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# Convergence of Picard-Mann Hybrid Iterative Process for Generalized Nonexpansive Mappings in CAT(0) Spaces

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**Abstract.** In this paper, we first prove existence of fixed points of generalized nonexpansive mappings in CAT(0) spaces. These are the mappings which satisfy the so-called condition (E). We then approximate them by the $\triangle$-convergence and strong convergence using Picard-Mann hybrid iterative process. Our results generalize the corresponding results of many authors.

*To the memory of Professor Lj. Ćirić (1935–2016)*

## 1. Introduction

Let $T$ be a self-mapping of $C$, a nonempty convex subset of a normed space $X$. The Picard or successive iterative process [12] is defined by the sequence $\{x_n\}$ :

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Tx_n, \; n \in \mathbb{N}. \end{cases} \tag{1}$$

The Mann iterative process [11] is defined by the sequence $\{x_n\}$ :

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \; n \in \mathbb{N} \end{cases} \tag{2}$$

where $\{\alpha_n\}$ is in $(0,1)$.

The sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \; n \in \mathbb{N} \end{cases} \tag{3}$$

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where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$, is known as the Ishikawa iterative process [6]. This process can be seen as a "Double Mann iterative process" or "a hybrid of Mann process with itself".

In 2013, Khan [7] introduced an iterative process and named it as Picard-Mann hybrid iterative process as:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)\, x_n + \alpha_n T x_n, \ n \in \mathbb{N} \end{cases} \tag{4}$$

where $\{\alpha_n\}$ is in $(0,1)$. This process is independent of all Picard, Mann and Ishikawa iterative processes since $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$. Even if it is allowed to take $\alpha_n = 1$ to make it a special case of Ishikawa iterative process, our process is faster than Ishikawa and "faster is better" rule should prevail. He also proved the strong convergence and weak convergence theorems for the class of nonexpansive mappings in the Banach spaces.

In 2008, Suzuki [17] defined the condition (C) which is as follows: Let $C$ be a nonempty subset of a CAT(0) space $X$ and $T : C \to X$be a mapping. $T$ is said to satisfy condition (C) if $\frac{1}{2}d(x,Tx) \leq d(x,y)$ implies $d(Tx,Ty) \leq d(x,y)$ for all $x, y \in C$.

**Lemma 1.1.** *[17] Every nonexpansive mapping satisfies the condition (C). But the converse is not true.*

**Example 1.2.** *[17] Define a mapping $T$ on* $[0,3]$ *by*

$$T(x) = \begin{matrix} 0 & if\ x \neq 3, \\ 1 & if\ x = 3 \end{matrix}$$

*Then $T$ is single-valued mapping satisfying the condition (C) but $T$ is not nonexpansive.*

In 2011, Falset et al. [5] introduced the following definition of generalized nonexpansive mappings, that is, the mappings which satisfy the so-called condition (E).

Let $C$ be a nonempty subset of a Banach space $X$ and $T : C \to X$ be a single-valued mapping. Then $T$ is said to satisfy condition $(E_\mu)$ on $C$, if there exists $\mu \geq 1$ such that

$$||x - Ty|| \leq \mu ||Tx - x|| + ||x - y||$$

holds for all $x, y \in C$. $T$ is said to satisfy condition (E) on $C$ whenever $T$ satisfies the condition $(E_\mu)$ for some $\mu \geq 1$.

**Proposition 1.3.** *Every nonexpansive mapping satisfies the condition (E), but the converse is not true.*

**Proposition 1.4.** *[17] Let C be a bounded closed convex subset of a complete CAT(0) space X and $T : C \to X$ satisfies the condition (C). Then*

$$d(x,Ty) \leq 3d(Tx,x) + d(x,y)$$

*holds for all $x, y \in C$.*

From the above Proposition, it follows that condition (C) is the special case for $\mu = 3$ in condition (E). So the Example 1.2 also shows that there are mappings which satisfy condition (E) but they are not nonexpansive.

And here is an example which shows that there are mappings which satisfy condition (E) but not condition (C). Thus the mappings satisfying the condition (E) are more generalized than the mappings satisfying condition (C).

**Example 1.5.** *[2]In the space $X = C[0,1]$ under the supremum norm, consider a nonempty subset $K$ of $X$ defined as follows*

$$K = \{f \in C[0,1] : 0 = f(0) \leq f(x) \leq f(1) = 1\}.$$

*To any $g \in K$, associate a function $F_g : K \to K$ defined by $F_g(h(t)) = (goh)(t) = g(h(t))$. It is easy to verify that $F_g$ satisfies condition $(E_1)$ but does not satisfy condition (C).*

The purpose of this paper is two-fold: we first prove existence of fixed points of generalized nonexpansive mappings in CAT(0) spaces, and then approximate them by the $\triangle$-convergence and strong convergence using Picard-Mann hybrid iterative process. Our results generalize the corresponding results of many authors. Our results generalize many results existing in the literature such as those of Laokul and Panyanak [9], Laowang and Panyanak [10], Razani and Salahifard [13], Razani and Shabani [14], Shabani and Ghoncheh [16], Suzuki [17], Takahashi and Kim [18].

Now we convert the Picard -Mann hybrid iterative process to the CAT(0) space setting as: Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a mapping. Then we define the sequence $\{x_n\}$in $C$ iteratively as:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)\, x_n \oplus \alpha_n T x_n, \ n \in \mathbb{N} \end{cases} \tag{5}$$

where $\{\alpha_n\}$ is in $(0,1)$.

## 2. Preliminaries

Now we collect some elementary facts and results to make our presentation self-contained. Let $(X,d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0,l] \subset \mathbb{R}$ to $X$ such that $c(0) = x, c(l) = y$ and $d(c(t), c(t\prime)) = |t - t\prime|$ for all $t, t\prime \in [0,l]$. In particular, $c$ is an isometry and $d(x,y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique this geodesic segment is denoted by $[x,y]$. The space $(X,d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(X,d)$ consists of three points $x_1, x_2, x_3$ in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X,d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) = \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom: Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

If $x, y_1, y_2$ are points in a CAT(0) space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2} d(x, y_1)^2 + (1/2) d(x, y_2)^2 - (1/4) d(y_1, y_2)^2 \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [1]. In fact, a geodesic space is a CAT(0) space if and only if it satisfy (CN) inequality.

**Definition 2.1.** *[2]: Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = inf\{r(x, \{x_n\}) : x \in X\}$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.*

**Remark 2.2.** *In a CAT(0) space, $A(\{x_n\})$ consists of exactly one point as far as $\{x_n\}$ is a bounded sequence (see, e.g., [3], Proposition 7).*

**Definition 2.3.** *[2] A sequence $\{x_n\}$ in a CAT(0) space $X$ is said to $\triangle$-converge to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\triangle\text{-}\lim x_n = x$ and call $x$ the $\triangle$-limit of $\{x_n\}$. Note that given $\{x_n\} \subset X$ such that $\{x_n\}$ $\triangle$-converges to $x$ and given $y \in X$ with $y \neq x$, by uniqueness of the asymptotic center, we have*

$$\limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d(x_n, y).$$

Thus every CAT(0) space satisfies the Opial property.

**Lemma 2.4.** *[4] Let $X$ be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z) \tag{6}$$

for all $x, y, z \in X$ and $t \in [0,1]$.

**Lemma 2.5.** *[4] Let $(X, d)$ be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2 \tag{7}$$

for all $t \in [0,1]$ and $x, y, z \in X$.

**Lemma 2.6.** *[8] Every bounded sequence in a complete CAT(0) space always has a $\triangle$-convergent subsequence.*

**Lemma 2.7.** *[2] If $C$ is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in $C$ then the asymptotic center of $\{x_n\}$ is in $C$.*

The following Lemma is a consequence of Lemma 2.9 of [10] which will be used to prove our main results.

**Lemma 2.8.** *[10] Let $X$ be a complete CAT(0) space and $x \in X$. Suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0,1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in $X$ such that $\limsup_{n\to\infty} d(x_n, x) \leq r, \limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d(t_n y_n \oplus (1-t_n)x_n, x) = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

## 3. Main Results

We prove some lemmas needed for the development of our main theorems in this section.

**Lemma 3.1.** *Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to X$ be a mapping satisfying condition (E) and has a fixed point $p$. Then $T$ is a quasi-nonexpansive.*

*Proof.* Since T satisfies the condition (E) and has a fixed point $p$, therefore

$$d(p, Tx) \leq \mu d(Tp, p) + d(p, x) = d(p, x)$$

for all $x \in C$, $p \in F(T)$ and for some $\mu \geq 1$. Thus $T$ is quasi-nonexpansive. □

Following result guarantees the existence of fixed point of the mappings satisfying condition (E) in CAT(0) spaces.

**Lemma 3.2.** *Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a mapping satisfying condition (E). If $\{x_n\}$ is a sequence defined by (5). If $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $F(T)$ is nonempty.*

*Proof.* $A(\{x_n\})$ is singleton by Remark 2.2, because $\{x_n\}$ is bounded. Suppose that $A(\{x_n\}) = \{p\}$. By Lemma 2.7, $\{p\} \subset C$. Since $T$ satisfies condition (E),

$$d(x_n, Tp) \leq \mu d(x_n, Tx_n) + d(x_n, p).$$

This implies that

$$\limsup_{n\to\infty} d(x_n, Tp) \leq \limsup_{n\to\infty} \mu d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, p)$$

and hence $r(Tp, \{x_n\}) \leq r(p, \{x_n\})$. Since asymptotic center is unique, therefore $Tp = p$ and hence $F(T)$ is nonempty. □

**Lemma 3.3.** *Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a mapping satisfying condition (E). If $\{x_n\}$ is a sequence defined by* (5) *. Then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

*Proof.* Let $p \in F(T)$. Consider

$$\begin{aligned}
d(x_{n+1}, p) &= d(Ty_n, p) \\
&\leq \mu d(Tp, p) + d(y_n, p) \\
&= d(y_n, p) \\
&= d((1-\alpha_n)\, x_n \oplus \alpha_n T x_n, p) \\
&\leq (1-\alpha_n)\, d(x_n, p) + \alpha_n d(Tx_n, p) \\
&\leq (1-\alpha_n)\, d(x_n, p) + \alpha_n d(x_n, p) \\
&= d(x_n, p)
\end{aligned}$$

which shows that the sequence $\{d(x_n, p)\}$ is decreasing and bounded below so that $\lim_{n\to\infty} d(x_n, p)$ exists. □

**Lemma 3.4.** *Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a mapping satisfying condition (E). If $\{x_n\}$ is a sequence defined by* (5) *. Then $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

*Proof.* Let $p \in F(T)$ and by Lemma 3.3, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Assume that $\lim_{n\to\infty} d(x_n, p) = c$. We first prove that $\lim_{n\to\infty} d(y_n, p) = c$. Since $d(x_{n+1}, p) \leq d(y_n, p)$, therefore

$$\liminf_{n\to\infty} d(x_{n+1}, p) \leq \liminf_{n\to\infty} d(y_n, p)$$

and so $c \leq \liminf_{n\to\infty} d(y_n, p)$.

On the other hand, $d(y_n, p) \leq d(x_n, p)$ implies that $\limsup_{n\to\infty} d(y_n, p) \leq c$. Therefore, we get $\lim_{n\to\infty} d(y_n, p) = c$. Now, $d(Tx_n, p) \leq d(x_n, p)$ implies that $\lim_{n\to\infty} d(Tx_n, p) \leq c$. By using Lemma 2.8, we get $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

**Theorem 3.5.** *Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a mapping satisfying condition (E) with $F(T) \neq \emptyset$.Then the sequence $\{x_n\}$ defined by* (5) *$\triangle$-converges to a fixed point of $T$.*

*Proof.* By Lemma 3.4, we observe that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. We now let $\omega_w(x_n) = \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

To show the $\triangle$-convergence of $\{x_n\}$ to a fixed point of $T$, we claim that $\omega_w(x_n) \subset F(T)$ and is a singleton set. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.6 and 2.7, there exists a subsequence$\{v_n\}$ of $\{u_n\}$ such that $\triangle$- $\lim_{n\to\infty} v_n = v \in C$.

Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$ and $T$ satisfy the condition (E), there exists a $\mu \geq 1$ such that

$$d(v_n, Tv) \leq \mu d(v_n, Tv_n) + d(v_n, v).$$

By taking the lim sup of both sides, we have

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, Tv) &\le \limsup_{n\to\infty} \mu d(v_n, Tv_n) + d(v_n, v))\\ &\le \limsup_{n\to\infty} d(v_n, v).\end{aligned}$$

As $\triangle$- $\lim_{n\to\infty} v_n = v$, by the Opial property $\limsup_{n\to\infty} d(v_n, v) \le \limsup_{n\to\infty} d(v_n, Tv)$. Hence $Tv = v$, that is, $v \in F(T)$.

We claim that $u = v$. Suppose not, by the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u)\\ &\le \limsup_{n\to\infty} d(u_n, u)\\ &< \limsup_{n\to\infty} d(u_n, v)\\ &= \limsup_{n\to\infty} d(x_n, v)\\ &= \limsup_{n\to\infty} d(v_n, v)\end{aligned}$$

which is a contradiction and hence $u = v \in F(T)$. To show that $\{x_n\}$ $\triangle$-converges to a fixed point of $T$, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 2.6 and 2.7, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle$- $\lim_{n\to\infty} v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and$A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in F(T)$. We can complete the proof by showing that $x = v$. If $x \ne v$, then in view of Lemma 3.3, $\{d(v_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x)\\ &\le \limsup_{n\to\infty} d(x_n, x)\\ &\le \limsup_{n\to\infty} d(x_n, v)\\ &= \limsup_{n\to\infty} d(v_n, v)\end{aligned}$$

which is a contradiction and hence the conclusion follows. □

We have the following corollaries of the preceding theorem.

**Corollary 3.6.** *Let $C$ be a nonempty bounded, closed and convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a mapping satisfying condition (C) with $F(T) \ne \emptyset$. Then the sequence $\{x_n\}$ defined by* (5) *$\triangle$-converges to a fixed point of $T$.*

**Corollary 3.7.** *Let $C$ be a nonempty bounded, closed and convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a nonexpansive mapping with $F(T) \ne \emptyset$. Then the sequence $\{x_n\}$ defined by* (5) *$\triangle$-converges to a fixed point of $T$.*

We now turn our attention towards strong convergence theorems.

**Theorem 3.8.** *Let $X, C, T$ and $\{x_n\}$ be as in Theorem* 3.5. *Then $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ *where* $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$.

*Proof.* Necessityis obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 3.3, we have $d(x_{n+1}, p) \le d(x_n, p)$ for all $p \in F(T)$. This implies that $d(x_{n+1}, F(T)) \le d(x_n, F(T))$ so that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$.
Next we show that $\{x_n\}$ is a Cauchy sequence in $C$. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a positive integer $n_0$ such that $d(x_n, F(T)) < \frac{\varepsilon}{4}$ for all $n \ge n_0$. In particular, $\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\varepsilon}{4}$. Thus there must exist $p^* \in F(T)$ such that $d(x_{n_0}, p^*) < \frac{\varepsilon}{2}$.

Now for all $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &< d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq 2d(x_{n_0}, p^*) \\ &\leq 2\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset $C$ of a complete CAT(0) space and so it must converge to a point $q$ in $C$ and $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$ and closedness of $F(T)$ forces $q$ to be in $F(T)$. □

In 1974, Senter and Dotson [15] introduced the condition (I).

A mapping $T : C \to X$ is said to satisfy the condition (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$

**Theorem 3.9.** *Let $X, C, T$ and $\{x_n\}$ be as in Theorem* 3.5. *Let $T$ satisfy the condition (I), then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* From Lemma 3.4, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from the condition (I) that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is, $\lim_{n\to\infty} f(d(x_n, F(T)) = 0$. Since$f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$, therefore we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Now all the conditions of Theorem 3.8 are satisfied, therefore, by its conclusion, $\{x_n\}$ converges strongly to a fixed point of $F(T)$. □

**Remark 3.10.** *In view of previous discussion, the cases when the mapping satisfies Condition (C) or else is nonexpansive are now special cases of Theorem* 3.9.

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