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Distance in the Absolute Plane and Cauchy Functional Equations

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Abstract. Let $\mathbb{A}$ denotes the absolute plane and d_a the distance function on it. Using a constructive approach which leads to the functional equations, we study which conditions on a "measure" of segments on a given half-line l in the absolute plane are essential to be the restriction of d_a on l.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

We consider two well-known models of the absolute plane which we call E-model and H-model. The E-model is actually the standard model of the Euclidean plane and the H-model is the Poincaré disk model of the hyperbolic plane. Proposition 1 states that there is a distance d_a (which is in accordance with the relations between and congruence) on the absolute plane and that it is unique up to a multiplicative constant. Let d_e and d_h be the interpretations of d_a in the E-model and in the H-model, respectively.

We consider a fixed half-line l ($l = [0, 1)$ in the H-model and $l = [0, +\infty)$ in the E-model) and a function $f : l \to [0, +\infty)$ with some additivity properties. Next, denote by f_e and f_h restrictions of the functions $d_e(0, \cdot)$ and $d_h(0, \cdot)$ on l.

In the H-model (E-model) we use inversions of the unit disk (translations of the Euclidean plane) which maps l into itself (a constructive approach which leads to the functional equations) and we consider some conditions which are at the first glance less restrictive than the standard properties of the Euclidean or hyperbolic distance function. More precisely, for example, we get Theorem 2 which shows the reduced additivity property on l related to the divisions of segments of the form $[0, x]$, where $x \in l$, only on two and three congruent parts is equivalent to full additivity property.

Roughly speaking, we show that the reduced additivity property on l, related to the divisions of segments only on two congruent parts, leads to consider the functional equation

$$F(2t) = 2F(t), \tag{1}$$

where $F : [0, +\infty) \to [0, +\infty)$ is some unknown function, which satisfies:

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(I1) F is strictly increasing and continuous function.

Thus, our consideration is related to the next problem (which is stated as Problem 2 in Section 3).

Problem 1. *Find a solution of equation* (1) *which satisfies* (I1).

One could check immediately that for arbitrary constant $c > 0$ the function $F : [0, +\infty) \to [0, +\infty)$, that is given with

$$F(t) = ct, \tag{2}$$

is a solution of the Problem 1.

In the Section 4 we will show that the function given by (2) is not the unique solution of the Problem 1 in general.

In particular we derive the formulae for distances in Euclidean and hyperbolic plane with some novelty (see Theorem 1, Section 3). For an approach using Schwartz lemma see [9] and [8].

In addition, in Section 5 we prove Theorem 3 which states: Let a_n, $n \geqslant 0$, be a decreasing sequence in $(0, 1)$ such that a_{n+1} is the H-midpoint of the segment with endpoints 0 and a_n, and let $A = \{a_n : n \geqslant 0\}$. If $f : [0, 1) \to [0, +\infty)$ is a strictly increasing continuous function satisfies reduced additivity iteration property

(AD1) $f(T_a^k(0)) = kf(a)$, for all $k \in \mathbb{N}$ and $a \in A$.

Then, there is a constant $c > 0$ such that $f = cf_{\mathrm{h}}$ on $[0, 1)$.

2. The E-Model and the H-Model of Absolute Plane Geometry

For the sake of completeness in this section we give a brief description of the E-model and the H-model of the absolute plane geometry. It is known that a model of absolute plane geometry is obtained if we interpret the primitive terms (points and straight lines) and the basic relations (incidence relation, between and congruence) with the request that the axioms of incidence, the axioms of order, the axioms of congruence and the axioms of continuity from Hilbert's axiom system of absolute plane geometry to be satisfied.

In the E-model we assume that the points are elements from the set $\mathbb{C}$, the straight lines are the sets of points in $\mathbb{C}$ defined by the equations of the form $\bar{a}z + a\bar{z} + c = 0$, where $a \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{R}$. Similarly, in the H-model as a point we consider any element of the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the straight lines consist of all arcs of Euclidean circles contained within the disk $\mathbb{D}$, that are orthogonal to the boundary of the disk $\mathbb{D}$, and all diameters of the disk $\mathbb{D}$.

Before we give an interpretation of the relations incidences, between and congruence we give the definition of isometries in the E-model and in the H-model. We define the family of isometries in the E-model (shortly E-isometries) as the family of all mappings of $\mathbb{C}$ onto itself of the forms

$$z \mapsto e^{i\theta} z + a \qquad \text{or} \qquad z \mapsto e^{i\theta}\bar{z} + a,$$

where $a \in \mathbb{C}$ and $\theta \in [0, 2\pi)$. An isometry in the H-model (shortly H-isometry) is any self mapping of the disk $\mathbb{D}$ that is of the form

$$z \mapsto e^{i\theta}\frac{z + a}{1 + \bar{a}z} \qquad \text{or} \qquad z \mapsto e^{i\theta}\frac{\bar{z} + a}{1 + \overline{az}},$$

where $a \in \mathbb{D}$ and $\theta \in [\,0, 2\pi)$. In particular, set $T_a(z) = \dfrac{z + a}{1 + \bar{a}z}$.

In the both models we say that a point z is incident with a straight line p if $z \in p$. Further, we will say that a point z is between the points z_1 and z_2 if in the corresponding model there is an isometry φ such that $\varphi(z)$, $\varphi(z_1)$ and $\varphi(z_2)$ are real numbers and $0 = \varphi(z_1) < \varphi(z) < \varphi(z_2)$. Finally, for a pair of points (z_1, z_2) we say that it is congruent to a pair of points (w_1, w_2) if in the corresponding model there exists an isometry φ with the properties $\varphi(z_1) = w_1$ and $\varphi(z_2) = w_2$.

It is clear that the E-model and the H-model are both the models of absolute plane geometry, but in the E-model holds the Playfair's parallel axiom, while in the H-model the Bolyai-Lobachevsky parallel axiom is satisfied. Thus, these models as the models of the absolute plane geometry are not isomorphic.

Using the stereographic projection from Poincaré disk model of absolute plane geometry one can get Beltrami-Klein model of absolute plane geometry. The Gans model [5] is obtained by use an orthographic projection of the hyperboloid model onto the plane.

This model utilizes the entire Euclidean plane and it is not as widely used as other models but nevertheless is quite useful in the understanding of hyperbolic geometry.

3. The Distance in the Absolute Plane Geometry

For the details related to the hyperbolic geometry see for example [2, 3, 6, 9].

Our starting point is the following statement of absolute geometry which is slightly modification of the Theorem 4.3.B in [6].

Proposition 1. *Let $\mathbb{A}$ be the set of all points in absolute plane. Then on $\mathbb{A}$ there is a distance function (a metric) $d_{\rm a} : \mathbb{A} \times \mathbb{A} \to [0, +\infty)$ with the following properties:*

(A1) *if $a, b, c, d \in \mathbb{A}$ such that the pair of points (a, b) is congruent with the pair of points (c, d), then $d_{\rm a}(a, b) = d_{\rm a}(c, d)$;*

(A2) *if $a, b, c \in \mathbb{A}$ such that the point b is between the points a and c, then $d_{\rm a}(a, c) = d_{\rm a}(a, b) + d_{\rm a}(b, c)$.*

Furthermore, if there is another distance function $d'_{\rm a} : \mathbb{A} \times \mathbb{A} \to [0, +\infty)$ which satisfies the properties (A1) *and* (A2), *then there is $\lambda > 0$ such that $d'_{\rm a} = \lambda d_{\rm a}$.*

Let $d_{\rm e}$ and $d_{\rm h}$ be the interpretations of $d_{\rm a}$ in the E-model and in the H-model, respectively. We define $\mathrm{Euc}(z) = d_{\rm e}(0, z)$, $z \in \mathbb{C}$, and $\mathrm{Hyp}(z) = d_{\rm h}(0, z)$, $z \in \mathbb{D}$, respectively. Then, as a corollary of the property (A2), the restriction $f_{\rm h}$ of Hyp on $[0, 1)$ (the restriction $f_{\rm e}$ of Euc on $[0, \infty)$) has $k-$hyp ($k-$euc) additivity property: if for all points $c_0 = 0 < a = c_1 < c_2 < \ldots < c_k = b < 1$ ($c_0 = 0 < a = c_1 < c_2 < \ldots < c_k = b < +\infty$), such that the segments c_0c_1, c_1c_2, …, $c_{k-1}c_k$ are H-congruent (E-congruent), then $f_{\rm h}(b) = kf_{\rm h}(a)$ ($f_{\rm e}(b) = kf_{\rm e}(a)$) holds. We say that a strictly increasing and continuous function $f : [0, 1) \to [0, +\infty)$ ($f : [0, +\infty) \to [0, +\infty)$) satisfies the reduced additivity property in the H-model (E-model) if it has $2-$hyp and $3-$hyp ($2-$euc and $3-$euc) additivity properties. In opposite direction, we show that if a function f satisfies the reduced additivity property in the H-model (E-model), then f equals $f_{\rm h}$ on $[0, 1)$ (f equals $f_{\rm e}$ on $[0, +\infty)$) up to a multiplicative constant. Note that this is not true in general if f only satisfies $2-$hyp additivity property (see Example 1 below).

Now, combining Proposition 1 with a constructive approach, among other things, we derive a new proof of the formulae for $d_{\rm e}$ and $d_{\rm h}$ (see Theorem 1 below). In particular, we reduce the problem on finding a solution of the corresponding functional equation.

For our approach it is convenient to consider the E-model and the H-model simultaneously. Let z_1 and z_2 be two arbitrary points in the corresponding model. Then, there exists an isometry φ from the family of isometries of that model such that $\varphi(z_1) = 0$ and $\varphi(z_2) = r$, where $0 \leqslant r < +\infty$ (in the case of the E-model), and $0 \leqslant r < 1$ (in the case of the H-model). Further, from (A1) we will have $d_{\rm e}(z_1, z_2) = d_{\rm e}(\varphi(z_1), \varphi(z_2)) = d_{\rm e}(0, r) = f_{\rm e}(r)$ and $d_{\rm h}(z_1, z_2) = d_{\rm h}(\varphi(z_1), \varphi(z_2)) = d_{\rm h}(0, r) = f_{\rm h}(r)$. Hence, the distance functions $d_{\rm e}$ and $d_{\rm h}$ are determined by $f_{\rm e}$ and $f_{\rm h}$, respectively. Therefore, we find the explicit formulae for the functions $f_{\rm e}$ and $f_{\rm h}$.

First we need some definitions and two preparatory lemmas.

Recall that the point c is the midpoint of a segment with endpoints a and b if c is element of the straight line determined by a and b and if the pair of points (a, c) is congruent to the pair of points (c, b).

If k is a Euclidean circle, recall that by J_k we denote the inversion with respect to k.

By $[a, b]$ we denote segment which endpoints are a and b.

The following lemma is well-known from the basic geometry courses.

Lemma 1. *If an Euclidean circle k is orthogonal on* $\partial\mathbb{D}$ *and if* $J_k(0) = a$*, then common point of the circle k and the H-segment* $[0, a]$ *is the H-midpoint of the H-segment* $[0, a]$.

Let r_1 be a point such that the point $r \in [0, 1)$ is midpoint (in the sense of the corresponding model) of the segment with the endpoints 0 and r_1. Then, regard to the notation above, the following lemma holds.

Lemma 2. *In the E-model holds* $r_1 = 2r$*, whereas in the H-model we have* $r_1 = T_r(r) = \dfrac{2r}{1+r^2}$.

A proof in the case of the E-model we leave to the interested reader and from didactically reasons we give two proofs in the case of the H-model.

Proof. The first proof. By the definition of midpoint it is sufficient to determine an H-isometry φ such that

1° $\varphi(0) = r$;

2° $\varphi(r) > r$.

Really, then r is the H-midpoint of the H-segment $[0, r_1]$, where $r_1 = \varphi(r)$. It is clear that the H-isometry T_r satisfies 1° and 2°. It yields the proof.

The second proof. By the Lemma 1 it suffices to determine an Euclidean circle k with the following properties

1° k is orthogonal on $\partial\mathbb{D}$;

2° $r \in k$;

3° $J_k(0) = r_1$.

An Euclidean circle k, with the center c and of the radius R, satisfies 1° − 3° if and only if $c - R = r$, $1^2 + R^2 = c^2$ and $(c - 0) \cdot (c - r_1) = R^2$ (see Figure 1).

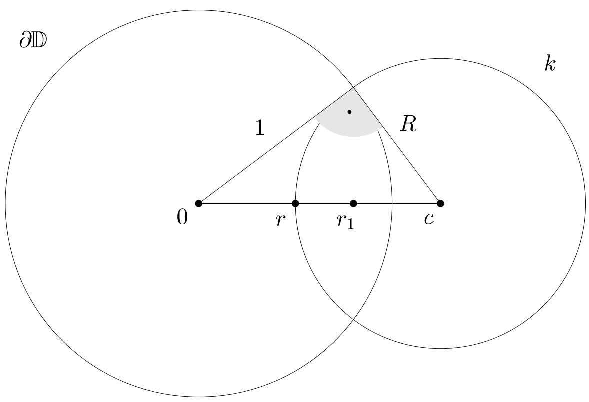

Figure 1: H-midpoint

Hence we have $c = \dfrac{r^2+1}{2r}$, $R = \dfrac{1-r^2}{2r}$ and $r_1 = \dfrac{2r}{1+r^2}$. □

Now, we return to the problem of determining the explicit formulae for the functions f_e and f_h.

Since f_e has 2−euc property and since f_h has 2−hyp property, taking into account that in the corresponding model the segment $[0, r]$ is congruent with the segment $[r, r_1]$, we have $f_e(r_1) = 2f_e(r)$ and $f_h(r_1) = 2f_h(r)$. Then, by using Lemma 2, for all $r \in [0, +\infty)$ we obtain

$$f_e(2r) = 2f_e(r) \tag{3}$$

and for all $r \in [0,1)$ we obtain

$$f_{\mathrm{h}}\left(\frac{2r}{1+r^2}\right) = 2f_{\mathrm{h}}(r). \tag{4}$$

Note that from (A2), for $j \in \{\mathrm{e},\mathrm{h}\}$, the function

(B2) f_j is strictly increasing and continuous function on its domain.

As we know that for all $t \in [0,+\infty)$ holds $\tanh 2t = \dfrac{2\tanh t}{1+\tanh^2 t}$ and if we introduce the substitution $r = \tanh t$, $t \in [0,+\infty)$, in the equality (4), we can get that for all $t \in [0,+\infty)$ holds

$$f_{\mathrm{h}}(\tanh 2t) = 2f_{\mathrm{h}}(\tanh t). \tag{5}$$

Thus, if $\widetilde{f_{\mathrm{h}}} = f_{\mathrm{h}} \circ \tanh$, we get

$$\widetilde{f_{\mathrm{h}}}(2t) = 2\widetilde{f_{\mathrm{h}}}(t), \tag{6}$$

for all $t \in [0,+\infty)$. Note that, taking into account the equations (3) and (6), we can conclude that the functions f_{e} and $\widetilde{f_{\mathrm{h}}}$ satisfy the same equation.

Hence, we are motivated to consider the functional equation

$$F(2t) = 2F(t), \tag{7}$$

where $F : [0,+\infty) \to [0,+\infty)$ is some unknown function. Taking into account the property (B2) we will seek a solution F of the equation (7) which satisfies

(C2) F is strictly increasing and continuous function.

Thus, our consideration is related to the next problem.

Problem 2. *Find all solutions of equation* (7) *which satisfy* (C2).

One could check immediately that for arbitrary constant $c > 0$ the function $F : [0,+\infty) \to [0,+\infty)$, that is given with

$$F(t) = ct, \tag{8}$$

is a solution of the Problem 2.

In the next section we will show that the function given by (8) is not the unique solution of the Problem 2 in general. However, if F is f_{e} or $\widetilde{f_{\mathrm{h}}}$, then F satisfies additional conditions which are corollaries of the Proposition 1. Namely, F satisfies the reduced additivity property and by Theorem 2 we get that all solutions of the Problem 2 are given by $F(t) = ct$, $t \geqslant 0$.

It is interesting that we can avoid an application of the Theorem 2.

For example, in the H-model if we suppose

(H1) $\widetilde{f_h}(t) = ct$, $t \geqslant 0$ and $c > 0$ is some constant,

then using $\widetilde{f_{\mathrm{h}}} = f_{\mathrm{h}} \circ \tanh$, we get $f_{\mathrm{h}}(r) = (\widetilde{f_{\mathrm{h}}} \circ \tanh^{-1})(r) = \dfrac{1}{2}c\ln\dfrac{1+r}{1-r}$, for all $r \in [0,1)$.

If $z_1, z_2 \in \mathbb{D}$, there are H-isometry φ and $r \in [0,1)$ such that $\varphi(z_1) = 0$ and $\varphi(z_2) = r$. Hence, we have $d_{\mathrm{h}}(z_1,z_2) = d_{\mathrm{h}}(0,r) = f_{\mathrm{h}}(r)$. Now, we introduce a function $d : \mathbb{D}\times\mathbb{D} \to [0,+\infty)$ on the following way. For $z_1, z_2 \in \mathbb{D}$ there are H-isometry φ and $r \in [0,1)$ such that $\varphi(z_1) = 0$ and $\varphi(z_2) = r$ and we define $d(z_1,z_2) = f_{\mathrm{h}}(r)$. It is well-known that d is a distance in the H-model which satisfies (A1) and (A2) (see for example [1], [8]). Hence, we conclude the hypothesis (H1) is true.

Thus, we get the formula for d_{h}. In a similar way we can determine $d_{\mathrm{e}}(z_1,z_2)$, for all $z_1, z_2 \in \mathbb{C}$.

If we summarize the above considerations we obtain the following theorem.

Theorem 1. *The distance function, whose existence and uniqueness (up to a multiplicative constant) is established by the Proposition 1, is given by:*

a) (*in the E-model*) $d_e(z_1, z_2) = c|z_1 - z_2|$, *i.e. for all* $z_1, z_2 \in \mathbb{C}$ *we have* $d_e(z_1, z_2) = c|z_1 - z_2|$, *where* $c > 0$ *is some constant.*

b) (*in the H-model*) $d_h(z_1, z_2) = \frac{1}{2} c \ln \frac{1 + \delta(z_1, z_2)}{1 - \delta(z_1, z_2)}$, *i.e. for all* $z_1, z_2 \in \mathbb{D}$ *we have* $d_h(z_1, z_2) = \frac{1}{2} c \ln \frac{1 + \delta(z_1, z_2)}{1 - \delta(z_1, z_2)}$, *where* $c > 0$ *is some constant and* $\delta(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \overline{z_2} z_1} \right|$.

Note that for $c = 1$ the distance function d_e is the ordinary Euclidean distance function in $\mathbb{C}$, whereas for $c = 2$ the distance function d_h is well-known Poincaré's distance function on the disk $\mathbb{D}$.

3.1. Connection with the hyperbolic density

Here we shortly discuss the derivation of the formula b) from the Theorem 1 using a Riemannian metric density which is in our case the hyperbolic density. In courses of complex analysis (for more details see for example [1], [8] and [2]) the usual way to derive the Poincaré distance function on $\mathbb{D}$ is to use the Riemannian metric density function $\rho : \mathbb{D} \to [0, +\infty)$ which is given by the formula

$$\rho(z) = \frac{2}{1 - |z|^2}.$$

The element of length ds (in conformal form) is then determined by

$$ds = \rho(z)|dz|.$$

Thus Poincaré's length of rectifiable curve $\gamma : [\,0, 1] \to \mathbb{D}$ is given by

$$l(\gamma) = \int_\gamma \rho(z)|dz|$$

and the Poincaré's distance of the points $z_1, z_2 \in \mathbb{D}$ is defined by

$$d_h(z_1, z_2) = \inf \int_\gamma \rho(z)|dz|, \tag{9}$$

where the infimum is taken over all rectifiable curves with endpoints z_1 and z_2. Finally from (9) it will holds

$$d_h(z_1, z_2) = \ln \frac{1 + \delta(z_1, z_2)}{1 - \delta(z_1, z_2)}, \tag{10}$$

where $\delta(z_1, z_2) = \left| \frac{z_2 - z_1}{1 - \overline{z_1} z_2} \right|$ is the pseudo-hyperbolic distance. That we leave as an exercise.

Note that we have derived the formula (10) using the construction of the distance function d_h "step by step" and using the interpretation of the Proposition 1 in the H-model.

4. Functional Equations

Now we will consider Problem 2 with more details. As we mentioned in previous section the function $F : [0, +\infty) \to [0, +\infty)$ that is given by $F(t) = ct$, where $c > 0$ is a constant, is not the unique solution of the Problem 2. The next example shows that there are another solutions of the Problem 2.

Example 1. *Let $n \in \mathbb{Z}$. Set $a_n = \frac{1}{2^n}$, $b_n = \frac{a_n + a_{n+1}}{2}$, $I_n = [a_{n+1}, a_n]$ and $0 < c_1 < c < c_2$. Then $\bigcup_{n\in\mathbb{Z}} I_n = (0, +\infty)$. Let $F : [0, +\infty) \to [0, +\infty)$ be a function defined by $F(0) = 0$, $F(a_n) = ca_n$ and, for fixed n, the restriction F_n of F on I_n is a polygonal line, with the slope c_1 on $[a_{n+1}, b_n]$ and the slope c_2 on $[b_n, a_n]$.*

One can show that the function F satisfies the equation (7) and (C2).

It is natural to ask the following questions:

(Q1) What additional conditions (except the condition (C2)) should satisfies the solution of the equation (7) to be unique?

(Q2) Whether those conditions follow from (A1) and (A2), i.e. whether the functions f_e and $\widetilde{f_h}$ satisfy those conditions?

We give the answers to these questions below.

Theorem 2. *Suppose that $F : [0, +\infty) \to [0, +\infty)$ is a function with the following properties:*

1° *F is continuous and strictly increasing;*

2° *$F(2t) = 2F(t)$, for all $t \in [0, +\infty)$;*

3° *$F(3t) = 3F(t)$, for all $t \in [0, +\infty)$.*

Then $F(t) = ct$, for all $t \in [0, +\infty)$, where $c > 0$ is a constant.

Before we give a proof of the Theorem 2 we introduce a lemma (whose proof is left as an exercise) that we will use in the proof of that theorem.

Lemma 3. *The set $\{3^m 2^n \mid m, n \in \mathbb{Z}\}$ is dense in $[\,0, +\infty)$.*

Proof. [Proof of the Theorem 2] From the assumptions 2° and 3° we have $F(3^m 2^n) = 3^m 2^n F(1)$ for all $m, n \in \mathbb{Z}$. Let $G : [\,0, +\infty) \to \mathbb{R}$ be the function defined by $G(t) = tF(1)$ and let $A = \{3^m 2^n \mid m, n \in \mathbb{Z}\}$. For all $t \in A$ we then have $F(t) = G(t)$. By the assumption 1° and since G is continuous on $[\,0, +\infty)$, we get that $F(t) = G(t) = tF(1)$, for all $t \in [\,0, +\infty)$, because they are identical on the dense subset A of $[\,0, +\infty)$. Hence, since F is strictly increasing function it follows that $c = F(1) > F(0) = 0$. □

It is easy to check that the functions f_e and $\widetilde{f_h}$ satisfy assumption 3° from the Theorem 2, i.e. that assumption 3° immediately follows from (A1) and (A2).

Related to the uniqueness of the solution of the equation (7) we get the following proposition.

Proposition 2. *Let $F : [0, +\infty) \to [0, +\infty)$ be a function such that*

1° *$F(2t) = 2F(t)$, for all $t \in [0, +\infty)$;*

2° *the second derivative F'' exists on* $[0, +\infty)$ *and is continuous at* 0.

Then $F(t) = ct$, for all $t \in [0, +\infty)$, where $c > 0$ is a constant.

Proof. From $F(2t) = 2F(t)$, it follows that $2F'(2t) = 2F'(t)$ and therefore $2F''(2t) = F''(t)$ for all $t \in [0, +\infty)$. Hence, $F''(t) = \frac{1}{2^n} F''\left(\frac{t}{2^n}\right)$ and therefore $F''(t) = 0$ for all $t \in [0, +\infty)$. Thus, $F(t) = ct + c_0$. Since $F(0) = 0$, we find that $F(t) = ct$, where $c > 0$ is a constant. □

For functional equations see for example we refer to [7] and the literature cited there.

5. Additivity Properties of Hyperbolic Distance

We advise the interested reader to recall the notation of d_{h} and T_a.

Roughly speaking, in previous sections, in connection with the hyperbolic distance function d_{h}, we have studied the functions which have some additivity property along a particular half line in the H-model.

In this section we study similar problem by using iterations of an H-isometry T_a, where $0 < a < 1$, instead of the k−hyp additivity property.

Let $0 < a < 1$. Then T_a is increasing function on $[0, 1)$ and it maps $[0, 1)$ into itself. Since T_a is the H-isometry and $T_a(0) = a$ then, by the k−hyp additivity property for f_{h}, we have

(AD) $f_{\mathrm{h}}(T_a^k(0)) = k f_{\mathrm{h}}(a)$, for all $k \in \mathbb{N}$.

Note that by T^k we denote $\underbrace{T \circ T \circ \ldots \circ T}_{k}$.

In particular, for $k = 2$, we have $f_{\mathrm{h}}\left(\dfrac{2a}{1 + a^2}\right) = 2 f_{\mathrm{h}}(a)$.

Motivated by property (AD) we say that a strictly increasing continuous function $f : [0, 1) \to [0, +\infty)$ satisfies the reduced additivity iteration property with respect to T_a if

(AD1) $f(T_a^k(0)) = k f(a)$, for all $k \in \mathbb{N}$.

Note that the reduced additivity iteration property is a "discrete variant" of the k−hyp additivity property.

We say that a function f satisfies the reduced additivity iteration property on a set $A \subset (0, 1)$ if f satisfies property (AD1), for all T_a, where $a \in A$.

Note that if A is a finite set and f satisfies the reduced additivity iteration property on A we cannot get any reasonable conclusion about f.

Now, it is a natural question which conditions on A enables us to determine f on whole set $[0, 1)$. The next theorem yields an answer to this question.

Theorem 3. *Let a_n, $n \geqslant 0$, be a decreasing sequence in $(0, 1)$ such that a_{n+1} is the H-midpoint of the segment with endpoints 0 and a_n, and let $A = \{a_n : n \geqslant 0\}$. If $f : [0, 1) \to [0, +\infty)$ is a strictly increasing continuous function satisfies reduced additivity iteration property on A. Then there is a constant $c > 0$ such that $f = c f_{\mathrm{h}}$ on $[0, 1)$.*

Proof. Let $c > 0$ is the constant such that $f(a_0) = c f_{\mathrm{h}}(a_0)$. Let $A_n = \{T_{a_n}^k(0) : k \in \mathbb{N}\}$ be the set of iterations and let $\underline{A} = \bigcup_{n \geqslant 0} A_n$. Since T_{a_n} is an H-isometry and since f satisfies the reduced additivity iteration property with respect to T_{a_n}, for all $n \geqslant 0$, it follows that f equals $c f_{\mathrm{h}}$ on $\underline{A}$. However, since f and f_{h} are continuous and since $\underline{A}$ is dense in $[0, 1)$, f equals $c f_{\mathrm{h}}$ on $[0, 1)$. □

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