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Approximate Controllability for Time-Dependent Impulsive Neutral Stochastic Partial Differential Equations with Memory

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Abstract. We establish results concerning the approximate controllability for time-dependent impulsive neutral stochastic partial differential equations with memory in Hilbert spaces. By using semigroup theory, stochastic analysis techniques and fixed point approach, we derive a new set of sufficient conditions for the approximate controllability of nonlinear stochastic system under the assumption that the corresponding linear system is approximately controllable. Further, the above results are generalized to cover a class of much more general impulsive neutral stochastic delay partial differential equations driven by Lévy noise in infinite dimensions. Finally, an example is provided to illustrate our results.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

This paper is concerned with the approximate controllability of the following time-dependent impulsive neutral stochastic partial differential equations with memory

$$\begin{cases} d[x(t) - G(t, x(t-\tau))] = A(t)[x(t) - G(t, x(t-\tau))]dt + \Big[F(t, x(t), x(t-\tau)) + Bu(t)\Big]dt \\ \qquad\qquad\qquad\qquad\qquad +\sigma(t, x(t), x(t-\tau))dW(t), \quad t_k \neq t \in J := [0, T], \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k \in \{1, 2, ..., m\}, \\ x(t) = \varphi(t) \in \mathcal{C}_\tau = \mathcal{C}^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{H}), \quad -\tau \leq t \leq 0, \quad \tau > 0, \end{cases} \tag{1}$$

where $x(\cdot)$ is a stochastic process taking values in a real separable Hilbert space $\mathbb{H}$; $A(t) : D \subset \mathbb{H} \to \mathbb{H}$, $t \in J$ is a family of unbounded operators defined on a common domain D, which is dense in the space $\mathbb{H}$ and generates a strong evolution operator $U(s, t)$, $0 \leq t \leq s \leq T$. Assume that the mappings $G : J \times \mathbb{H} \to \mathbb{H}$, $F : J \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$, $\sigma : J \times \mathbb{H} \times \mathbb{H} \to \mathcal{L}_2^0$ are Borel measurable functions and $I_k : \mathbb{H} \to \mathbb{H}$, $k = 1, 2, ..., m$

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are continuous functions. The control function $u(\cdot)$ takes values in $L_2^{\mathcal{F}}(J, \mathbb{U})$ of admissible control functions for a separable Hilbert space $\mathbb{U}$ and B is a bounded linear operator from $\mathbb{U}$ into $\mathbb{H}$. Furthermore, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ be prefixed points, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump of the function x at time t_k with I_k determining the size of the jump, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Let $\varphi(t) : [-\tau, 0] \to \mathbb{H}$ is an $\mathcal{F}_0$-measurable random variables independent of the Wiener process W with $\mathbf{E}[\sup_{-\tau\leq s\leq 0} \|\varphi\|_{\mathbb{H}}^2] < \infty$.

Approximate controllability is one of the important fundamental concepts in mathematical control theory and plays an important role in both deterministic and stochastic control systems. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the case of infinite-dimensional systems, two basic concepts of controllability can be distinguished which are exact and approximate controllability. This is strongly related to the fact that in infinite dimensional spaces there exist linear subspaces, which are not closed. Exact controllability enables to steer the system to arbitrary final state while approximate controllability means that system can be steered to arbitrary small neighborhood of final state. In other words, approximate controllability gives the possibility of steering the system to states which form the dense subspace in the state space. However, the concept of exact controllability is usually too strong and, indeed has limited applicability in infinite-dimensional spaces (see [25]). Further, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications (see [15, 17] and the references therein). Therefore, it is important, in fact necessary to study the weaker concept of controllability, namely approximate controllability for nonlinear systems.

As we all known, controllability of deterministic systems are widely used in many fields of science and technology (for instance, see [6, 8, 13, 20, 24, 27] and the references therein). In practice, deterministic systems often fluctuate due to environmental noise. Therefore, the study of stochastic problems are more applicable in dynamical system theory. In [18], Mao studied the asymptotic properties of neutral stochastic differential delay equations. Sathya and Balachandran [23] proved sufficient conditions for controllability of nonlocal impulsive stochastic quasilinear integrodifferential systems by means of the Banach fixed point theorem. By using the Sadovskii fixed point theorem, Muthukumar and Rajivganthi [19] investigated the approximate controllability of fractional order neutral stochastic integro-differential systems with nonlocal conditions and infinite delay. Besides the environmental noise, sometimes, we have to consider the impulsive effects, which exist in many evolution processes, because the impulsive effects may bring an abrupt change at certain moments of times (see, e.g. [14, 27]). Moreover, for most research about nonlinear stochastic systems, the control function $u_\alpha(t, x)$ was always constructed by its corresponding linear systems and stochastic maximum principle [3], however, the stochastic maximum principle is not available in impulsive stochastic systems as a result of its linear form. Therefore, there is a real need to discuss impulsive differential control systems with memory (delay). In [10], Huan derived a set of sufficient conditions for the controllability results of nonlocal second-order impulsive neutral stochastic integro-differential equations with infinite delay in Hilbert spaces by means of the Banach fixed point theorem combined with theories of a strongly continuous cosine families of bounded linear operators. Recently, Huan and Gao [11] have extended the results of the paper [10] for a class of nonlocal second-order impulsive neutral stochastic integro-differential equations with infnite delay and Poisson jumps. For more detail on the well-posedness and controllability of stochastic systems with impulsive effect, we refer the reader to [1, 2, 5, 12, 23] and the references therein.

However, to the best of our knowledge, it seems that little is known about approximate controllability for time-dependent impulsive neutral stochastic partial differential equations with memory. Under the Lipschitz conditions, the linear growth conditions are weakened and under the condition that the corresponding linear system is approximately controllable, the aim of this paper is to discuss this interesting problem.

The main tools used in this paper include semigroup theory, stochastic analysis techniques and the Banach contraction principle.

This paper is organized as follows: In Section 2, we recall some basic notations, definitions, preliminaries. In Section 3 we prove the approximate controllability of system (1). An example is shown in the last section to illustrate the viability of the abstract result of this work.

2. Preliminaries

Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle_{\mathbb{K}})$ denote two real separable Hilbert spaces, with their vectors norms and their inner products, respectively. We denote by $\mathcal{L}(\mathbb{K};\mathbb{H})$ be the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, which is equipped with the usual operator norm $\|\cdot\|$. In this paper, we use the symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises. Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual condition (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets). Let $W = (W(t))_{t\geq 0}$ be a Q-Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ with the covariance operator Q such that $Tr(Q) < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, ...,$ and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that $\langle W(t), e\rangle_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k, e\rangle_{\mathbb{K}}\beta_k(t)$, $e \in \mathbb{K}$, $t \geq 0$.

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K};\mathbb{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ into $\mathbb{H}$ with the inner product $\langle\Psi, \phi\rangle_{\mathcal{L}_2^0} = Tr[\Psi Q\phi^*]$, where ϕ^* is the adjoint of the operator ϕ.

Let $\tau > 0$ and $C := C([-\tau, 0];\mathbb{H})$ denotes the family of all continuous functions from $[-\tau, 0]$ to $\mathbb{H}$. The space C is assumed to be equipped with the norm $\|\varsigma\|_C := \sup_{-\tau\leq t\leq 0} \|\varsigma(t)\|_{\mathbb{H}}$, $\varsigma(t) \in C$.

We also assume that $C^b_{\mathcal{F}_0}([-\tau, 0];\mathbb{H})$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0];\mathbb{H})$-valued random variables. For all $t \geq 0$, $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$ is regarded as $C([-\tau, 0];\mathbb{H})$-valued stochastic process. Further, let $\mathcal{PC}(J, L^2(\Omega;\mathbb{H})) = \{x(t)$ is continuous everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k), k = 1, 2, ..., m\}$ be the Banach space of piece-wise continuous function from J into $L^2(\Omega;\mathbb{H})$ with the norm $\|x\|_{\mathcal{PC}} = \sup_{t\in J} |x(t)| < \infty$. Let $\mathbf{PC} = \mathbf{PC}(J; L^2)$ be the closed subspace of $\mathcal{PC}(J, L^2(\Omega;\mathbb{H}))$ consisting of measurable and $\mathcal{F}_t$-adapted $\mathbb{H}$-valued process $x(\cdot) \in \mathcal{PC}(J, L^2(\Omega;\mathbb{H}))$ endowed with the norm $\|x\|^2_{\mathbf{PC}} = \mathbf{E}\sup_{t\in J} \|x(t)\|^2_{\mathbb{H}}$.

In what follows, we assume that $\{A(t), t \geq 0\}$ is a family of closed densely defined linear unbounded operators on $\mathbb{H}$ and with domain $D = D(A(t))$ independent of t.

Definition 2.1. *A family of bounded linear operators* $\{U(t,s)\}_{(t,s)\in\Delta} : U(t,s) : \mathbb{H} \to \mathbb{H}$ *for* $(t,s) \in \Delta := \{(t,s) \in J\times J : 0 \leq s \leq t \leq T\}$ *is called an evolution system if the following properties are satisfied:*

(1) $U(t,t) = I$ *where* I *is the identity operator in* $\mathbb{H}$.

(2) $U(t,s)U(s,\tau) = U(t,\tau)$ *for* $0 \leq \tau \leq s \leq t \leq T$.

(3) $U(t,s) \in \mathcal{L}(\mathbb{H})$ *the space of bounded linear operators on* $\mathbb{H}$, *where for every* $(t,s) \in \Delta$ *and for each* $x \in \mathbb{H}$, *the mapping* $(t,s) \to U(t,s)x$ *is strongly continuous.*

Remark 2.2. *If* $A(t), t \geq 0$ *is a second-oder differential operator* A, *i.e.* $A(t) = A$ *for each* $t \geq 0$. *Then,* A *generates a* C_0*-semigroup* $\{e^{At}, t \geq 0\}$.

More details on evolution systems and their properties could be found on the books of Ahmed [4] and Pazy [21].

Next, let us recall the definition of mild solution for (1).

Definition 2.3. *An $\mathcal{F}_t$-adapted stochastic process $x : J \to \mathbb{H}$ is called a mild solution of (1) if for each $u \in L_2^{\mathcal{F}}(J, U)$ and for arbitrary $t \in J$, $\mathbf{P}\{\omega : \int_J \|x(s)\|_{\mathbb{H}}^2 ds < +\infty\} = 1$ it satisfies the integral equation*

$$\begin{aligned} x(t) =& U(t,0)[\varphi(0) - G(0,\varphi)] + G(t, x(t-\tau)) + \int_0^t U(t,s)[F(s, x(s), x(s-\tau)) + Bu(s)]ds \\ &+ \int_0^t U(t,s)\sigma(s, x(s), x(s-\tau))dW(s) + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-)) \end{aligned} \tag{2}$$

for any $x_0(\cdot) = \varphi(\cdot) \in C_\tau$.

It is convenient to introduce the relevant operators and the basic controllability condition.

(i) The operator $L_0^T \in \mathcal{L}(L_2^{\mathcal{F}}(J, \mathbb{H}), L_2(\Omega, \mathcal{F}_T, \mathbb{H}))$ is defined by

$$L_0^T u = \int_0^T U(T,s)Bu(s)ds,$$

where $L_2^{\mathcal{F}}(J, \mathbb{H})$ is the space of all $\mathcal{F}_t$-adapted, H-valued measurable square integrable processes on $J \times \Omega$. Clearly the adjoint $(L_0^T)^* : L_2(\Omega, \mathcal{F}_T, \mathbb{H}) \to L_2^{\mathcal{F}}(J, \mathbb{H})$ is defined by

$$[(L_0^T)^* z](t) = B^* U^*(T,t)\mathbf{E}\{z \mid \mathcal{F}_t\}.$$

(ii) The linear controllability operator Π_0^T which is associated with the operator L_0^T is defined by

$$\Pi_0^T\{\cdot\} = L_0^T (L_0^T)^*\{\cdot\} = \int_0^T U(T,t)BB^*U^*(T,t)\mathbf{E}\{\cdot \mid \mathcal{F}_t\}dt.$$

which belongs to $\mathcal{L}(L_2(\Omega, \mathcal{F}_T, \mathbb{H}), L_2(\Omega, \mathcal{F}_T, \mathbb{H}))$ and the controllability operator $\Gamma_s^T \in \mathcal{L}(\mathbb{H}, \mathbb{H})$ is

$$\Gamma_s^T = \int_s^T U(T,t)BB^*U^*(T,t)dt, \quad 0 \le s < t.$$

Lemma 2.4. *([15]) For any $h \in L_2(\Omega, \mathcal{F}_T, H)$, there exists $z \in L_2^{\mathcal{F}}(J, \mathcal{L}_2^0)$ such that*

$$h = \mathbf{E}h + \int_J z(s)dW(s).$$

Let $x(t; \varphi, u)$ denotes state value of the system (1) at time t corresponding to the control $u \in L_2^{\mathcal{F}}(J, \mathbb{U})$ and the initial value φ. In particular, the state of system (1) at $t = T$, $x(T; \varphi, u)$ is called the terminal state with control u. $\mathcal{R}_T := \mathcal{R}(T, \varphi) = \{x(T; \varphi, u) : u(\cdot) \in L_2^{\mathcal{F}}(J, \mathbb{U})\}$ is called the reachable set of the system (1).

Definition 2.5. *The stochastic system (1) is said to be approximately controllable on the interval J if*

$$\overline{\mathcal{R}}_T = L_2(\Omega, \mathcal{F}_T, \mathbb{H}),$$

where $\overline{\mathcal{R}}_T$ is the closure of the reachable set.

3. Controllability Result

To study the approximate controllability of system (1), we will work under the following assumptions:

(**H**1) $U(t,s)$ is a compact operator for $t-s>0$ and there exists a constant $C\geq 1$ such that

$$\|U(t,s)\|_{\mathcal{L}(\mathbb{H})}\leq C \quad \text{for every } (t,s)\in\Delta.$$

(**H**2) There exists a positive constant C_0 such that for all $t\in J$, $x,y\in\mathbb{H}$

$$\|G(t,x)-G(t,y)\|^2_{\mathbb{H}}\leq C_0\Big(\|x-y\|^2_{\mathbb{H}}\Big).$$

(**H**3) There exists a positive constant C_1 such that for all $t\in J$, $x_1,y_1,x_2,y_2\in\mathbb{H}$

$$\|F(t,x_1,y_1)-F(t,x_2,y_2)\|^2_{\mathbb{H}}\vee\|\sigma(t,x_1,y_1)-\sigma(t,x_2,y_2)\|^2_{\mathcal{L}_2^0}\leq C_1\Big(\|x_1-x_2\|^2_{\mathbb{H}}+\|y_1-y_2\|^2_{\mathbb{H}}\Big).$$

(**H**4) There exists some positive constants Q_k, $k=1,2,...,m$ such that for all $t\in J$, $x,y\in\mathbb{H}$

$$\|I_k(x)-I_k(y)\|^2_{\mathbb{H}}\leq Q_k\|x-y\|^2_{\mathbb{H}}.$$

(**H**5) For all $t\in J$, there exists a positive constant M such that

$$\|G(t,0)\|^2_{\mathbb{H}}\vee\|F(t,0,0)\|^2_{\mathbb{H}}\vee\|\sigma(t,0,0)\|^2_{\mathbb{H}}\vee\|I_k(0)\|^2_{\mathbb{H}}\leq M.$$

(**H**6) For $0\leq t<T$, the operator
$\alpha\mathcal{R}(\alpha,\Gamma_t^T):=\alpha(\alpha I+\Gamma_t^T)^{-1}\to 0$ as $\alpha\to 0^+$ in the strong operator topology.

(**H**7) The functions F,σ are uniformly bounded.

Remark 3.1. *In view of [17], the assumption (*$\mathbf{H}$*6) is equivalent to the linear system of (1) is approximately controllable.*

Let $\alpha>0$ and $h\in L_2(\Omega,\mathcal{F}_T,H)$. Define the control function in the following form:

$$\begin{aligned}
&u_\alpha(t,x)\\
&=B^*U^*(T,t)\Bigg\{R(\alpha,\Pi_0^T)\Big[\mathbf{E}h-U(T,0)[\varphi(0)-G(0,\varphi)]-G(t,x(t-\tau))-\sum_{0<t_k<T}U(T,t_k)I_k(x(t_k^-))\Big]\\
&\quad+\int_0^T R(\alpha,\Pi_s^T)z(s)dW(s)-\int_0^T R(\alpha,\Pi_s^T)U(T,s)F(s,x(s),x(s-\tau))ds\\
&\quad-\int_0^T R(\alpha,\Pi_s^T)U(T,s)\sigma(s,x(s),x(s-\tau))dW(s)\Bigg\}. \qquad (3)
\end{aligned}$$

Consider the operator $\Upsilon:\mathcal{PC}(J,L^2(\Omega;\mathbb{H}))\to\mathcal{PC}(J,L^2(\Omega;\mathbb{H}))$ defined by

$$\begin{aligned}
(\Upsilon x)(t)=&U(t,0)[\varphi(0)-G(0,\varphi)]+G(t,x(t-\tau))+\int_0^t U(t,s)[F(s,x(s),x(s-\tau))+Bu_\alpha(s,x)]ds\\
&+\int_0^t U(t,s)\sigma(s,x(s),x(s-\tau))dW(s)+\sum_{0<t_k<t}U(t,t_k)I_k(x(t_k^-)). \qquad (4)
\end{aligned}$$

In what follows, we shall show that system (1) is approximately controlable if for all $\alpha>0$ there exists a fixed point of the operator Υ.

Theorem 3.2. *Let the assumptions* (**H1**) − (**H6**) *be hold. Then the operator* Υ *has a fixed point in* **PC** *provided that*

$$5\Big(\frac{\alpha^2 + 4T^2C^4C_B^4}{\alpha^2}\Big)\Big[C_0 + 2T(T+1)C^2C_1 + mC^2\sum_{k=1}^{m} Q_k\Big] < 1, \ \textit{where}\ C_B = \|B\|. \tag{5}$$

Proof By our assumptions, the basic inequality $\left(\sum_{i=1}^{n} x_i\right)^p \le n^{(p-1)\vee 0}\sum_{i=1}^{n} x_i^p, p > 0$, Hölder's inequality, and the Doob martingale inequality, we obtain that for $x \in \mathbf{P}C$

$$\begin{aligned}
&\mathbf{E}\|u_\alpha(t,x)\|^2\\
&\le \frac{7}{\alpha^2}C^2C_B^2\Big\{\|\mathbf{E}h\|^2 + 2C^2(1+C_0)\mathbf{E}\|\varphi\|_C^2 + 2\Big[M + C_0(\mathbf{E}\|\varphi\|_C^2 + \|x\|_{\mathbf{P}C}^2)\Big] \\
&\quad + 2mC^2\sum_{k=1}^{m} Q_k(M + \|x\|_{\mathbf{P}C}^2) + C^2\mathbf{E}\int_J \|z(s)\|_{\mathcal{L}_2^0}^2 ds + 2C^2(T+1)\Big[MT + C_1(\tau\mathbf{E}\|\varphi\|_C^2 + 2T\|x\|_{\mathbf{P}C}^2)\Big]\Big\}.
\end{aligned} \tag{6}$$

Thanks to (6) we have

$$\begin{aligned}
&\|(\Upsilon x)(t)\|_{\mathbf{P}C}^2\\
&\le 6\Big\{C^2\|\varphi(0) - G(0,\varphi)\|^2\Big) + \mathbf{E}\|G(t, x(t-\tau)) - G(t,0) + G(t,0)\|^2\\
&\quad + C^2T\mathbf{E}\int_0^t \|F(s,x(s),x(s-\tau)) - F(t,0,0) + F(t,0,0)\|^2 ds + C^2C_B^2T^2\mathbf{E}\|u_\alpha(s,x)\|^2\\
&\quad + C^2\mathbf{E}\int_0^t \|\sigma(s,x(s),x(s-\tau)) - \sigma(t,0,0) + \sigma(t,0,0)\|^2 ds + C^2\mathbf{E}\sum_{0<t_k<t}\|I_k(x(t_k^-) - I_k(0) + I_k(0)\|^2\Big\}\\
&\le 6\Big\{\frac{7}{\alpha^2}T^2C^4C_B^4\Big(\|\mathbf{E}h\|^2 + C^2\mathbf{E}\int_J \|z(s)\|_{\mathcal{L}_2^0}^2 ds\Big) + \Big(1 + \frac{7}{\alpha^2}T^2C^4C_B^4\Big)\Big(2C^2(1+C_0)\mathbf{E}\|\varphi\|_C^2\\
&\quad + 2\Big[M + C_0(\mathbf{E}\|\varphi\|_C^2 + \|x\|_{\mathbf{P}C}^2)\Big] + 2mC^2\sum_{k=1}^{m} Q_k(M + \|x\|_{\mathbf{P}C}^2)\\
&\quad + 2C^2(T+1)\Big[MT + C_1(\tau\mathbf{E}\|\varphi\|_C^2 + 2T\|x\|_{\mathbf{P}C}^2)\Big]\Big)\Big\} < \infty,
\end{aligned}$$

that is, $\Upsilon(\mathbf{P}C) \subset \mathbf{P}C$.

Now, we are going to use the Banach fixed point theorem to prove that Υ has a unique fixed point in **P**C. We claim that Υ is a contraction on **P**C. For any $x, y \in \mathbf{P}C$, $t \in J$ by the same ways as above, we can get

$$\begin{aligned}
&\|(\Upsilon x)(t) - (\Upsilon y)(t)\|_{\mathbf{P}C}^2\\
&\le 5\Big[C_0 + 2T(T+1)C^2C_1 + mC^2\sum_{k=1}^{m} Q_k\Big]\|x-y\|_{\mathbf{P}C}^2 + 5T^2C^2C_B^2\mathbf{E}\|u_\alpha(t,x) - u_\alpha(t,y)\|_{\mathbb{H}}^2\\
&\le 5\Big(\frac{\alpha^2 + 4T^2C^4C_B^4}{\alpha^2}\Big)\Big[C_0 + 2T(T+1)C^2C_1 + mC^2\sum_{k=1}^{m} Q_k\Big]\|x-y\|_{\mathbf{P}C}^2.
\end{aligned}$$

By assumption (5), we conclude that Υ is a contraction mapping on **P**C. Thus by the Banach fixed point theorem, has a unique fixed point $x(\cdot) \in \mathbf{P}C$. The proof is completed.

Theorem 3.3. *Assume the condition in Theorem 3.2 and the assumption* (**H7**) *are satisfied, then system (1) is approximately controllable on* $[0, T]$.

Proof By Theorem 3.2, Υ has a unique fixed point x^*_α in $\mathbf{P}C$. By substituting (3) into (4) and using the stochastic Fubini theorem [9], it can be easily seen that

$$\begin{aligned} x^*_\alpha(T) = h - \alpha R(\alpha, \Pi_0^T)\Big[\mathbf{E}h - U(T,0)[\varphi(0) - G(0,\varphi)] - G(t, x^*(t-\tau)) - \sum_{0<t_k<T} U(T,t_k)I_k(x^*(t_k^-))\Big] \\ + \int_0^T \alpha R(\alpha, \Pi_s^T)U(T,s)F(s, x^*(s), x^*(s-\tau))ds \\ + \int_0^T \alpha R(\alpha, \Pi_s^T)U(T,s)[\sigma(s, x^*(s), x^*(s-\tau)) - z(s)]dW(s). \end{aligned} \tag{7}$$

It follows from the assumption (**H7**) that there exists $C_{F\sigma} > 0$ such that

$$\|F(s, x^*(s), x^*(s-\tau))\|^2_{\mathbb{H}} + \|\sigma(s, x^*(s), x^*(s-\tau))\|^2_{\mathcal{L}_2^0} \le C_{F\sigma}.$$

Then there is a subsequence still denoted by $\{F(s, x^*(s), x^*(s-\tau)), \sigma(s, x^*(s), x^*(s-\tau))\}$ which converges weakly to, say, $\{F(s), \sigma(s)\}$ in $\mathbb{H} \times \mathcal{L}_2^0$. On the other hand, by assumption (**H6**), for all $0 \le t < T$, $\alpha R(\alpha, \Pi_s^T) \xrightarrow{\alpha \to 0^+} 0$ strongly and moreover $\|\alpha R(\alpha, \Pi_s^T)\| \le 1$. Therefore, by the Lebesgue dominated convergence theorem and the compactness of $U(\cdot,\cdot)$ it follows that

$$\begin{aligned} &\mathbf{E}\|x^*_\alpha(T) - h\|^2 \\ &\le 6\mathbf{E}\Big\|\alpha R(\alpha, \Pi_0^T)\Big[\mathbf{E}h - \alpha R(\alpha, \Pi_0^T)\Big[\mathbf{E}h - U(T,0)[\varphi(0) - G(0,\varphi)] - G(t, x^*(t-\tau)) \\ &\quad - \sum_{0<t_k<T} U(T,t_k)I_k(x^*(t_k^-))\Big] \\ &\quad + 6\mathbf{E}\Big(\int_0^T \|\alpha R(\alpha, \Pi_s^T)\|\|U(T,s)\|\|F(s, x^*(s), x^*(s-\tau)) - F(s)\|ds\Big)^2 \\ &\quad + 6\mathbf{E}\Big(\int_0^T \alpha R(\alpha, \Pi_s^T)U(T,s)F(s)ds\Big)^2 \\ &\quad + 6\mathbf{E}\int_0^T \|\alpha R(\alpha, \Pi_s^T)\|^2\|U(T,s)\|^2\|\sigma(s, x^*(s), x^*(s-\tau)) - \sigma(s)\|^2_{\mathcal{L}_2^0}ds \\ &\quad + 6\mathbf{E}\int_0^T \|\alpha R(\alpha, \Pi_s^T)U(T,s)\sigma(s)\|^2_{\mathcal{L}_2^0}ds + 6\mathbf{E}\int_0^T \|\alpha R(\alpha, \Pi_s^T)U(T,s)z(s)\|^2_{\mathcal{L}_2^0}ds \xrightarrow{\alpha \to 0^+} 0. \end{aligned}$$

Thus, $x^*_\alpha(T) \to h$ holds in $\mathbb{H}$ and consequently we obtain the approximate controllability of system (1). Theorem 3.3 is proved.

Remark 3.4. *Stochastic partial differential equations driven by Wiener processes has been studied extensively by many authors, and is an active research field. However, more recently, stochastic evolution equations and stochastic differential equations with the perturbations of Poisson noise or Lévy noise have attracted more and more attentions, for instances, [7, 22, 26], etc. In this remark, we will study the approximate controllability of following time-dependent impulsive neutral stochastic delay partial differential equations driven by Lévy noise*

$$\begin{cases} d[x(t) - G(t, x(t-\tau))] = A(t)[x(t) - G(t, x(t-\tau))]dt + \Big[F(t, x(t), x(t-\tau)) + Bu(t)\Big]dt \\ \qquad +\sigma(t, x(t), x(t-\tau))dW(t) + \int_{\mathbf{Z}} L(t, x(t), x(t-\tau), z)\widetilde{N}(dt, dz), \quad t_k \ne t \in J := [0,T], \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k \in \{1, 2, ..., m\}, \\ x(t) = \varphi(t) \in C_\tau = C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{H}), \quad -\tau \le t \le 0, \end{cases} \tag{8}$$

where the functions G, F, σ are defined as in Theorem 3.3; $L : J \times \mathbb{H} \times \mathbb{H} \times \mathbf{Z} \to \mathbb{H}$. Let $p = p(t)$, $t \in D_p$ (the domain of $p(t)$) be a stationary $\mathcal{F}_t$-Poisson point process taking its value in a measurable space $(\mathbf{Z}, \mathcal{B}(\mathbf{Z}))$ with a σ-finite intensity *measure $\lambda(dz)$ by $N(dt,dz)$ the Poisson counting measure associated with p, that is, $N(t,\mathbf{Z}) = \sum_{s\in D_p, s\le t} \mathbb{I}_{\mathbf{Z}}(p(s))$ for any measurable set $\mathbf{Z} \in \mathcal{B}(\mathbb{K}-\{0\})$, which denotes the Borel σ-field of $(\mathbb{K}-\{0\})$. Let*

$$\widetilde{N}(dt,dz) := N(dt,dz) - \lambda(dz)dt$$

be the compensated Poisson measure that is independent of $W(t)$.

An $\mathcal{F}_t$-adapted càdlàg stochastic process $x: J \to \mathbb{H}$ is called a mild solution of (8) if for each $u \in L_2^{\mathcal{F}}(J,\mathbb{U})$ and for arbitrary $t \in J$, $\mathbf{P}\{\omega : \int_J \|x(s)\|^2_{\mathbb{H}} ds < +\infty\} = 1$ it satisfies the integral equation

$$\begin{aligned} x(t) =& U(t,0)[\varphi(0) - G(0,\varphi)] + G(t,x(t-\tau)) + \int_0^t U(t,s)[F(s,x(s),x(s-\tau)) + Bu(s)]ds \\ &+ \int_0^t U(t,s)\sigma(s,x(s),x(s-\tau))dW(s) + \sum_{0<t_k<t} U(t,t_k)I_k(x(t_k^-)) \\ &+ \int_0^t \int_{\mathbf{Z}} U(t,s)L(s,x(s),x(s-\tau),z)\widetilde{N}(ds,dz) \end{aligned} \tag{9}$$

for any $x_0(\cdot) = \varphi(\cdot) \in C_\tau$.

By implementing appropriate assumptions on the functions, one can easily prove that by adapting and employing the techniques used in Theorem 3.3, the system (8) is approximately controllable on $[0,T]$.

4. An Example

In this section, an example is provided to illustrate the obtained theory. We consider the following stochastic classical heat equation with memory of the form:

$$\begin{cases} d[v(t,y) - g(t,v(t-\tau,y))] = \Big[\dfrac{\partial^2}{\partial y^2}v(t,y) + a(t,y)v(t,y) - g(t,v(t-\tau,y)\Big]dt + f(t,v(t,y)),v(t-\tau,y)dt \\ \qquad\qquad + \mu(t,y)dt + \widehat{\sigma}(t,v(t,y)),v(t-\tau,y)dW(t), \quad y\in[0,\pi], t_k \neq t \in J=[0,T], \\ v(t_k^+,y) - v(t_k^-,y) = I_k(v(t_k^-,y)), \quad k\in\{1,2,...,m\}, \\ v(t,0) = v(t,\pi) = 0 \quad t\in J, \\ v(\theta,\cdot) = \varphi(\theta,\cdot) \in \mathbb{H} = L^2[0,\pi], \quad \varphi(\cdot,y)\in C([-\tau,0];\mathbb{R}), \quad \theta\in[0,\pi], \end{cases} \tag{10}$$

where $W(t)$ is a real standard Wiener process in $\mathbb{H}$ defined on a stochastic basis $(\Omega,\mathcal{F},\mathbf{P})$.

We take $\mathbb{H} = \mathbb{K} = \mathbb{U} = L^2([0,\pi])$ with the norm $\|\cdot\|$. Define $A: \mathbb{H} \to \mathbb{H}$ by $Ax = x''$ with domain

$$D(A) = \{x(\cdot) \in \mathbb{H} : x, x' \text{ are absolutely continuous}, x'' \in \mathbb{H}, x(0) = x(\pi) = 0\}.$$

The spectrum of A consists of the eigenvalues $-n^2$ for $n \in \mathbb{N}$, with associated eigenvectors $e_n(y) := \sqrt{\frac{2}{\pi}}\sin ny$, $n = 1,2,3,....$ Furthermore, the set $\{e_n : n\in\mathbb{N}\}$ is an orthogonal basics in $\mathbb{H}$. Then

$$Ax = \sum_{n=1}^{\infty} n^2\langle x, e_n\rangle e_n, \quad x\in D(A).$$

It is wellknown that A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\ge 0}$ on $\mathbb{H}$ and is given (see Pazy [21], page 70) by

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n\rangle e_n, \quad x\in\mathbb{H}.$$

It follows from this representation that $S(t)$ is compact for every $t > 0$ and $\|S(t)\| \leq e^{-t}$ for every $t \geq 0$.

On the domain $D(A)$, we define the operators $A(t) : D(A) \subset \mathbb{H} \to \mathbb{H}$ by $A(t)z(y) = Az(y) + a(t, y)z(y)$. Let $a(\cdot,\cdot)$ be continuous and $a(t, y) \leq -\lambda$, $\lambda > 0$ for all $t \in J$, $y \in [0, \pi]$, it follows that the system

$$\begin{cases} dv(t) = A(t)v(t)dt, \quad t \geq s, \\ v(s) = z \in \mathbb{H}, \end{cases}$$

has an associated evolution family $\{U(t, s)\}_{t\geq s}$ as

$$U(t, s)z(y) = \left(S(t-s)e^{\int_s^t a(\rho,y)d\rho}z\right)(y).$$

From the above expression, it follows that $U(t, s)$ is a compact operator and every $t, s \in J$ with $t > s$

$$\|U(t, s)\| \leq e^{-(1+\lambda)(t-s)}.$$

Thus, (**H**1) is true.

Now, we define the linear continuous mapping B from

$$\mathbb{U} = \left\{u = \sum_{n=2}^{\infty} u_n e_n : \|u\|^2_{\mathbb{U}} := \sum_{n=2}^{\infty} u_n^2 < \infty\right\}$$

to $\mathbb{H}$ as follows:

$$Bu = 2u_2e_1 + \sum_{n=2}^{\infty} u_n e_n.$$

Put $x(t)(\cdot) = v(t, \cdot)$ and $u(t) = \mu(t, y)$ where $\mu(t, y) : J \times [0, \pi] \to [0, \pi]$ is continuous.

We choose $B = I$ the identity operator and define the operators $G(t, v)(\cdot) = g(t, v(\cdot))$, $F(t, v, v)(\cdot) = f(t, v(\cdot), v(\cdot))$, $\sigma(t, v, v)(\cdot) = \widehat{\sigma}(t, v(\cdot), v(\cdot))$. Then, under the above conditions, we can represent the stochastic control system (10) in the abstract form (1).

Assume that the linear operator L_0^T be defined by

$$(L_0^T u)(y) = \int_0^T S(T-s)e^{\int_s^t a(\rho,y)d\rho}\mu(s, y)ds.$$

On the other hand, because of the compactness of $U(t, s)$ generated by $A(t)$, the associated linear system of (10) is not exactly controllable but it is approximately controllable (see [16]). Therefore, we can conclude that the stochastic control system (10) is approximately controllable on $[0, T]$ provided that all the conditions of Theorem 3.3 are satisfied.

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