X}, \sigma_b)$ is neither a $b$-metric, nor a metric-like space). Define mappings $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\mathcal{J} : \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \qquad \alpha(u, v) = 1 \text{ for all } u, v \in \mathcal{X}.$$

Moreover, take $F(t) = G(t) = \ln t$ and $\beta(t) = \frac{5}{6}$ for $t > 0$ (Remark 3.5, Case (II)). Then it is easy to see that all the conditions of Theorem 3.4 are fulfilled—just the condition $\mathcal{J} \in \Upsilon(\mathcal{X}, \alpha, FG)$ needs to be checked. In this case it reduces to

$$\frac{4}{3}\sigma_b(\mathcal{J}u, \mathcal{J}v) \le \frac{5}{6}\Theta_{\mathcal{J}}(u, v), \tag{24}$$

for all $u, v \in \mathcal{X}$ with $\sigma_b(\mathcal{J}u, \mathcal{J}v) > 0$, where $\Theta_{\mathcal{J}}(u, v)$ is defined by (3).

Now $\sigma_b(\mathcal{J}0, \mathcal{J}0) = \sigma_b(\mathcal{J}0, \mathcal{J}2) = \sigma_b(\mathcal{J}2, \mathcal{J}2) = \sigma_b(\mathcal{J}2, \mathcal{J}0) = 0$, so only the following three cases have to be considered:

**Case I:** For $u = 0$ and $v = 1$ (similarly, for $u = 1$ and $v = 0$),

$$\sigma_b(\mathcal{J}u, \mathcal{J}v) = \sigma_b(\mathcal{J}0, \mathcal{J}1) = \sigma_b(0, 2) = \frac{3}{2}$$

and

$$\begin{aligned}\Theta_{\mathcal{J}}(0,1) &= \max\left\{ \sigma_b(0,1), \sigma_b(0,\mathcal{J}0), \sigma_b(1,\mathcal{J}1), \tfrac{\sigma_b(0,\mathcal{J}1)+\sigma_b(1,\mathcal{J}0)}{4\cdot\frac{4}{3}}, \tfrac{\sigma_b(0,\mathcal{J}0)\sigma_b(1,\mathcal{J}1)}{1+\sigma_b(0,1)}, \tfrac{\sigma_b(0,\mathcal{J}0)\sigma_b(1,\mathcal{J}1)}{1+\sigma_b(\mathcal{J}0,\mathcal{J}1)} \right\} \\ &= \max\left\{\frac{3}{4}, 0, 3, \frac{27}{64}, 0, 0\right\} = 3.\end{aligned}$$

Hence, (24) reduces to $\frac{4}{3} \cdot \frac{3}{2} \le \frac{5}{6} \cdot 3$.

**Case II:** For $u = v = 1$,

$$\sigma_b(\mathcal{J}u, \mathcal{J}v) = \sigma_b(\mathcal{J}1, \mathcal{J}1) = \sigma_b(2, 2) = \frac{15}{4}$$

and

$$\begin{aligned}\Theta_{\mathcal{J}}(1,1) &= \max\left\{ \sigma_b(1,1), \sigma_b(1,\mathcal{J}1), \sigma_b(1,\mathcal{J}1), \tfrac{\sigma_b(1,\mathcal{J}1)+\sigma_b(1,\mathcal{J}1)}{4\cdot\frac{4}{3}}, \tfrac{\sigma_b(1,\mathcal{J}1)\sigma_b(1,\mathcal{J}1)}{1+\sigma_b(1,1)}, \tfrac{\sigma_b(1,\mathcal{J}1)\sigma_b(1,\mathcal{J}1)}{1+\sigma_b(\mathcal{J}1,\mathcal{J}1)} \right\} \\ &= \max\left\{\frac{1}{2}, 3, 3, \frac{9}{4}, 6, \frac{24}{19}\right\} = 6.\end{aligned}$$

Hence, (24) reduces to $\frac{4}{3} \cdot \frac{15}{4} \le \frac{5}{6} \cdot 6$.

Note that this result could not be obtained without rational terms in contractive condition (whatever functions $F, G$ and $\beta$ are chosen).

**Case III:** For $u = 2$ and $v = 1$ (or $u = 1$ and $v = 2$) the result follows similarly as in Case I.

This implies that (24) holds in all the cases, thus $\mathcal{J} \in \Upsilon(\mathcal{X}, \alpha, FG)$. Therefore, all the conditions of Theorem 3.4 are satisfied. Thus we can conclude that $\mathcal{J}$ has a unique fixed point (which is $u^* = 0$).

Note also that the conclusion in this example could not be obtained using the standard metric $d(u, v) = |u - v|$ nor the usual $b$-metric $d_b(u, v) = (u - v)^2$. For example, if the standard metric were used, for $u = 0$, $v = 1$, we would get $d(\mathcal{J}u, \mathcal{J}v) = 2 > 1 = \Theta_{\mathcal{J}}(u, v)$ and no $\lambda \in (0, 1)$ could be chosen in order that (24) holds.

The next example demonstrates the use of functions $\alpha$ and $\beta$, and again the importance of rational terms.

**Example 4.2.** Consider the set $\mathcal{X} = \{a, b, c, d, e\}$ and choose real numbers $p, q > 0$ such that $p + 50q < \log \frac{10}{9}$, i.e., $e^{-(p+50q)} > \frac{9}{10}$. Define a mapping $\sigma_b : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ by

$$\begin{aligned}
&\sigma_b(d, d) = 0,\\
&\sigma_b(b, d) = \sigma_b(b, a) = \sigma_b(c, c) = \sigma_b(c, d) = \sigma_b(d, a) = \sigma_b(e, e) = q,\\
&\sigma_b(c, e) = \sigma_b(c, a) = \sigma(e, a) = 4q,\\
&\sigma_b(b, c) = 9q,\\
&\sigma_b(b, b) = \sigma_b(b, e) = \sigma_b(d, e) = \sigma_b(a, a) = 10q,
\end{aligned}$$

and $\sigma_b(u, v) = \sigma_b(v, u)$ for all $u, v \in \mathcal{X}$. It is clear that $(\mathcal{X}, \sigma_b)$ is a complete $b$-metric-like space with parameter $s = \frac{9}{2}$. Define mappings $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ by

$$\mathcal{J} : \begin{pmatrix} a & b & c & d & e \\ c & d & a & d & b \end{pmatrix}, \qquad \alpha(u, v) = \begin{cases} 1, & u, v \in \{b, d, e\} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that just the condition $\mathcal{J} \in \Upsilon(\mathcal{X}, \alpha, FG)$ needs to be checked—we will show that it holds in the Geraghty-form (23) with $\beta$ defined as $\beta(t) = e^{-(p+t)}$ for $t > 0$ and $\beta(0) < 1$.

The only two cases when $\alpha(u, v) \geq 1$ and $\sigma_b(\mathcal{J}u, \mathcal{J}v) > 0$ are the following:

**Case I**: $u = b$, $v = e$ (or $u = e$, $v = b$). We have

$$s\sigma_b(\mathcal{J}u, \mathcal{J}v) = s\sigma_b(d, b) = \frac{9}{2}\, q,$$

and

$$\begin{aligned}
\Theta_{\mathcal{J}}(u, v) &= \max\left\{ \sigma_b(b, e), \sigma_b(b, d), \sigma_b(e, b), \tfrac{\sigma_b(b,b)+\sigma_b(d,e)}{4\cdot\frac{9}{2}}, \tfrac{\sigma_b(b,d)\sigma_b(e,b)}{1+\sigma_b(b,e)}, \tfrac{\sigma_b(b,d)\sigma_b(e,b)}{1+\sigma_b(d,b)} \right\}\\
&= \max\left\{10q, q, 10q, \frac{20}{18}q, \frac{10q^2}{1 + 10q}, \frac{10q^2}{1 + v}\right\} = 10q.
\end{aligned}$$

The inequality (23) reduces to

$$\frac{9}{2}\, q \leq e^{-(p+10q)} \cdot 10q$$

and holds true by the way $p, q$ have been chosen.

**Case II**: $u = v = e$. We have

$$s\sigma_b(\mathcal{J}u, \mathcal{J}v) = s\sigma_b(b, b) = 45\, q,$$

and

$$\begin{aligned}
\Theta_{\mathcal{J}}(u, v) &= \max\left\{ \sigma_b(e, e), \sigma_b(e, b), \sigma_b(e, b), \tfrac{\sigma_b(e,b)+\sigma_b(e,b)}{4\cdot\frac{9}{2}}, \tfrac{\sigma_b(e,b)\sigma_b(e,b)}{1+\sigma_b(e,e)}, \tfrac{\sigma_b(e,b)\sigma_b(e,b)}{1+\sigma_b(b,b)} \right\}\\
&= \max\left\{q, 10q, 10q, \frac{20}{18}q, \frac{100q^2}{1 + q}, \frac{100q^2}{1 + 10q}\right\} > 50\, q \quad \text{(since } q < 1).
\end{aligned}$$

The inequality (23) reduces to

$$45\,q \le e^{-(p+50q)} \cdot 50\,q$$

and again holds true by the way $p, q$ have been chosen.

Thus, all the conditions of Theorem 3.4 are fulffilled and the mapping $\mathcal{J}$ has a unique fixed point (which is $d$).

Note that the condition would not be satisfied without the rational terms in $\Theta_{\mathcal{J}}$ (in the last case we would get $\Theta_{\mathcal{J}}(u, v) = 10q$ and no appropriate function $\beta$ could be found). Also, the condition is not satisfied on the whole space (that is, the use of function $\alpha$ is crucial).

## 5. Periodic Point Results

It is an obvious fact that, if $\mathcal{J}$ is a self-map which has a fixed point $u$, then $u$ is also a fixed point of $\mathcal{J}^n$ for arbitrary $n \in \mathbb{N}$. However, the converse is obviously false, i.e., a self-map can have a "periodic" point (a point $u$ satisfying $\mathcal{J}^n u = u$ for some $n \in \mathbb{N}$) which is not its fixed point. In this section, we prove a periodic point result for self-mappings on a complete $b$-metric-like space, thus modifying a result from [17].

**Definition 5.1.** *[14] A mapping $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ is said to have the property (P) if it has no periodic points, i.e., if $Fix(\mathcal{J}^n) = Fix(\mathcal{J})$ for every $n \in \mathbb{N}$, where $Fix(\mathcal{J}) := \{u \in \mathcal{X} : \mathcal{J}u = u\}$.*

**Theorem 5.2.** *Let $(\mathcal{X}, \sigma_b)$ be a complete b-metric-like space with coefficient $s \ge 1$ and let $\alpha : \mathcal{X}^2 \to [0, \infty)$, and $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be given mappings satisfying the following conditions:*

$(\widehat{FG1})$ *there exist $F \in \Delta_F$ and $(G, \beta) \in \Delta_{G,\beta}$ such that*

$$\begin{aligned} &u \in \mathcal{X} \text{ with } \alpha(u, \mathcal{J}u) \ge 1 \text{ and } \sigma_b(\mathcal{J}u, \mathcal{J}^2 u) > 0 \text{ implies} \\ &F(\sigma_b(\mathcal{J}u, \mathcal{J}^2 u)) \le F(\Theta_{\mathcal{J}}(u, \mathcal{J}u)) + G(\beta(\Theta_{\mathcal{J}}(u, \mathcal{J}u))) \end{aligned} \tag{25}$$

*where*

$$\Theta_{\mathcal{J}}(u, \mathcal{J}u) = \max \left\{ \begin{array}{c} \sigma_b(u, \mathcal{J}u), \sigma_b(\mathcal{J}u, \mathcal{J}^2 u), \dfrac{\sigma_b(u, \mathcal{J}^2 u) + \sigma_b(\mathcal{J}u, \mathcal{J}u)}{4s}, \\ \dfrac{\sigma_b(u, \mathcal{J}u)\sigma_b(\mathcal{J}u, \mathcal{J}^2 u)}{1 + \sigma_b(u, \mathcal{J}u)}, \dfrac{\sigma_b(u, \mathcal{J}u)\sigma_b(\mathcal{J}u, \mathcal{J}^2 u)}{1 + \sigma_b(\mathcal{J}u, \mathcal{J}^2 u)} \end{array} \right\}$$

$(\widehat{FG2})$ *there exists $u_0 \in \mathcal{X}$ such that $\alpha(u_0, \mathcal{J}u_0) \ge 1$;*

$(\widehat{FG3})$ *$\mathcal{J} \in \mathcal{WA}(\mathcal{X}, \alpha)$;*

$(\widehat{FG4})$ *if $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $\alpha(u_n, u_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $u_n \to u$ as $n \to \infty$, then $\mathcal{J}u_n \to \mathcal{J}u$ as $n \to \infty$;*

$(\widehat{FG5})$ *if $w \in Fix(\mathcal{J}^n)$ and $w \notin Fix(\mathcal{J})$, then $\alpha(\mathcal{J}^{n-1}w, \mathcal{J}^n w) \ge 1$.*

*Then $\mathcal{J}$ has the property (P).*

*Proof.* Starting from the given $u_0 \in \mathcal{X}$ satisfying $\alpha(u_0, \mathcal{J}u_0) \ge 1$, define the sequence $\{u_n\}$ by the rule $u_n = \mathcal{J}^n u_0 = \mathcal{J}u_{n-1}$, $n \in \mathbb{N}$. Using $(\widehat{FG3})$, we get by induction

$$\alpha(u_n, u_{n+1}) \ge 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1} = \mathcal{J}u_{n_0}$, then $u_{n_0}$ is a fixed point of $\mathcal{J}$. Hence, we assume $u_n \ne u_{n+1}$, i.e., $\sigma_b(\mathcal{J}u_{n-1}, \mathcal{J}^2 u_{n-1}) > 0$ for all $n \in \mathbb{N}$.

From $(\widehat{FG1})$, we have

$$\begin{aligned} F(\sigma_b(u_n, u_{n+1})) &= F(\sigma_b(\mathcal{J}u_{n-1}, \mathcal{J}^2 u_{n-1})) \\ &\le F(\Theta_{\mathcal{J}}(u_{n-1}, \mathcal{J}u_{n-1})) + G(\beta(\Theta_{\mathcal{J}}(u_{n-1}, \mathcal{J}u_{n-1}))), \end{aligned} \tag{26}$$

where

$$\begin{aligned}\Theta_{\mathcal{J}}(u_{n-1},\mathcal{J}u_{n-1}) &= \max\left\{\begin{array}{c}\sigma(u_{n-1},\mathcal{J}u_{n-1}),\sigma(\mathcal{J}u_{n-1},\mathcal{J}^2u_{n-1}),\frac{\sigma(u_{n-1},\mathcal{J}^2u_{n-1})+\sigma(\mathcal{J}u_{n-1},\mathcal{J}u_{n-1})}{4s},\\ \frac{\sigma(u_{n-1},\mathcal{J}u_{n-1})\sigma(\mathcal{J}u_{n-1},\mathcal{J}^2u_{n-1})}{1+\sigma(u_{n-1},\mathcal{J}u_{n-1})},\frac{\sigma(u_{n-1},\mathcal{J}u_{n-1})\sigma(\mathcal{J}u_{n-1},\mathcal{J}^2u_{n-1})}{1+\sigma(\mathcal{J}u_{n-1},\mathcal{J}^2u_{n-1})}\end{array}\right\}\\ &\le \max\left\{\sigma_b(u_{n-1},\mathcal{J}u_{n-1}),\sigma_b(\mathcal{J}u_{n-1},\mathcal{J}^2u_{n-1}),\frac{3s\sigma_b(u_{n-1},\mathcal{J}u_{n-1})+s\sigma_b(\mathcal{J}u_{n-1},\mathcal{J}^2u_{n-1})}{4s}\right\}\\ &= \max\left\{\sigma_b(u_{n-1},\mathcal{J}u_{n-1}),\sigma_b(\mathcal{J}u_{n-1},\mathcal{J}^2u_{n-1})\right\}.\end{aligned}$$

If there exists $n \in \mathbb{N}$ such that

$$\max\{\sigma_b(u_{n-1},\mathcal{J}u_{n-1}),\sigma_b(\mathcal{J}u_{n-1},\mathcal{J}^2u_{n-1})\} = \sigma_b(\mathcal{J}u_{n-1},\mathcal{J}^2u_{n-1}),$$

then (26) becomes

$$F(\sigma_b(u_n,u_{n+1})) \le F(\sigma_b(u_n,u_{n+1})) + G(\beta(\sigma_b(u_n,u_{n+1}))),$$

which implies that $G(\beta(\sigma_b(u_n,u_{n+1}))) \ge 0$ and so $\beta(\sigma_b(u_n,u_{n+1})) \ge 1$, a contradiction. Thus, we conclude that

$$F(\sigma_b(u_n,u_{n+1})) \le F(\sigma_b(u_{n-1},u_n)) + G(\beta(\sigma_b(u_{n-1},u_n))).$$

By using a similar interpretation as in the proof of Theorem 3.2, we get that the sequence $\{u_n\}$ is a $\sigma_b$-Cauchy sequence and hence the $\sigma_b$-completeness of $(\mathcal{X},\sigma_b)$ ensures that there exists $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

From $(\widehat{FG4})$, we get $u_{n+1} = \mathcal{J}u_n \to \mathcal{J}u^*$ as $n \to \infty$, that is $u^* = \mathcal{J}u^*$. Thus, $\mathcal{J}$ has a fixed point and $Fix(\mathcal{J}^n) = Fix(\mathcal{J})$ is true for $n = 1$. Let $n > 1$ and assume, contrary to what has to be proved, that $w \in Fix(\mathcal{J}^n)$ and $w \notin Fix(\mathcal{J})$; then $\sigma_b(w,\mathcal{J}w) > 0$. Now applying $(\widehat{FG5})$ and $(\widehat{FG1})$, we get

$$\begin{aligned}F(\sigma_b(w,\mathcal{J}w)) &= F(\sigma_b(\mathcal{J}(\mathcal{J}^{n-1}w),\mathcal{J}^2(\mathcal{J}^{n-1}w))\\ &\le F(\Theta_{\mathcal{J}}(\mathcal{J}^{n-1}w,\mathcal{J}^{n-1}w)) + G(\beta(\Theta_{\mathcal{J}}(\mathcal{J}^{n-1}w,\mathcal{J}^{n-1}w)))\end{aligned}$$

where

$$\begin{aligned}&\Theta_{\mathcal{J}}(\mathcal{J}^{n-1}w,\mathcal{J}^{n-1}w)\\ &= \max\left\{\begin{array}{c}\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}^nw),\sigma_b(\mathcal{J}(\mathcal{J}^{n-1}w),\mathcal{J}^2(\mathcal{J}^{n-1}w)),\frac{\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}^2(\mathcal{J}^{n-1}w))+\sigma_b(\mathcal{J}(\mathcal{J}^{n-1}w),\mathcal{J}(\mathcal{J}^{n-1}w))}{4s},\\ \frac{\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}(\mathcal{J}^{n-1}w))\sigma_b(\mathcal{J}(\mathcal{J}^{n-1}w),\mathcal{J}^2(\mathcal{J}^{n-1}w))}{1+\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}(\mathcal{J}^{n-1}w))},\frac{\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}(\mathcal{J}^{n-1}w))\sigma_b(\mathcal{J}(\mathcal{J}^{n-1}w),\mathcal{J}^2(\mathcal{J}^{n-1}w))}{1+\sigma_b(\mathcal{J}(\mathcal{J}^{n-1}w),\mathcal{J}^2(\mathcal{J}^{n-1}w))}\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\sigma_b(\mathcal{J}^{n-1}w,w),\sigma_b(w,\mathcal{J}w),\frac{\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}w)+\sigma_b(w,w)}{4s},\\ \frac{\sigma_b(\mathcal{J}^{n-1}w,w)\sigma_b(w,\mathcal{J}w)}{1+\sigma_b(\mathcal{J}^{n-1}w,w)},\frac{\sigma_b(\mathcal{J}^{n-1}w,w)\sigma_b(w,\mathcal{J}w)}{1+\sigma_b(w,\mathcal{J}w)}\end{array}\right\}\\ &\le \max\left\{\sigma_b(\mathcal{J}^{n-1}w,w),\sigma_b(w,\mathcal{J}w),\frac{s\sigma_b(\mathcal{J}^{n-1}w,w)+3s\sigma_b(w,\mathcal{J}w)}{4}\right\}\\ &\le \max\{\sigma_b(\mathcal{J}^{n-1}w,w),\sigma_b(w,\mathcal{J}w)\}\\ &= \sigma_b(\mathcal{J}^{n-1}w,w).\end{aligned}$$

Consequently, we write

$$\begin{aligned}&F(\sigma_b(w,\mathcal{J}w))\\ &\le F(\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}^nw)) + G(\beta(\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}^nw)))\\ &\le F(\sigma_b(\mathcal{J}^{n-2}w,\mathcal{J}^{n-1}w)) + G(\beta(\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}^nw))) + G(\beta(\sigma_b(\mathcal{J}^{n-2}w,\mathcal{J}^{n-1}w)))\\ &\le \cdots\\ &\le F(\sigma_b(w,\mathcal{J}w)) + G(\beta(\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}^nw))) + G(\beta(\sigma_b(\mathcal{J}^{n-2}w,\mathcal{J}^{n-1}w))) + \cdots + G(\beta(\sigma_b(w,\mathcal{J}w))),\end{aligned}$$

that is,

$$F(\sigma_b(w,\mathcal{J}w)) \leq F(\sigma_b(w,\mathcal{J}w)) + \sum_{i=1}^{n} G(\beta(\sigma_b(\mathcal{J}^{n-1}w,\mathcal{J}^{n}w))).$$

By passing to the limit as $n \to \infty$ in the above inequality and using properties of $(G,\beta) \in \Delta_{G,\beta}$, we have $F(\sigma_b(w,\mathcal{J}w)) = -\infty$, which is a contradiction and hence we deduce that $\sigma_b(w,\mathcal{J}w) = 0$. Therefore, $Fix(\mathcal{J}^n) = Fix(\mathcal{J})$ for all $n \in \mathbb{N}$. $\square$

## 6. Ulam-Hyers Stability in *b*-Metric-Like Space

Extending the known definitions in *b*-metric space (see, e.g., [7, 19]), we introduce the notion of generalized Ulam-Hyers stability of fixed point problems in *b*-metric-like spaces.

**Definition 6.1.** *Let $(\mathcal{X},\sigma_b)$ be a b-metric-like space with coefficient $s \geq 1$, and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a given mapping. The fixed point equation*

$$u = \mathcal{J}u, \ u \in \mathcal{X} \tag{27}$$

*is said to be generalized Ulam-Hyers stable in the framework of b-metric-like spaces if there exists an increasing function $\psi : [0,\infty) \to [0,\infty)$, continuous at 0, with $\psi(0) = 0$, such that for each $\epsilon > 0$ and an $\epsilon$-solution $v \in \mathcal{X}$, that is,*

$$\sigma_b(v,\mathcal{J}v) \leq \epsilon,$$

*there exists a solution $w \in \mathcal{X}$ of the fixed point equation* (27) *such that*

$$\sigma_b(v,w) \leq \psi(s\epsilon). \tag{28}$$

*If $\psi(t) = ct$ for all $t \in [0,\infty)$, where $c > 0$, then* (27) *is said to be Ulam-Hyers stable in the framework of b-metric-like spaces.*

**Remark 6.2.** *If $s = 1$, then Definition 6.1 reduces to the definition of generalized Ulam-Hyers stability in metric-like spaces. Also, if $\psi(t) = ct$ for all $t \in [0,\infty)$, where $c > 0$, then it reduces to the definition of Ulam-Hyers stability in metric-like spaces. In addition, if the distance function is defined by $\sigma_b(u,v) = |u - v|$, then it is converted to the classical Ulam-Hyers stability.*

**Theorem 6.3.** *Let $(\mathcal{X},\sigma_b)$ be a complete b-metric-like space with coefficient $s \geq 1$. Suppose that all the hypotheses of Theorem 3.4 hold (considering contraction condition of the Geraghty-form* (23)*) and also that the function $\varphi : [0,\infty) \to [0,\infty)$ defined by $\varphi(t) := t[1 - \beta(t)]$ is strictly increasing and onto. If $\alpha(u,v) \geq 1$ for all $\epsilon$-solutions $u,v \in \mathcal{X}$ of the fixed point equation* (27)*, then this equation is generalized Ulam-Hyers stable.*

*Proof.* Following Theorem 3.4, we have $\mathcal{J}u^* = u^*$, that is, $u^* \in \mathcal{X}$ is a solution of the fixed point equation (27) with $\sigma_b(u^*,u^*) = 0$. Let $\epsilon > 0$ and $v^* \in \mathcal{X}$ be an $\epsilon$-solution of (27), that is,

$$\sigma_b(v^*,\mathcal{J}v^*) \leq \epsilon.$$

Since $\sigma_b(u^*,\mathcal{J}u^*) = \sigma_b(u^*,u^*) = 0 \leq \epsilon$, $u^*$ and $v^*$ are $\epsilon$-solutions. By hypothesis, we get $\alpha(u^*,v^*) \geq 1$ and so

$$\begin{aligned} \sigma_b(u^*,v^*) = \sigma_b(\mathcal{J}u^*,v^*) &\leq s[\sigma_b(\mathcal{J}u^*,\mathcal{J}v^*) + \sigma_b(\mathcal{J}v^*,v^*)] \\ &\leq \beta(\Theta_{\mathcal{J}}(u^*,v^*))\Theta_{\mathcal{J}}(u^*,v^*) + s\epsilon, \end{aligned} \tag{29}$$

where

$$\begin{aligned}
\Theta_{\mathcal{J}}(u^*,v^*) &= \max\left\{\begin{array}{c} \sigma_b(u^*,v^*),\sigma_b(u^*,\mathcal{J}u^*),\sigma_b(v^*,\mathcal{J}v^*),\frac{\sigma_b(u^*,\mathcal{J}v^*)+\sigma_b(v^*,\mathcal{J}u^*)}{4s}, \\ \frac{\sigma_b(u^*,\mathcal{J}u^*)\sigma_b(v^*,\mathcal{J}v^*)}{1+\sigma_b(u^*,v^*)},\frac{\sigma_b(u^*,\mathcal{J}u^*)\sigma_b(v^*,\mathcal{J}v^*)}{1+\sigma_b(\mathcal{J}u^*,\mathcal{J}v^*)} \end{array}\right\} \\
&\le \max\left\{ \sigma_b(u^*,v^*),0,\epsilon,\frac{2s\sigma_b(u^*,v^*)+s\sigma_b(u^*,\mathcal{J}u^*)+s\sigma_b(v^*,\mathcal{J}v^*)}{4s},0,0 \right\} \\
&= \max\left\{ \sigma_b(u^*,v^*),0,\epsilon,\frac{2s\sigma_b(u^*,v^*)+s\epsilon}{4s},0,0 \right\} \\
&= \max\left\{ \sigma_b(u^*,v^*),\epsilon \right\}.
\end{aligned}$$

Consider the two possible cases.

1° If $\Theta_{\mathcal{J}}(u^*,v^*) = \sigma_b(u^*,v^*)$, then we get

$$\sigma_b(u^*,v^*) \le \beta(\sigma_b(u^*,v^*))\sigma_b(u^*,v^*) + s\epsilon,$$

which implies that

$$\sigma_b(u^*,v^*)[1-\beta(\sigma_b(u^*,v^*))] \le s\epsilon.$$

Since $\varphi(t) = t[1-\beta(t)]$, we have $\varphi(\sigma_b(u^*,v^*)) \le s\epsilon$ which implies that

$$\sigma_b(u^*,v^*) \le \psi(s\epsilon),$$

where $\psi := \varphi^{-1} : [0,\infty) \to [0,\infty)$ exists, is increasing, continuous at 0 and $\varphi^{-1}(0) = 0$. Observe also that, since $0 \le \beta(t) < 1$, it is $0 \le \varphi(t) \le t$, and so $\psi(t) \ge t$ for $t \in [0,\infty)$.

2° If $\Theta_{\mathcal{J}}(u^*,v^*) = \epsilon$, then (29) gives that

$$\sigma_b(u^*,v^*) \le \epsilon \le s\epsilon \le \psi(s\epsilon).$$

Thus, the inequality (28) holds in all cases and, therefore, the fixed point equation (27) is generalized Ulam-Hyers stable. □

## 7. An Application to Cantilever Beam Problem

Consider the following fourth-order two-point boundary value problem which is an example of beam problem when uniform load is distributed, that is, the so-called Cantilever Beam Problem:

$$\begin{cases} u''''(t) = K(t,u(t)), \ \ 0 < t < 1; \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{30}$$

with $I = [0,1]$ and $K \in C(I \times \mathbb{R}, \mathbb{R})$. This problem is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,r)K(r,u(r))\,dr, \ \text{ for } \ t \in I, \tag{31}$$

where $G : I \times I \to [0,\infty)$ is the Green function given by

$$G(t,r) = \frac{1}{6}\begin{cases} r^2(3t-r), & 0 \le r \le t \le 1 \\ t^2(3r-t), & 0 \le t \le r \le 1. \end{cases}$$

Consider the set $\mathcal{X} = C(I,\mathbb{R}) := \{x : I \to \mathbb{R} \mid x \text{ is continuous on } I\}$ and define a $b$-metric-like $\sigma_b : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ by

$$\sigma_b(u,v) = \max_{t\in I} \max\{|u(t)|,|v(t)|\}^p \ \text{ for } \ u,v \in \mathcal{X},$$

where $p > 1$. Then $(\mathcal{X},\sigma_b,s = 2^{p-1})$ is a $\sigma_b$-complete $b$-metric like space.

**Theorem 7.1.** *Let $\mathcal{J} : \mathcal{X} \times \mathcal{X}$ be the operator defined by*

$$\mathcal{J}u(t) = \int_0^1 G(t,r)K(r,u(r))\,dr \ \text{ for } \ t \in I.$$

*Also, let $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function. Suppose the following assertions hold:*

*(C1) $K : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, non-decreasing in the second variable;*

*(C2) there exists $u_0 \in \mathcal{X}$ such that $\xi(u_0(t), \mathcal{J}u_0(t)) \geq 0$ for all $t \in I$;*

*(C3) for all $u, v, w \in \mathcal{X}$, $\xi(u(t), v(t)) \geq 0$ and $\xi(v(t), w(t)) \geq 0$ for all $t \in I$ implies that $\xi(u(t), w(t)) \geq 0$ for all $t \in I$;*

*(C4) if $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $u_n \to u$ in $\mathcal{X}$ and $\xi(u_n(t), u_{n+1}(t)) \geq 0$ for all $n \in \mathbb{N}$ and $t \in I$, then $\xi(u_n(t), u(t)) \geq 0$ for all $n \in \mathbb{N}$ and $t \in I$;*

*(C5) $u \in \mathcal{X}$ and $\xi(u(t), \mathcal{J}u(t)) \geq 0$ for all $t \in I$ implies that $\xi(\mathcal{J}u(t), \mathcal{J}\mathcal{J}u(t)) \geq 0$ for all $t \in I$;*

*(C6) there exists $p > 1$ such that for $u, v \in \mathcal{X}$ with $\xi(u, v) \geq 0$, and $r \in I$ we have*

$$\max\{\,|K(r,u(r))|, |K(r,v(r))|\,\} \leq \left[\frac{1}{2^{2p-2}} \exp(-\Theta_1(u,v)(r))\Theta_1(u,v)(r)\right]^{\frac{1}{p}}, \tag{32}$$

*where*

$$\Theta_1(u,v)(r) = \max\left\{\begin{array}{c} (\max\{u(r),v(r)\})^p, (\max\{u(r),\mathcal{J}u(r)\})^p, (\max\{v(r),\mathcal{J}v(r)\})^p, \\ \frac{(\max\{u(r),\mathcal{J}v(r)\})^p + (\max\{v(r),\mathcal{J}u(r)\})^p}{2^{p+1}}, \\ \frac{(\max\{u(r),\mathcal{J}u(r)\})^p (\max\{v(r),\mathcal{J}v(r)\})^p}{1+(\max\{u(r),v(r)\})^p}, \frac{(\max\{u(r),\mathcal{J}u(r)\})^p (\max\{v(r),\mathcal{J}v(r)\})^p}{1+(\max\{\mathcal{J}u(r),\mathcal{J}v(r)\})^p} \end{array}\right\}^{\frac{1}{p}};$$

*(C7) $\sup_{t\in I} \int_0^1 G(t,r)\,dr \leq \dfrac{1}{2}$.*

*Then there exists a solution of the integral equation* (31)*, and hence, there exists a solution of the problem* (30)*. Moreover, the fixed point problem* (31) *is generalized Ulam-Hyers stable.*

*Proof.* Define a function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t), y(t)) \geq 0, \text{ for all } t \in I \\ \gamma, & \text{otherwise,} \end{cases} \tag{33}$$

where $\gamma \in (0,1)$. It is easy to see that the assumption (C2) readily implies the condition (FG2) of Theorem 3.2. Also, the assumption (C3) implies that $\alpha$ has a transitive property (FG3). Finally, (C4) implies the regularity condition (FG4') of Theorem 3.3.

The assumption (C5) easily implies that $\mathcal{J} \in \mathcal{WA}(\mathcal{X}, \alpha)$.

We are going to check that $\mathcal{J} \in \Upsilon(\mathcal{X}, \alpha, FG)$. For this, let $u, v \in \mathcal{X}$ be such that $\alpha(u,v) \geq 1$, i.e., $\xi(u(t), v(t)) \geq 0$ for all $t \in I$. For each $t \in I$, by the definition of operator $\mathcal{J}$, we have

$$\begin{aligned}
\sigma_b(\mathcal{J}u, \mathcal{J}v) &= \left(\max_{t\in I}\left\{\left|\int_0^1 G(t,r)K(r,u(r))dr\right|, \left|\int_0^1 G(t,r)K(r,v(r))dr\right|\right\}\right)^p \\
&\leq \left(\max_{t\in I}\left\{\int_0^1 G(t,r)\,|K(r,u(r))|\,dr, \int_0^1 G(t,r)\,|K(r,v(r))|\,dr\right\}\right)^p \\
&= \left(\max_{t\in I}\int_0^1 G(t,r)\max_{t\in I}(|K(r,u(r))|, |K(r,v(r))|)\,dr\right)^p \\
&\leq \left(\max_{t\in I}\int_0^1 G(t,r)\left[\frac{1}{2^{2p-2}}\exp(-\Theta_1(u,v)(r))\Theta_1(u,v)(r)\right]^{\frac{1}{p}} dr\right)^p \\
&\leq \left(\max_{t\in I}\int_0^1 G(t,r)\left[\frac{1}{2^{2p-2}}\exp(-\Theta_{\mathcal{J}}(u,v))\Theta_{\mathcal{J}}(u,v)\right]^{\frac{1}{p}} dr\right)^p \\
&\leq \frac{1}{2^{2p-2}}\exp(-\Theta_{\mathcal{J}}(u,v))\Theta_{\mathcal{J}}(u,v).
\end{aligned}$$

Now, by considering the functions $F, G : \mathbb{R}^+ \to \mathbb{R}$ and $\beta : [0, +\infty) \to [0, 1)$ given by:

$$F(t) = G(t) = \ln t, \quad \beta(t) = \frac{1}{2^{p-1}} \exp(-t), \text{ for } t \geq 0,$$

we get

$$F(2^{p-1}\sigma_b(\mathcal{J}u, \mathcal{J}v))) \leq F(\Theta_{\mathcal{J}}(u, v)) + G(\beta(\Theta_{\mathcal{J}}(u, v))).$$

Thus all the hypotheses of Theorem 3.3 are fulfilled for $s = 2^{p-1}$. Thus there exists a fixed point of $\mathcal{J}$, that is, a continuous function $u^* \in C(I, \mathbb{R})$ such that $\mathcal{J}u^* = u^*$, that is,

$$u^*(t) = \mathcal{J}u^*(t) = \int_0^1 G(t, r)K(r, u^*(r))\, dr.$$

Consequently, $u^*$ is a solution of the boundary value problem (30).

Finally, by virtue of $\beta(t) = \frac{1}{2^{p-1}} \exp(-t)$, we define $\varphi(t) := t[1 - \beta(t)]$. Since $\varphi$ is strictly increasing and onto, all the hypotheses of Theorem 6.3 hold, so the fixed point of (31) is $\varphi^{-1}$-generalized Ulam-Hyers stable. □

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