Filomat 31:11 (2017), 3553–3558
https://doi.org/10.2298/FIL1711553O

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Essential Maps and Coincidence Principles for General Classes of Maps

Donal O'Regan[a]

[a] *School of Mathematics, Statistics and Applied Mathematics*
National University of Ireland, Galway
Ireland

Abstract. We introduce the notion of an essential map for a general class of maps. In addition we present homotopy and normalization type properties for these maps.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

In this paper we introduce the notion of a Φ–essential map for a variety of new classes of maps. First we present a homotopy type property i.e. if two maps G and F are in a class and G is Φ–essential and F is homotopic to G (in a certain sense) then F is Φ–essential. Also we present a normalization type property i.e. we present conditions which guarantee that a map is Φ–essential. The notion of an essential map was introduced by Granas [4] in 1976 and the notion was extended in a variety of settings by many authors (see [3, 6, 7] and the references therein). In particular the theory here is partly motivated by continuation thoerems for *DKT* and *PK* maps in [1, 2].

2. Main Results

Let E be a completely regular topological space and U an open subset of E.

We will consider classes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ of maps.

Definition 2.1. *We say $F \in D(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{D}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$); here 2^E denotes the family of nonempty subsets of E.*

Definition 2.2. *We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U}, E)$ and there exists a selection $\Psi \in D(\overline{U}, E)$ of F.*

In this section we <u>fix</u> a $\Phi \in B(\overline{U}, E)$.

2010 *Mathematics Subject Classification*. Primary 54H25, 47H10
Keywords. essential maps, coincidence points, topological principles
Received: 15 September 2016; Accepted: 18 April 2017
Communicated by Vladimir Rakočević
Email address: donal.oregan@nuigalway.ie (Donal O'Regan)

Definition 2.3. *We say $F \in A_{\partial U}(\overline{U}, E)$ (respectively $F \in D_{\partial U}(\overline{U}, E)$) if $F \in A(\overline{U}, E)$ (respectively $F \in D(\overline{U}, E)$) with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.*

Definition 2.4. *Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$ if for any selection $\Psi \in D(\overline{U}, E)$ of F and any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.*

Remark 2.5. *(i). Note in Definition 2.1 that Ψ is a selection of F if $\Psi(x) \subseteq F(x)$ for $x \in \overline{U}$.*

(ii). Note if $F \in A_{\partial U}(\overline{U}, E)$ is Φ–essential in $A_{\partial U}(\overline{U}, E)$ and if $\Psi \in D(\overline{U}, E)$ is any selection of F then there exists an $x \in U$ with $\Psi(x) \cap \Phi(x) \neq \emptyset$. To see this take $J = \Psi$ in Definition 2.4; note for $x \in \partial U$ that $\Psi(x) \cap \Phi(x) \subseteq F(x) \cap \Phi(x) = \emptyset$. Finally we note if $\Psi(x) \cap \Phi(x) \neq \emptyset$ for $x \in U$ then $\emptyset \neq \Psi(x) \cap \Phi(x) \subseteq F(x) \cap \Phi(x)$.

(iii). If $F \in \mathbf{A}(\overline{U}, E)$ and if $F \in D(\overline{U}, E)$ then an example of a selection Ψ of F in Definition 2.2 is F itself. In applications we see by appropriately choosing U, E, and D then automatically there exists a selection Ψ of F in Definition 2.2 (see for example Remark 2.8).

Theorem 2.6. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$ and let $G \in A_{\partial U}(\overline{U}, E)$ be Φ–essential in $A_{\partial U}(\overline{U}, E)$. For any selection $\Psi \in D(\overline{U}, E)$ (respectively $\Lambda \in D(\overline{U}, E)$) of F (respectively G) and for any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$ assume there exists a map $H^{\Psi,\Lambda,J}$ defined on $\overline{U} \times [0,1]$ with values in E with $H^{\Psi,\Lambda,J}(\,.\,, \eta(\,.\,)) \in D(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t^{\Psi,\Lambda,J}(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $\left\{x \in \overline{U} : \ \Phi(x) \cap H^{\Psi,\Lambda,J}(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$ is compact (respectively closed) and $H_0^{\Psi,\Lambda,J} = \Lambda$, $H_1^{\Psi,\Lambda,J} = J$; here $H_t^{\Psi,\Lambda,J}(x) = H^{\Psi,\Lambda,J}(x,t)$. Then F is Φ–essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. Let $\Psi \in D(\overline{U}, E)$ (respectively $\Lambda \in D(\overline{U}, E)$) be any selection of F (respectively G). Now consider any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$. We must show there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$. Choose the map $H^{\Psi,\Lambda,J}$ as in the statement of Theorem 2.6. Consider

$$\Omega = \left\{x \in \overline{U} : \ \Phi(x) \cap H^{\Psi,\Lambda,J}(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Note $\Omega \neq \emptyset$ since $H_0^{\Psi,\Lambda,J} = \Lambda$ and G is Φ–essential in $A_{\partial U}(\overline{U}, E)$; to see this note $\Lambda(x) \cap \Phi(x) \subseteq G(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$, and from Remark 2.5 (ii) there exists $y \in U$ with $\Lambda(y) \cap \Phi(y) \neq \emptyset$. Also Ω is compact (respectively closed) if E is a completely regular (respectively normal) topological space. Next note $\Omega \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map R_μ by

$$R_\mu(x) = H^{\Psi,\Lambda,J}(x, \mu(x)) = H_{\mu(x)}^{\Psi,\Lambda,J}(x).$$

Note $R_\mu \in D_{\partial U}(\overline{U}, E)$ and $R_\mu|_{\partial U} = H_0^{\Psi,\Lambda,J}|_{\partial U} = \Lambda|_{\partial U}$. Now since G is Φ–essential in $A_{\partial U}(\overline{U}, E)$ then there exists $x \in U$ with $R_\mu(x) \cap \Phi(x) \neq \emptyset$ (i.e. $H_{\mu(x)}^{\Psi,\Lambda,J}(x) \cap \Phi(x) \neq \emptyset$) and so $x \in \Omega$. As a result $\mu(x) = 1$ so $\emptyset \neq H_1^{\Psi,\Lambda,J}(x) \cap \Phi(x) = J(x) \cap \Phi(x)$, and we are finished. □

Note Theorem 2.6 is a homotopy type result. Next we present a number of normalization type results i.e. results which guarantee that a map is Φ–essential in $A_{\partial U}(\overline{U}, E)$. In our first result E is a topological vector space and $\Phi \in B(E, E)$ is fixed (we say $\Phi \in B(E, E)$ if $\Phi : E \to 2^E$ and $\Phi \in \mathbf{B}(E, E)$).

Theorem 2.7. *Let E be a topological vector space, U an open subset of E and $\Phi \in B(E, E)$. Assume the following conditions hold:*

$$0 \in A_{\partial U}(\overline{U}, E) \text{ and } 0 \in D(\overline{U}, E) \text{ where } 0 \text{ denotes the zero map} \tag{2.1}$$

$$\begin{cases} \text{for any map } J \in D_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = \{0\} \text{ and} \\ R^J(x) = \begin{cases} J(x), & x \in \overline{U} \\ \{0\}, & x \in E \setminus \overline{U}, \end{cases} \\ \text{there exists } y \in E \text{ with } \Phi(y) \cap R^J(y) \neq \emptyset \end{cases} \tag{2.2}$$

and

$$\text{(2.3)} \qquad \textit{there is no } \ z \in E \setminus \overline{U} \ \textit{ with } \ \Phi(z) \cap \{0\} \neq \emptyset.$$

Then the zero map is Φ*–essential in* $A_{\partial U}(\overline{U}, E)$.

Proof. Let $F(x) = 0$ for $x \in \overline{U}$ (i.e. F is the zero map) and let $\Psi \in D(\overline{U}, E)$ be any selection of F. Note Ψ is the zero map (note $\Psi : \overline{U} \to 2^E$, $\Psi(x) \subseteq F(x)$ for $x \in \overline{U}$ and $0 \in D(\overline{U}, E)$). Consider any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \{0\}$. To show that the zero map is Φ–essential in $A_{\partial U}(\overline{U}, E)$ we must show there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$. Let

$$R^J(x) = \begin{cases} J(x), & x \in \overline{U} \\ \{0\}, & x \in E \setminus \overline{U}. \end{cases}$$

Now (2.2) guarantees that there exists $y \in E$ with $\Phi(y) \cap R^J(y) \neq \emptyset$. There are two cases to consider, namely $y \in U$ and $y \in E \setminus U$. If $y \in U$ then $\Phi(y) \cap J(y) \neq \emptyset$, and we are finished. If $y \in E \setminus U$ then since $R^J(y) = \{0\}$ (note $J|_{\partial U} = \{0\}$) we have $\Phi(y) \cap \{0\} \neq \emptyset$, and this contradicts (2.3). □

Remark 2.8. *We first recall the PK maps from the literature. Let Z and W be subsets of Hausdorff topological vector spaces* Y_1 *and* Y_2 *and F a multifunction. We say* $F \in PK(Z, W)$ *if W is convex and there exists a map* $S : Z \to W$ *with* $Z = \cup \{int\, S^{-1}(w) : \ w \in W\}$, $co\,(S(x)) \subseteq F(x)$ *for* $x \in Z$ *and* $S(x) \neq \emptyset$ *for each* $x \in Z$*; here* $S^{-1}(w) = \{z : \ w \in S(z)\}$.

Let E be a locally convex Hausdorff topological vector space, U an open subset of E, $0 \in U$, $\overline{U}$ *paracompact and* $\Phi = I$ *(the identity map). In this case we let* $\mathbf{D} = D$ *and* $\mathbf{A} = A$*. We say* $Q \in D(\overline{U}, E)$ *if* $Q : \overline{U} \to E$ *is a continuous compact map. We say* $F \in A(\overline{U}, E)$ *if* $F \in PK(\overline{U}, E)$ *and F is a compact map (the existence of a continuous selection* Ψ *of F is guaranteed from [5, Theorem 1.3] and note* Ψ *is compact since* Ψ *is a selection of F and F is compact, so* $\Psi \in D(\overline{U}, E)$*). Notice* (2.1), (2.2) *(see [2, 5] or note that it is immediate from Schauder's fixed point theorem) and* (2.3) *(note* $0 \in U$ *and* $\Phi = I$*) hold. We note that a "compact" map above could be replaced by a more general "compactness type" map; see [2].*

Remark 2.9. *We note here that an assumption was inadvertently left out in [1, 2]. In [1] the continuous selection* Ψ *of F should be required to satisfy Property (A) (this was inadvertently left out) i.e. if F satisfies Property (A) then it should be assumed that any continuous selection* Ψ *of F satisfies Property (A) (of course this assumption is automatically satisfied for the type of map considered in the literature i.e. Property (A) usually means that the map is compact or condensing).*

Theorem 2.10. *Let E be a Hausdorff topological space, U an open subset of E,* $\Phi \in B(E, E)$ *and* $F \in A_{\partial U}(\overline{U}, E)$*. Assume the following conditions hold:*

$$\text{(2.4)} \qquad \begin{cases} \textit{there exists a retraction } \ r : E \to \overline{U} \\ \textit{with } \ r(w) \in \partial U \ \textit{ if } \ w \in E \setminus U \end{cases}$$

$$\text{(2.5)} \qquad \begin{cases} \textit{for any selection } \ \Psi \in D(\overline{U}, E) \ \textit{ of } \ F \ \textit{ and any} \\ \textit{map } \ J \in D_{\partial U}(\overline{U}, E) \ \textit{ with } \ J|_{\partial U} = \Psi|_{\partial U} \ \textit{ there} \\ \textit{exists } \ y \in E \ \textit{ with } \ J\,r(y) \cap \Phi(y) \neq \emptyset \end{cases}$$

and

$$\text{(2.6)} \qquad \begin{cases} \textit{for any selection } \ \Psi \in D(\overline{U}, E) \ \textit{ of } \ F \ \textit{there} \\ \textit{is no } \ y \in E \setminus U \ \textit{ and } \ z \in \partial U \ \textit{ with } \ z = r(y) \\ \textit{and } \ \Psi(z) \cap \Phi(y) \neq \emptyset. \end{cases}$$

Then F is Φ*–essential in* $A_{\partial U}(\overline{U}, E)$.

Proof. Let $\Psi \in D(\overline{U}, E)$ be any selection of F and consider any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$. Now (2.5) guarantees that there exists a $y \in E$ with $J\,r(y) \cap \Phi(y) \neq \emptyset$. Let $z = r(y)$ and note $J(z) \cap \Phi(y) \neq \emptyset$. There are two cases to consider, namely $y \in U$ and $y \in E \setminus U$. If $y \in U$ then $z = r(y) = y$ so $J(y) \cap \Phi(y) \neq \emptyset$, and we are finished. If $y \in E \setminus U$ then $z \in \partial U$ and note

$$\emptyset \neq J(z) \cap \Phi(y) = \Psi(z) \cap \Phi(y) \text{ since } J|_{\partial U} = \Psi|_{\partial U},$$

and this contradicts (2.6). □

Remark 2.11. *Note there is a "dual" version of Theorem 2.10 if we consider $r\,J$ instead of $J\,r$ (above). Let $\Phi \in B(\overline{U}, E)$, $F \in A_{\partial U}(\overline{U}, E)$ and suppose* (2.4) *holds. In addition assume the following conditions hold:*

$$\begin{cases} \textit{for any selection } \Psi \in D(\overline{U}, E) \textit{ of } F \textit{ and any} \\ \textit{map } J \in D_{\partial U}(\overline{U}, E) \textit{ with } J|_{\partial U} = \Psi|_{\partial U} \textit{ there} \\ \textit{exists } w \in \overline{U} \textit{ with } r\,J(w) \cap \Phi(w) \neq \emptyset \end{cases}$$

and

$$\begin{cases} \textit{for any selection } \Psi \in D(\overline{U}, E) \textit{ of } F \textit{ and any} \\ \textit{map } J \in D_{\partial U}(\overline{U}, E) \textit{ with } J|_{\partial U} = \Psi|_{\partial U} \textit{ there} \\ \textit{is no } z \in E \setminus U \textit{ and } y \in \overline{U} \textit{ with } z \in J(y) \\ \textit{and } r(z) \in \Phi(y). \end{cases}$$

Then F is Φ–essential in $A_{\partial U}(\overline{U}, E)$.

Theorem 2.12. *Let E be a topological vector space (so automatically completely regular), U an open subset of E, $\Phi \in B(E, E)$ and $F \in A_{\partial U}(\overline{U}, E)$. Assume the following conditions hold:*

$$\textit{there exists } x \in \overline{U} \textit{ with } \Phi(x) \cap \{0\} \neq \emptyset \tag{2.7}$$

$$\begin{cases} \textit{for any selection } \Psi \in D(\overline{U}, E) \textit{ of } F \textit{ and any} \\ \textit{map } J \in D_{\partial U}(\overline{U}, E) \textit{ with } J|_{\partial U} = \Psi|_{\partial U} \textit{ then} \\ \Phi(x) \cap \lambda\,\Psi(x) = \emptyset \textit{ for } x \in \partial U \textit{ and } \lambda \in [0,1] \textit{ and} \\ \Omega = \left\{x \in \overline{U} : \Phi(x) \cap \lambda\,J(x) \neq \emptyset \textit{ for some } \lambda \in [0,1]\right\} \\ \textit{is compact} \end{cases} \tag{2.8}$$

$$\textit{there exists a retraction } r : E \to \overline{U} \tag{2.9}$$

$$\begin{cases} \textit{for any continuous map } \mu : E \to [0,1] \textit{ with } \mu(E \setminus U) = 0 \\ \textit{and any selection } \Psi \in D(\overline{U}, E) \textit{ of } F \textit{ and any map} \\ J \in D_{\partial U}(\overline{U}, E) \textit{ with } J|_{\partial U} = \Psi|_{\partial U} \textit{ there exists} \\ x \in E \textit{ with } \Phi(x) \cap \mu(x)\,J(r(x)) \neq \emptyset \end{cases} \tag{2.10}$$

and

$$\textit{there is no } z \in E \setminus U \textit{ with } \Phi(z) \cap \{0\} \neq \emptyset. \tag{2.11}$$

Then F is Φ–essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Let $\Psi \in D(\overline{U}, E)$ be any selection of F and consider any map $J \in D_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = \Psi|_{\partial U}$. Let

$$\Omega = \left\{x \in \overline{U} : \Phi(x) \cap \lambda\,J(x) \neq \emptyset \text{ for some } \lambda \in [0,1]\right\}.$$

Now (2.7) and (2.8) guarantee that $\Omega \neq \emptyset$ is compact and $\Omega \subseteq \overline{U}$. We claim $\Omega \subseteq U$. To see this let $x \in \Omega$ and $x \in \partial U$. Then since $J|_{\partial U} = \Psi|_{\partial U}$ we have $\Phi(x) \cap \lambda\,\Psi(x) \neq \emptyset$, and this contradicts (2.8). Now there exists a continuous map $\mu : E \to [0,1]$ with $\mu(E \setminus U) = 0$ and $\mu(\Omega) = 1$. Let r be as in (2.9) and (2.10) guarantees that there exists $x \in E$ with $\Phi(x) \cap \mu(x)\,J(r(x)) \neq \emptyset$. If $x \in E \setminus U$ then $\mu(x) = 0$ so $\Phi(x) \cap \{0\} \neq \emptyset$, and this contradicts (2.11). Thus $x \in U$ so $\Phi(x) \cap \mu(x)\,J(x) \neq \emptyset$. Hence $x \in \Omega$ so $\mu(x) = 1$ and consequently $\Phi(x) \cap J(x) \neq \emptyset$. □

Remark 2.13. *If in Theorem 2.12 the space E is normal then the assumption that* Ω *is compact in* (2.8) *can be replaced by* Ω *is closed.*

Remark 2.14. *We say* $F \in MA(\overline{U}, E)$ *if* $F : \overline{U} \to 2^E$ *and* $F \in \mathbf{A}(\overline{U}, E)$ *and we say* $F \in MA_{\partial U}(\overline{U}, E)$ *if* $F \in MA(\overline{U}, E)$ *with* $F(x) \cap \Phi(x) = \emptyset$ *for* $x \in \partial U$. *Now we say* $F \in MA_{\partial U}(\overline{U}, E)$ *is* Φ*–essential in* $MA_{\partial U}(\overline{U}, E)$ *if for every map* $J \in MA_{\partial U}(\overline{U}, E)$ *with* $J|_{\partial U} = F|_{\partial U}$ *there exists* $x \in U$ *with* $J(x) \cap \Phi(x) \neq \emptyset$. *There are obvious analogues of Theorem's 2.6, 2.7, 2.10, 2.12 for MA maps (these statements are left to the reader). For example the analogue of Theorem 2.10 is: Suppose* $\Phi \in B(E, E)$, $F \in MA_{\partial U}(\overline{U}, E)$ *with* (2.4) *and the following conditions holding:*

$$\begin{cases} \text{for any map } \ J \in MA_{\partial U}(\overline{U}, E) \ \text{ with } \ J|_{\partial U} = F|_{\partial U} \\ \text{there exists } \ y \in E \ \text{ with } \ J\,r(y) \cap \Phi(y) \neq \emptyset \end{cases}$$

and

$$\begin{cases} \text{there is no } \ y \in E \setminus U \ \text{ and } \ z \in \partial U \\ \text{with } \ z = r(y) \ \text{and } \ F(z) \cap \Phi(y) \neq \emptyset. \end{cases}$$

Then F *is* Φ*–essential in* $MA_{\partial U}(\overline{U}, E)$.

We now show that the ideas in this section can be applied to other natural situations. Let E be a Hausdorff topological vector space, Y a topological vector space, and U an open subset of E. Also let $L : dom\,L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\,L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear single valued map with $L + T : dom\,L \to Y$ a bijection; for convenience we say $T \in H_L(E, Y)$.

Definition 2.15. *We say* $F \in D(\overline{U}, Y; L, T)$ *(respectively* $F \in B(\overline{U}, Y; L, T)$*) if* $F : \overline{U} \to 2^Y$ *and* $(L + T)^{-1}(F + T) \in D(\overline{U}, E)$ *(respectively* $(L + T)^{-1}(F + T) \in B(\overline{U}, E)$*).*

Definition 2.16. *We say* $F \in A(\overline{U}, Y; L, T)$ *if* $F : \overline{U} \to 2^Y$ *and* $(L + T)^{-1}(F + T) \in \mathbf{A}(\overline{U}, E)$ *and there exists a selection* $\Psi \in D(\overline{U}, Y; L, T)$ *of* F.

In this section we fix a $\Phi \in B(\overline{U}, Y; L, T)$.

Definition 2.17. *We say* $F \in A_{\partial U}(\overline{U}, Y; L, T)$ *(respectively* $F \in D_{\partial U}(\overline{U}, Y; L, T)$*) if* $F \in A(\overline{U}, Y; L, T)$*) (respectively* $F \in D(\overline{U}, Y; L, T)$*)) with* $(L + T)^{-1}(F + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ *for* $x \in \partial U$.

Definition 2.18. *Let* $F \in A_{\partial U}(\overline{U}, Y; L, T)$. *We say* $F : \overline{U} \to 2^Y$ *is L–*Φ*–essential in* $A_{\partial U}(\overline{U}, Y; L, T)$ *if for any selection* $\Psi \in D(\overline{U}, Y; L, T)$ *of* F *and any map* $J \in D_{\partial U}(\overline{U}, Y; L, T)$ *with* $J|_{\partial U} = \Psi|_{\partial U}$ *there exists* $x \in U$ *with* $(L + T)^{-1}(J + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$.

Theorem 2.19. *Let* E *be a topological vector space (so automatically completely regular),* Y *a topological vector space,* U *an open subset of* E, $L : dom\,L \subseteq E \to Y$ *a linear single valued map and* $T \in H_L(E, Y)$. *Let* $F \in A_{\partial U}(\overline{U}, Y; L, T)$ *and let* $G \in A_{\partial U}(\overline{U}, Y; L, T)$ *be L–*Φ*–essential in* $A_{\partial U}(\overline{U}, Y; L, T)$. *For any selection* $\Psi \in D(\overline{U}, Y; L, T)$ *(respectively* $\Lambda \in D(\overline{U}, Y; L, T)$*) of* F *(respectively* G*) and for any map* $J \in D_{\partial U}(\overline{U}, Y; L, T)$ *with* $J|_{\partial U} = \Psi|_{\partial U}$ *assume there exists a map* $H^{\Psi,\Lambda,J}$ *defined on* $\overline{U} \times [0,1]$ *with values in* Y *with* $(L + T)^{-1}(H^{\Psi,\Lambda,J}(\,.\,, \eta(\,.\,)) + T(\,.\,)) \in D(\overline{U}, E)$ *for any* *continuous function* $\eta : \overline{U} \to [0,1]$ *with* $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t^{\Psi,\Lambda,J} + T)(x) \cap (L+T)^{-1}(\Phi + T)(x) = \emptyset$ *for any* $x \in \partial U$ *and* $t \in (0,1)$, $H_0^{\Psi,\Lambda,J} = \Lambda$, $H_1^{\Psi,\Lambda,J} = J$ *(here* $H_t^{\Psi,\Lambda,J}(x) = H^{\Psi,\Lambda,J}(x,t)$*) and*

$$\left\{x \in \overline{U} :\ (L+T)^{-1}(\Phi + T)(x) \cap (L+T)^{-1}(H_t^{\Psi,\Lambda,J} + T)(x) \neq \emptyset \ \textit{ for some } \ t \in [0,1]\right\}$$

is compact. Then F is L–Φ–essential in $A_{\partial U}(\overline{U}, Y; L, T)$.

Proof. Let $\Psi \in D(\overline{U}, Y; L, T)$ (respectively $\Lambda \in D(\overline{U}, Y; L, T)$) be any selection of F (respectively G) and consider the map $J \in D_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = \Psi|_{\partial U}$. Choose the map $H^{\Psi,\Lambda,J}$ as in the statement of Theorem 2.19. Consider

$$\Omega = \left\{x \in \overline{U} :\ (L+T)^{-1}(\Phi + T)(x) \cap (L+T)^{-1}(H_t^{\Psi,\Lambda,J} + T)(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Note $\Omega \neq \emptyset$ is compact and $\Omega \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map R_μ by $R_\mu(x) = H^{\Psi,\Lambda,J}(x, \mu(x)) = H_{\mu(x)}^{\Psi,\Lambda,J}(x)$. Note $R_\mu \in D_{\partial U}(\overline{U}, Y; L, T)$ and $R_\mu|_{\partial U} = H_0^{\Psi,\Lambda,J}|_{\partial U} = \Lambda|_{\partial U}$. Now since G is L–Φ–essential in $A_{\partial U}(\overline{U}, Y; L, T)$ there exists $x \in U$ with $(L+T)^{-1}(R_\mu + T)(x) \cap (L+T)^{-1}(\Phi + T)(x) \neq \emptyset$, and so $x \in \Omega$. As a result $\mu(x) = 1$ so

$$\begin{aligned} \emptyset &\neq (L+T)^{-1}(H_1^{\Psi,\Lambda,J} + T)(x) \cap (L+T)^{-1}(\Phi + T)(x) \\ &= (L+T)^{-1}(J + T)(x) \cap (L+T)^{-1}(\Phi + T)(x), \end{aligned}$$

and we are finished. □

Remark 2.20. *If in Theorem 2.19 the space E is additionally normal then the assumption that*

$$\Omega = \left\{x \in \overline{U} :\ (L+T)^{-1}(\Phi + T)(x) \cap (L+T)^{-1}(H_t^{\Psi,\Lambda,J} + T)(x) \neq \emptyset \ \textit{ for some } \ t \in [0,1]\right\}$$

is compact in the statement of Theorem 2.19 can be replaced by Ω *is closed.*

Remark 2.21. *We say* $F \in MA(\overline{U}, Y; L, T)$ *if* $F : \overline{U} \to 2^Y$ *and* $(L+T)^{-1}(F + T) \in \mathbf{A}(\overline{U}, E)$ *and we say* $F \in MA_{\partial U}(\overline{U}, Y; L, T)$ *if* $F \in MA(\overline{U}, Y; L, T)$) *with* $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ *for* $x \in \partial U$. *Now we say* $F \in MA_{\partial U}(\overline{U}, Y; L, T)$ *is L–Φ–essential in* $MA_{\partial U}(\overline{U}, Y; L, T)$ *if for every map* $J \in MA_{\partial U}(\overline{U}, Y; L, T)$ *with* $J|_{\partial U} = F|_{\partial U}$ *there exists* $x \in U$ *with* $(L+T)^{-1}(J+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) \neq \emptyset$. *There is an obvious analogue of Theorem 2.19 for MA maps.*

References

[1] R.P. Agarwal, D. O'Regan, Continuation methods for closed, weakly closed, DKT and WDKT maps, Computers and Mathematics with Applications 38 (1999) 81–88.

[2] R.P. Agarwal and D. O'Regan, An essential map theory for $\mathcal{U}_c^\kappa$ and PK maps, Topological Methods in Nonlinear Analysis 21 (2003) 375–386.

[3] G. Gabor, L. Gorniewicz and M. Slosarski, Generalized topological essentiality and coincidence points of multivalued maps, Set-Valued Analysis 17 (2009) 1–19.

[4] A. Granas, Sur la méthode de continuité de Poincaré, C.R. Acad. Sci. Paris 282 (1976) 983–985.

[5] D. O'Regan, Fixed point theorems for the $\mathcal{B}^\kappa$–admissible maps of Park, Applicable Analysis 79 (2001) 173–185.

[6] D. O'Regan, Coincidence points for multivalued maps based on Φ-epi and Φ-essential maps, Dynamic Systems and Applications 24 (2015) 143–154.

[7] R. Precup, On the topological transversality principle, Nonlinear Analysis 20 (1993) 1–9.