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A Generalization of Reich's Fixed Point Theorem for Multi-Valued Mappings

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Abstract. Motivated by a problem concerning multi-valued mappings posed by Reich [S. Reich, Some fixed point problems, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 57 (1974) 194–198] and a paper of Jleli and Samet [M. Jleli, B. Samet, A new generalization of the Banach contraction principle, J. Inequal. Appl. 2014:38 (2014) 1–8], we consider a new class of multi-valued mappings that satisfy a Φ-contractive condition in complete metric spaces and prove some fixed point theorems. These results generalize Reich's and Mizoguchi-Takahashi's fixed point theorems. Some examples are given to show the usability of the obtained results.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

It is well-known that the notion of fixed point is of great interest in mathematics as well as in many areas of applied sciences. Formally, this concept can be introduced as follows:

Definition 1.1. *Let X be a nonempty set and $T : X \to N(X)$ be a multi-valued mapping, where $N(X)$ denotes the family of all nonempty subsets of X. A point $x \in X$ is said to be a fixed point of T if $x \in Tx$. If X is endowed with a topology, then $x \in X$ is said to be an approximate fixed point of T if $x \in \overline{Tx}$, where $\overline{Tx}$ denotes the closure of Tx. Moreover, the set of all fixed points is denoted by $Fix(T)$.*

Remark 1.2. *Note that the term "approximate fixed point" is not used here in the usual sense, for example as in the paper by Matoušková and Reich [7].*

Metric fixed point theory is the branch of mathematical analysis which focuses on the existence (and uniqueness) of fixed points under metric conditions on both the domain of the mapping and the mapping itself. In the setting of multi-valued mappings, the fundamental result of this theory is Nadler's theorem [9].

Theorem 1.3 (Nadler's theorem). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be such that $H(Tx, Ty) \leq k_0 d(x, y)$, for all $x, y \in X$, and some $k_0 \in [0, 1[$, where $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X. Then $Fix(T)$ is nonempty, that is, there exists $x \in X$ such that $x \in Tx$.*

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In the last decades, this result was generalized and extended by many authors in several directions (see, for example, [1, 2, 4, 6, 11, 12, 14–18] and the references therein).

Now, we point out the following result given by Reich [12].

Theorem 1.4. *Let (X, d) be a complete metric space. Suppose $T : X \to K(X)$ satisfies $H(Tx, Ty) \leq k(d(x, y))d(x, y)$, for all $x, y \in X$, $x \neq y$, where $K(X)$ denotes the family of all nonempty compact subset of X and $k :]0, +\infty[\to [0, 1[$ satisfies*

$$\limsup_{t \to r^+} k(t) < 1 \quad \text{for all } r > 0.$$

Then T has a fixed point.

In 1974, Reich [13] asked if it is possible to take $CB(X)$ instead of $K(X)$ in Theorem 1.4. The closest answer to this problem was given by Mizoguchi and Takahashi [8] by considering a function $k :]0, +\infty[\to [0, 1[$ such that

$$\limsup_{t \to r^+} k(t) < 1 \quad \text{for all } r \geq 0.$$

In this paper, we continue this study by establishing the existence of approximate fixed points for multi-valued mappings in the setting of complete metric spaces. Here, we start from looking at the paper of Jleli and Samet [5], who introduced a new concept of contraction. In so doing, we present a new weaker contractive condition (say Φ-contractive condition) with an unifying power over many contractive conditions existing in the literature. Some examples are given to show the usability of the obtained theorems.

2. Preliminaries

The aim of this section is to present some notions and results used in the paper. We denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers.

In the sequel, for a function $\Phi : \mathbb{R}^+ \to]1, +\infty[$, we consider the following properties:

(ϕ_1) Φ is non-decreasing;

(ϕ_2) for each sequence $\{t_n\}$ of positive numbers, $\lim_{n\to+\infty} t_n = 0$ if and only if $\lim_{n\to+\infty} \Phi(t_n) = 1$;

(ϕ_3) for each sequence $\{t_n\}$ of positive numbers with $\lim_{n\to+\infty} t_n = 0$ there exist $\gamma \in]0, 1[$ and $\lambda \in]0, +\infty]$ such that $\lim_{n\to+\infty} \frac{\Phi(t_n) - 1}{(t_n)^\gamma} = \lambda$.

We denote by Ψ the family of all functions Φ satisfying the conditions (ϕ_1)-(ϕ_3).

Definition 2.1 ([5]). *Let (X, d) be a metric space and let $T : X \to X$ be a self-mapping. T is called a Φ-contraction if there exists $\Phi \in \Psi$ and $k \in [0, 1[$ such that*

$$\Phi(d(Tx, Ty)) \leq [\Phi(d(x, y))]^k \quad \textit{for all } x, y \in X \textit{ with } d(Tx, Ty) > 0. \tag{1}$$

By using the contractive condition (1) with $\Phi \in \Psi$, Jleli and Samet in [5] established a unique fixed point result which generalizes the Banach fixed point theorem in [3]. If we choose the function Φ opportunely, then it is possible to obtain some classes of contractions known in the literature; see again [5].

Let (X, d) be a metric space and $CL(X)$ denote the family of all nonempty closed subsets of X. It is obvious that $K(X) \subset CB(X) \subset CL(X) \subset N(X)$. For $A, B \in CB(X)$, define

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

where

$$\delta(A, B) = \sup_{a \in A} d(a, B) \quad \text{and} \quad d(a, B) = \inf_{b \in B} d(a, b).$$

It is known that H is a metric on $CB(X)$, which is called Hausdorff metric induced by d, and a generalized metric on $CL(X)$.

In this paper, see [10], we denote by $\mathcal{I}$ the family of multi-valued mappings $T : X \to N(X)$ that have the following property of approximation: if $d(x, Tx) > 0$ for some $x \in X$, then there exists $y \in Tx$ such that $d(x, y) = d(x, Tx)$. Clearly, each multi-valued mapping with compact values belongs to $\mathcal{I}$.

Let (X, d) be a metric space, $\alpha : X \times X \to [0, +\infty[$ be a function and let $T : X \to N(X)$. The multi-valued mapping T is called weakly α-admissible if, for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$. Clearly, if $\alpha(x, y) = 1$ for all $x, y \in X$, then T is weakly α-admissible. In the sequel, we denote by α_0 the function from $X \times X$ to $[0, +\infty[$ defined by $\alpha_0(x, y) = 1$ for all $x, y \in X$.

Here, we obtain some extensions of the result of Jleli and Samet [5] for multi-valued mappings in complete metric space. These results, in some way, generalize Reich's and Mizoguchi-Takahashi's theorems.

3. New Fixed Point Theorems in Complete Metric Spaces

In the framework of complete metric spaces, we establish some approximate fixed point results for multi-valued mappings satisfying a Φ-contractive condition.

Let (X, d) be a metric space and let $T : X \to N(X)$. The graph $Gr(T)$ of the multi-valued mapping T is the subset $\{(x, y) : x \in X, y \in Tx\}$ of $X \times X$. Then T is a closed multi-valued mapping if $Gr(T)$ is a closed subset of $(X \times X, d^*)$, where

$$d^*((x, y), (u, v)) = d(x, u) + d(y, v) \qquad \text{for all } (x, y), (u, v) \in X \times X.$$

As well as, with $\overline{T}$ we denote the multi-valued mapping defined by $\overline{T}x = \overline{Tx}$ for all $x \in X$.

In the sequel, we use the quantity:

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\} \tag{2}$$

and the following conditions:

(a) $d(x, Tx) = 0$ if there exists a sequence $\{x_n\} \subset X$ convergent to $x \in X$ such that $d(x_{n-1}, x_n) = d(x_{n-1}, Tx_{n-1})$, $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$ and $d(x_{n-1}, Tx_{n-1}) \to 0$ as $n \to +\infty$;

(b) the function $f : X \to [0, +\infty[$ defined by $fx = d(x, Tx)$ for all $x \in X$ is lower semicontinuous;

(c) for each $y \in X$ with $y \notin \overline{Ty}$, we have $\inf\{d(x, y) + d(x, Tx) : x \in X\} > 0$;

(d) $\overline{T}$ is a closed multi-valued mapping.

Let (X, d) be a metric space, let $T : X \to N(X)$ and $\alpha : X \times X \to [0, +\infty[$. The multi-valued mapping T is a Φ-contraction with respect to the function α if there exist two functions $k : \mathbb{R}^+ \to [0, 1[$ and $\Phi \in \Psi$ such that

$$\Phi(H(Tx, Ty)) \leq [\Phi(M(x, y))]^{k(d(x,y))} \quad \text{for all } x, y \in X \text{ such that } \alpha(x, y) \geq 1 \text{ and } H(Tx, Ty) > 0, \tag{3}$$

where $M(x, y)$ is as in (2) and $\limsup_{t \to r^+} k(t) < 1$ for all $r \geq 0$.

The following example shows that the family of Φ-contractions with respect to the function α is not empty.

Example 3.1. *Let (X, d) be a metric space and let $T : X \to N(X)$ be a multi-valued mapping. Assume that there exist two functions $\gamma : \mathbb{R}^+ \to [0, 1[$ and $\alpha : X \times X \to [0, +\infty[$ such that*

$$H(Tx, Ty) \leq \gamma(d(x, y))\, M(x, y) \quad \textit{for all } x, y \in X \textit{ with } \alpha(x, y) \geq 1 \textit{ and } H(Tx, Ty) > 0, \tag{4}$$

where $M(x, y)$ is as in (2) and $\limsup_{t \to r^+} \gamma(t) < 1$ for all $r \geq 0$. Then T is a Φ-contraction with respect to the function α.

In fact, from (4), *we deduce*

$$e^{\sqrt{H(Tx,Ty)}} \leq e^{\sqrt{\gamma(d(x,y))M(x,y)}} = \left[e^{\sqrt{M(x,y)}}\right]^{\sqrt{\gamma(d(x,y))}}$$

and hence T is a Φ-contraction with respect to the function α if we choose the functions $k : \mathbb{R}^+ \to [0,1[$ defined by $k(t) = \sqrt{\gamma(t)}$ and $\Phi \in \Psi$ defined by $\Phi(t) = e^{\sqrt{t}}$ for all $t \in \mathbb{R}^+$.

The following theorem is the main result in the framework of complete metric spaces.

Theorem 3.2. *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0,+\infty[$ and $T : X \to N(X)$ with $T \in \mathcal{I}$. Assume that T is a Φ-contraction with respect to the function α such that the following conditions are satisfied:*

(i) T is a weakly α-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$.

Then T has an approximate fixed point $z \in X$ if one of the conditions (a)-(d) holds.

Proof. Proceeding by absurd, we suppose that T has not approximate fixed points. This implies that $d(x,Tx) = d(x,\overline{Tx}) > 0$ for all $x \in X$. Now, the condition (ii) ensures that there exists $x_0 \in X$ such that $\alpha(x_0,x_1) \geq 1$ for some $x_1 \in Tx_0$. From $d(x_1,Tx_1) > 0$ and $T \in \mathcal{I}$, we obtain that there exists $x_2 \in Tx_1$ such that $d(x_1,x_2) = d(x_1,Tx_1)$. Since T is a weakly α-admissible multi-valued mapping, we have $\alpha(x_1,x_2) \geq 1$. Proceeding by induction, we can construct a sequence $\{x_n\} \subset X$ such that $x_{n+1} \in Tx_n$, $d(x_n,x_{n+1}) = d(x_n,Tx_n) > 0$ and $\alpha(x_{n-1},x_n) \geq 1$ for all $n \in \mathbb{N}$. From

$$0 < d(x_n,x_{n+1}) = d(x_n,Tx_n) \leq H(Tx_{n-1},Tx_n),$$

we deduce that $H(Tx_{n-1},Tx_n) > 0$ for all $n \in \mathbb{N}$. Using (3) with $x = x_{n-1}$ and $y = x_n$, we get

$$\Phi(H(Tx_{n-1},Tx_n)) \leq [\Phi(M(x_{n-1},x_n))]^{k(d(x_{n-1},x_n))},$$

where

$$\begin{aligned} M(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n), \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}\right\} \\ &\leq \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1})}{2}\right\} \\ &= \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\}. \end{aligned}$$

Now, from

$$\Phi(d(x_n,x_{n+1})) \leq \Phi(H(Tx_{n-1},Tx_n)) \leq [\Phi(M(x_{n-1},x_n))]^{k(d(x_{n-1},x_n))},$$

we have

$$\Phi(d(x_n,x_{n+1})) \leq [\Phi(M(x_{n-1},x_n))]^{k(d(x_{n-1},x_n))}.$$

If for some $m \in \mathbb{N}$, we have $M(x_{m-1},x_m) = d(x_m,x_{m+1})$, then we get

$$\Phi(d(x_m,x_{m+1})) \leq [\Phi(M(x_{m-1},x_m))]^{k(d(x_{m-1},x_m))} < \Phi(d(x_m,x_{m+1})),$$

which is a contradiction. Consequently,

$$\Phi(d(x_n,x_{n+1})) \leq [\Phi(d(x_{n-1},x_n))]^{k(d(x_{n-1},x_n))} < \Phi(d(x_{n-1},x_n)) \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Furthermore, from (5) we deduce that $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence. Hence there exists $r \geq 0$ such that $d(x_{n-1}, x_n) \to r$ as $n \to +\infty$. We claim that $r = 0$. Assume on the contrary that $r > 0$, then from

$$\limsup_{n\to+\infty} k(d(x_{n-1}, x_n)) < 1,$$

we deduce that there exists $s \in]0, 1[$ such that $k(d(x_{n-1}, x_n)) \leq s$ for all $n \in \mathbb{N}$. Then by (5), we get

$$\Phi(d(x_n, x_{n+1})) \leq [\Phi(d(x_{n-1}, x_n))]^s \quad \text{for all } n \in \mathbb{N}$$

and hence

$$\Phi(d(x_n, x_{n+1})) \leq [\Phi(d(x_0, x_1))]^{s^n} \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

From (6), we obtain

$$\lim_{n\to+\infty} \Phi(d(x_n, x_{n+1})) = 1$$

and, by property (ϕ_2), we deduce that $d(x_{n-1}, x_n) \to 0$ as $n \to +\infty$.

We claim that $\{x_n\}$ is a Cauchy sequence, by showing that the series $\sum_{j=0}^{+\infty} d(x_j, x_{j+1})$ is convergent. To this aim, we use condition (ϕ_3). This condition ensures that there exists $\gamma \in]0, 1[$ and $\lambda \in]0, +\infty]$ such that

$$\lim_{n\to+\infty} \frac{\Phi(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^\gamma} = \lambda.$$

Let $\beta^{-1} \in]0, \lambda[$. From the definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$[d(x_n, x_{n+1})]^\gamma \leq \beta[\Phi(d(x_n, x_{n+1})) - 1] \quad \text{for all } n \geq n_0.$$

Using (6) and the above inequality, we get:

$$n[d(x_n, x_{n+1})]^\gamma \leq \beta n[(\Phi(d(x_0, x_1)))^{s^n} - 1] \quad \text{for all } n \in \mathbb{N},\ n \geq n_0.$$

Letting $n \to +\infty$ in the previous inequality, we get $\lim_{n\to+\infty} n[d(x_n, x_{n+1})]^\gamma = 0$. Thus there exists $n_1 \in \mathbb{N}$ with $n_1 \geq n_0$ such that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\gamma}}} \quad \text{for all } n \geq n_1.$$

This inequality ensures that the series

$$\sum_{j=0}^{+\infty} d(x_j, x_{j+1})$$

is convergent and so $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to +\infty$. If $z \in \overline{Tz}$ the proof is complete, and hence suppose $z \notin \overline{Tz}$. We distinguish the following four cases:

Case 1. Assume that condition (a) holds. Then we deduce that $d(z, Tz) = 0$, since the sequence $\{x_n\}$ that converges to z satisfies the hypotheses of condition (a). This implies that $z \in \overline{Tz}$, that is, z is an approximate fixed point of T.

Case 2. Assume that condition (b) holds. From

$$d(z, Tz) \leq \liminf_{n\to+\infty} d(x_n, Tx_n) = 0,$$

it follows that condition (a) holds and so T has an approximate fixed point.

Case 3. Assume that condition (c) holds. From $z \notin \overline{Tz}$, it follows

$$\begin{aligned} 0 &< \inf\{d(x,z) + d(x,Tx) : x \in X\} \\ &\leq \inf\{d(x_n,z) + d(x_n,Tx_n) : n \in \mathbb{N}\} \\ &= 0, \end{aligned}$$

which is a contradiction. Thus $z \in \overline{Tz}$, that is, z is an approximate fixed point of T.

Case 4. Assume that condition (d) holds. From

$$\lim_{n\to+\infty} d^*((x_n,x_{n+1}),(z,z)) = \lim_{n\to+\infty}[d(x_n,z) + d(x_{n+1},z)] = 0,$$

we obtain that $(z,z) \in Gr(\overline{T})$, that is, $z \in \overline{Tz}$. Hence z is an approximate fixed point of T. □

From Theorem 3.2, we deduce some results of fixed point as the following corollaries.

Corollary 3.3. *Let (X,d) be a complete metric space, $\alpha : X\times X \to [0,+\infty[$ and $T : X \to CL(X)$ with $T \in \mathcal{I}$. Assume that T is a Φ-contraction with respect to the function α such that the following conditions are satisfied:*

(i) T is a weakly α-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$.

Then T has a fixed point $z \in X$ if one of the conditions (a)-(d) holds.

Corollary 3.4. *Let (X,d) be a complete metric space, $\alpha : X\times X \to [0,+\infty[$ and $T : X \to K(X)$. Assume that T is a Φ-contraction with respect to the function α such that the following conditions are satisfied:*

(i) T is a weakly α-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$.

Then T has a fixed point $z \in X$ if one of conditions (a)-(d) holds.

Example 3.5. *Let $X = [0,1]$ and $d : X\times X \to [0,+\infty[$ be given by $d(x,y) = |x-y|$ for all $x,y \in X$. Clearly, (X,d) is a complete metric space. Let $T : X \to N(X)$ be a multi-valued mapping defined by*

$$Tx = \begin{cases} \{0\} & \text{if } x = 0, \\ ]0,\frac{x}{2}] & \text{if } x \in]0,1]. \end{cases}$$

Note that $T \in \mathcal{I}$. From

$$H(Tx,Ty) \leq \frac{1}{2}d(x,y) \quad \text{for all } x,y \in X,$$

we deduce that T is a Φ-contraction with respect to the function α_0 if we choose $\Phi \in \Psi$ defined by $\Phi(t) = e^{\sqrt{t}}$ and $k : \mathbb{R}^+ \to [0,1[$ defined by $k(t) = 1/2$ for all $t \in \mathbb{R}^+$. Now, if $\{x_n\} \subset [0,1]$ converges to $x \in [0,1]$ and $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$, then

$$d(x,Tx) \leq d(x,x_n) + H(Tx_{n-1},Tx) \leq d(x,x_n) + d(x_{n-1},x) \to 0 \quad \text{as } n \to +\infty.$$

This ensures that condition (a) holds and so T has an approximate fixed point that is also a fixed point. Now, we note that Theorem 1.3 is not applicable in this case since Tx is not closed for all $x \in]0,1]$. The same holds for Theorem 1.4.

We show that Theorem 3.2 is a generalization of Nadler's fixed point theorem for multi-valued mappings belonging to $\mathcal{I}$. In fact, if $T : X \to CB(X)$ satisfies the condition of Nadler's theorem, that is, $H(Tx, Ty) \leq k_0 d(x, y)$ for all $x, y \in X$, some $k_0 \in [0, 1[$ and $T \in \mathcal{I}$, then we have

$$e^{\sqrt{H(Tx,Ty)}} \leq [e^{\sqrt{d(x,y)}}]^{\sqrt{k_0}}.$$

If we choose $\Phi \in \Psi$ defined by $\Phi(t) = e^{\sqrt{t}}$ and $k : \mathbb{R}^+ \to [0, 1[$ defined by $k(t) = \sqrt{k_0}$ for all $t \in \mathbb{R}^+$, we get

$$\Phi(H(Tx, Ty) \leq [\Phi(d(x, y))]^{k(d(x,y))} \leq [\Phi(M(x, y))]^{k(d(x,y))} \quad \text{for all } x, y \in X \text{ with } H(Tx, Ty) > 0.$$

Thus T is a Φ-contraction with respect to the function α_0. Now, if $\{x_n\} \subset X$ converges to $x \in X$ and $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$, then

$$d(x, Tx) \leq d(x, x_n) + H(Tx_{n-1}, Tx) \leq d(x, x_n) + k_0 d(x_{n-1}, x) \to 0 \quad \text{as } n \to +\infty.$$

This ensures that condition (a) holds. Since T satisfies all the hypotheses of Theorem 3.2, we deduce that T has an approximate fixed point that is also a fixed point.

Clearly, Theorem 3.2 is an extension of Nadler's fixed point theorem as shown in the following example.

Example 3.6. *Let $X = \mathbb{N} \cup \{0\}$ and consider the metric $d : X \times X \to [0, +\infty[$ defined by*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ x + y & \text{if } x \neq y. \end{cases}$$

Clearly, (X, d) is a complete metric space. Let $T : X \to K(X)$ be a multi-valued mapping defined by

$$Tx = \begin{cases} \{0\} & \text{if } x \in \{0, 1\}, \\ \{0, 1, \ldots, x - 1\} & \text{if } x \geq 2. \end{cases}$$

Note that $T \in \mathcal{I}$, since $Tx \in K(X)$ for all $x \in X$, and that $Gr(T)$ is a closed subset of $(X \times X, d^*)$, where

$$d^*((x, y), (u, v)) = d(x, u) + d(y, v) \quad \text{for all } (x, y), (u, v) \in X \times X.$$

Thus T is a closed multi-valued mapping that satisfies the following condition:

$$\frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx,Ty)-M(x,y)} < e^{-1} \quad \text{for all } x, y \in X \text{ with } H(Tx, Ty) > 0.$$

Finally, we distinguish the following two cases:

Case 1. If $y \in \{0, 1\}$ and $x \geq 2$, then

$$\frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx,Ty)-M(x,y)} \leq \frac{x - 1}{x} e^{x-1-x} < e^{-1}.$$

Case 2. If $2 \leq y < x$, then

$$\frac{H(Tx, Ty)}{M(x, y)} e^{H(Tx,Ty)-M(x,y)} = \frac{x - 1}{x + y} e^{x-1-x-y} < e^{-1}.$$

Consequently, for all $x, y \in X$ with $H(Tx, Ty) > 0$, we get

$$e^{\sqrt{H(Tx,Ty)e^{H(Tx,Ty)}}} \leq [e^{\sqrt{M(x,y)e^{M(x,y)}}}]^{\sqrt{e^{-1}}}.$$

This ensures that T is a Φ-contraction with respect to the function α_0, if we choose $\Phi \in \Psi$ defined by $\Phi(t) = e^{\sqrt{te^t}}$ and $k : \mathbb{R}^+ \to [0, 1[$ defined by $k(t) = \sqrt{e^{-1}}$ for all $t \in \mathbb{R}^+$.

Since all the hypotheses of Theorem 3.2 are satisfied, we can conclude that T has a fixed point.

Now, we note that Theorem 1.3 is not applicable in this case. Proceeding by contradiction, suppose that there exists $k \in]0,1[$ such that $H(Tx,Ty) \le kd(x,y)$ holds true for all $x,y \in X$. From $H(Tx,T0) = x-1 \le kx$, for all $x \ge 1$, we get

$$1 = \lim_{x\to+\infty} \frac{x-1}{x} \le k,$$

which is a contradiction.

Remark 3.7. *Note that the multi-valued mapping in Example 3.6 satisfies the following condition:*

$$H(Tx,Ty) \le \frac{d(x,y)}{1+d(x,y)} M(x,y) \quad \text{for all } x,y \in X,\ x \ne y.$$

Thus, for Example 3.1, we deduce again that T is a Φ-contraction with respect to the function α_0 if we choose $\gamma : \mathbb{R}^+ \to [0,1[$ defined by $\gamma(t) = \frac{1+2t}{2+2t}$ for all $t \in \mathbb{R}^+$.

4. A Generalization of Mizoguchi-Takahashi's Fixed Point Theorem

In this section, we give a generalization of Mizoguchi-Takahashi's fixed point theorem.

Theorem 4.1. *Let (X,d) be a complete metric space and let $T : X \to N(X)$ a given mapping. Assume that there exists a right continuous function $\Phi \in \Psi$ and a function $\alpha : X \times X \to [0,+\infty[$ such that the following conditions are satisfied:*

(i) T is a weakly α-admissible mapping;

(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \ge 1$;

(iii) T is a Φ-contraction with respect to the function α.

Then T has an approximate fixed point $z \in X$ if one of the conditions (a)-(d) holds.

Proof. Proceeding by absurd, we suppose that T has not approximate fixed points. This implies that $d(x,Tx) = d(x,\overline{Tx}) > 0$ for all $x \in X$. Now, the condition (ii) ensures that there exists $x_0 \in X$ such that $\alpha(x_0,x_1) \ge 1$ for some $x_1 \in Tx_0$. From $d(x_1,Tx_1) > 0$ and the right continuity of Φ, we get

$$\begin{aligned}\inf_{w\in Tx_1} \Phi(d(x_1,w)) &\le \Phi(d(x_1,Tx_1)) \le \Phi(H(Tx_0,Tx_1)) \\ &\le \Phi(M(x_0,x_1))^{k(d(x_0,x_1))} < \Phi(M(x_0,x_1))^{(k(d(x_0,x_1))+1)/2}.\end{aligned}$$

this ensures that there exists $x_2 \in Tx_1$ such that

$$\Phi(d(x_1,x_2)) \le \Phi(M(x_0,x_1))^{(k(d(x_0,x_1))+1)/2}.$$

Since T is a weakly α-admissible multi-valued mapping, we have $\alpha(x_1,x_2) \ge 1$. Proceeding by induction, we can construct a sequence $\{x_n\} \subset X$ such that $x_{n+1} \in Tx_n$,

$$\Phi(d(x_n,x_{n+1})) \le \Phi(M(x_{n-1},x_n))^{(k(d(x_{n-1},x_n))+1)/2} \tag{7}$$

and $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$. Now, for $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned}M(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n), \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}\right\} \\ &\le \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1})}{2}\right\} \\ &= \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\}.\end{aligned}$$

Next, if for some $m \in \mathbb{N}$, we have $M(x_{m-1}, x_m) = d(x_m, x_{m+1})$, then by (7), we obtain

$$\Phi(d(x_m, x_{m+1})) \leq [\Phi(d(x_m, x_{m+1}))]^{(k(d(x_{m-1},x_m))+1)/2} < \Phi(d(x_m, x_{m+1})),$$

which is a contradiction. This ensures that $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ and hence

$$\Phi(d(x_n, x_{n+1})) \leq [\Phi(d(x_{n-1}, x_n))]^{(k(d(x_{n-1},x_n))+1)/2} \quad \text{for all } n \in \mathbb{N}. \tag{8}$$

Furthermore, from (8) we deduce that $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence. Hence there exists $r \geq 0$ such that $d(x_{n-1}, x_n) \to r$ as $n \to +\infty$. We claim that $r = 0$. Assume on the contrary that $r > 0$, then from

$$\limsup_{n\to+\infty} \frac{k(d(x_{n-1}, x_n)) + 1}{2} < 1,$$

we deduce that there exists $s \in]0, 1[$ such that $k(d(x_{n-1}, x_n)) + 1)/2 \leq s$ for all $n \in \mathbb{N}$. Then by (8), we get

$$\Phi(d(x_n, x_{n+1})) \leq [\Phi(d(x_{n-1}, x_n))]^s \quad \text{for all } n \in \mathbb{N}$$

and hence

$$\Phi(d(x_n, x_{n+1})) \leq [\Phi(d(x_0, x_1))]^{s^n} \quad \text{for all } n \in \mathbb{N}.$$

Now, proceeding as in the proof of Theorem 3.2, we obtain the existence of an approximate fixed point of T if one of the conditions (a)-(d) holds. In order to avoid repetition, we omit the details. □

We show that Theorem 4.1 is a generalization of the following Mizoguchi-Takahashi's fixed point theorem.

Theorem 4.2. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ satisfy*

$$H(Tx, Ty) \leq k_0(d(x, y))d(x, y) \quad \textit{for all } x, y \in X,\ x \neq y,$$

where $k_0 : \mathbb{R}^+ \to [0, 1[$ is a function such that $\limsup_{t\to r^+} k_0(t) < 1$ for every $r \in [0, +\infty[$. Then T has a fixed point.

Proof. For all $x, y \in X$ such that $H(Tx, Ty) > 0$, we have

$$e^{\sqrt{H(Tx,Ty)}} \leq [e^{\sqrt{d(x,y)}}]^{\sqrt{k_0(d(x,y))}}.$$

If we choose $\Phi \in \Psi$ defined by $\Phi(t) = e^{\sqrt{t}}$ and $k : \mathbb{R}^+ \to [0, 1[$ defined by $k(t) = \sqrt{k_0(t)}$ for all $t \in \mathbb{R}^+$, we get

$$\Phi(H(Tx, Ty)) \leq [\Phi(d(x, y))]^{k(d(x,y))} \leq [\Phi(M(x, y))]^{k(d(x,y))} \quad \text{for all } x, y \in X \text{ with } H(Tx, Ty) > 0.$$

Thus T is a Φ-contraction with respect to the function α_0. Now, if $\{x_n\} \subset X$ converges to $x \in X$ and $x_n \in Tx_{n-1}$ for all $n \in \mathbb{N}$, then

$$d(x, Tx) \leq d(x, x_n) + H(Tx_{n-1}, Tx) \leq d(x, x_n) + k_0(d(x_{n-1}, x))d(x_{n-1}, x) \to 0 \quad \text{as } n \to +\infty.$$

This ensures that condition (a) holds. Since T satisfies all the hypotheses of Theorem 4.1, we deduce that T has an approximate fixed point that is also a fixed point. □

Example 4.3. *Let $X = [0, +\infty[$ and consider the metric $d : X \times X \to [0, +\infty[$ defined by*

$$d(x, y) = \begin{cases} 0 & \textit{if } x = y, \\ x + y & \textit{if } x \neq y. \end{cases}$$

Clearly, (X, d) is a complete metric space. Let $T : X \to N(X)$ be a multi-valued mapping defined by

$$Tx = \begin{cases} \{0\} & \text{if } x \in [0,1], \\ \{3/4\} & \text{if } x \in]1,2], \\ [0, x-1[& \text{if } x > 2. \end{cases}$$

We distinguish the following three cases:

Case 1. If $y \in [0,1]$ and $x \in]1,2]$, then

$$H(Tx, Ty) = H(\{0\}, \{3/4\}) = 3/4.$$

Case 2. If $y \in [0,1]$ and $x > 2$, then

$$H(Tx, Ty) = H(\{0\}, [0, x-1[) = \sup\{d(z, 0) : z \in [0, x-1[\} = x - 1.$$

Case 3. If $y \in]1,2]$ and $x > 2$, then

$$H(Tx, Ty) = H(\{3/4\}, [0, x-1[) = \sup\{d(z, 3/4) : z \in [0, x-1[\} = x - 5/4.$$

Now, consider the function $\gamma : \mathbb{R}^+ \to [0,1[$ defined by

$$\gamma(t) = \begin{cases} 3/4 & \text{if } t \in [0, 9/4], \\ \dfrac{4t}{3+4t} & \text{if } t > 9/4. \end{cases}$$

It is easy to verify that

$$H(Tx, Ty) \leq \gamma(d(x,y)) M(x,y) \quad \text{for all } x, y \in X,\ x \neq y,$$

where $M(x,y)$ is as in (2) and $\limsup_{t \to r^+} \gamma(t) < 1$ for all $r \geq 0$. Then T is a Φ-contraction with respect to the function α_0, if we choose $\Phi \in \Psi$ defined by $\Phi(t) = e^{\sqrt{t}}$ and $k : \mathbb{R}^+ \to [0,1[$ defined by $k(t) = \sqrt{\gamma(t)}$ for all $t \in \mathbb{R}^+$. Since all the hypotheses of Theorem 4.1 are satisfied, we can conclude that T has an approximate fixed point that is a fixed point.

Finally, we note that Theorem 4.2 is not applicable in this case since Tx is not closed for all $x \in X$.

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